



Mathematical Modelling and Controller Design for 3 Degrees of Freedom Helicopter System

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Abstract. This paper presents a comprehensive study on the mathematical modeling and control design for a three degree of freedom helicopter system. The mathematical modeling, linearization, and control design of a three degree of freedom helicopter system are the main focus of this study. The initial phase focuses on system dynamics, beginning with the derivation of the nonlinear equations of motion using fundamental principles captured in free-body diagrams. These equations, which describe the complex, coupled dynamics of the system, are then simplified into a linearized model. This linear model provides the essential framework for subsequent controller synthesis. The second phase involves the design control strategy using ultimate gain technique. Techniques, including the classic Proportional-Integral-Derivative control, the optimal Linear Quadratic Regulator, and the advanced constraint handling Model Predictive Control, will be developed. The performance of these controllers will first be validated via detailed simulations in MATLAB/Simulink before being implemented on the actual hardware.

Keywords: Dynamic system, State space, Ultimate gain, Controller.

1 Introduction

Helicopter dynamics and control have long been a subject of great interest in aerospace engineering and control systems research due to their complex, nonlinear, and coupled nature. Helicopters, in contrast to fixed-wing aircraft, have the ability to move in multiple directions without forward velocity and to hover, take off, and land vertically [1]. These benefits come at the expense of much more complex dynamics, which makes accurate modeling and control a difficult but necessary task. In addition to improving the development of autonomous aerial vehicles, a better understanding of helicopter behavior aids in the creation of reliable and effective control systems that can be applied to a variety of mechatronic and robotic platforms [2]. Simplified laboratory-scale helicopter models, like the Three-Degree-of-Freedom (3DOF) helicopter, are useful resources for investigating basic control concepts in a realistic setting in control engineering research. Pitch, yaw, and elevation are the three independent axes along

which the 3DOF helicopter system normally permits rotational motion [3]. The 3DOF system captures the key features of multivariable coupling and nonlinear behavior found in actual flight systems, despite having a simpler mechanical structure than a fullscale helicopter. Because of this, it is a perfect place to test, evaluate, and validate sophisticated control algorithms before putting them into use on larger or more intricate aerial platforms [4].

One of the most important steps in the design of any control system is creating an accurate mathematical model. These formulas take into consideration the setup's geometric limitations, the forces and moments produced by the propellers, and the effects of gravity. However, the nonlinear model needs to be linearized around an operating point, usually the equilibrium or hover condition, because the majority of traditional control design techniques, rely on linear system representations [5]. Stability analysis and methodical controller design are based on the resulting linear model.

The next stage of the study concentrates on creating and putting into practice appropriate control strategies to attain precise and stable flight performance after the system model has been established. Because they are straightforward and reliable for single-input single-output systems, traditional controllers like proportional-integral-derivative (PID) controllers are frequently utilized. However, more sophisticated control techniques like the Linear Quadratic Regulator (LQR) and Model Predictive Control (MPC) are frequently used to achieve better performance and robustness due to the multivariable and coupled nature of the helicopter dynamics [6]. Prior to experimental implementation, simulation is essential for validating the control laws that have been designed. The derived mathematical models and controllers can be tested in a variety of scenarios to assess their responsiveness, stability, and capacity to reject disturbances using environments like MATLAB/Simulink. The controllers can be installed and adjusted on the actual 3DOF helicopter hardware after simulation results are satisfactory, enabling performance evaluation and real-time validation [7].

The mathematical modeling, linearization, and control design of a three-degree-of-freedom helicopter system are the main topics of this study. Deriving the nonlinear equations of motion, creating a linearized model appropriate for controller design, and creating control algorithms like PID, LQR, and MPC to accomplish accurate and stable motion control are among the goals. The performance of these controllers will be first validated via detailed simulations in MATLAB/Simulink before being implemented on the actual hardware. As a first step in this paper the state space model is derived and demonstrated in for 3 DOF Helicopter system in section III. The ultimate gain method using Zeigler Nichols technique is adopted and Controller gains are designed and simulated. The simulation results are shown in section IV.

2 Mathematical Modelling of 3 DOF Helicopter System

The Three-Degree-of-Freedom (3-DOF) helicopter is a simplified mechanical setup designed to emulate the fundamental dynamic characteristics of a real helicopter. The system features a rigid arm attached to a base, allowing rotation around three axes:

- Pitch (ρ): rotation about the lateral axis of the helicopter body.
- Elevation (ϵ): vertical angular movement of the entire arm assembly.

- Travel (τ): horizontal rotation of the entire system around the vertical axis at the base.

This configuration enables rotational motion along three axes while remaining fixed at a point, thereby eliminating any translational movement.

The helicopter is an underactuated system, meaning the number of control inputs is fewer than the number of degrees of freedom. It operates with two inputs and three outputs:

Inputs (Control Signals):

$$V_1 = F_B + F_F V_1 = F_F + F_B \quad (1)$$

Total thrust generated by the front and back rotors, mainly controlling the elevation angle.

$$V_2 = F_B - F_F V_2 = F_F - F_B \quad (2)$$

Thrust difference between the two rotors, primarily controlling pitch and indirectly influencing travel.

Outputs (Measured States): Pitch angle (ρ), Elevation angle (ε), Travel angle (τ)

Assumptions

1. Small angle linearization will be used for regulation design.
2. Thrust is proportional to command via K.
3. Rigid body with constant inertia and lever arms as given.

Some design parameters are Rise Time: 4 sec, Settling Time: 20 sec, Overshoot: 15%, Steady State Error: 3-5%. The parameters for three degree of freedom Helicopter system is shown in Table 1.

Using Newton–Euler approach, Sum of moments = moment of inertia \times angular acceleration.

Pitch (ρ) Derivation

Differential thrust torque:

$$M = l_p (F_f - F_b) = l_p K_t u_2 \quad (3)$$

Newton–Euler:

$$J_p \rho'' = l_p K_t u_2 \quad (4)$$

Elevation (ε) Derivation

Total thrust torque:

$$M_{thrust} = l_h (F_f + F_b) \cos(\rho) = l_h K_t u_1 \cos(\rho) \quad (5)$$

Gravity torque:

$$M_{grav} = -(m_h g l_c - m_w g l_w) \sin(\varepsilon) \quad (6)$$

Newton–Euler:

$$J_e \varepsilon'' = l_h K_t u_1 \cos(\rho) - (m_h g l_c - m_w g l_w) \sin(\varepsilon) \quad (7)$$

Travel (τ) Derivation

Travel is driven by the horizontal component of total thrust when pitched:

$$M_{travel} = l_h (F_f + F_b) \sin(\rho) \cos(\varepsilon) \quad (8)$$

Newton–Euler:

$$J_t \tau'' = l_h K_t u_1 \sin(\rho) \cos(\varepsilon) \quad (9)$$

All Equations are linearized under the small-angle approximation.

3 State-Space Model

A state-space model is a mathematical representation of a physical system that describes how the system's internal state evolves over time and how it produces outputs based on inputs. It offers a compact and general framework for analyzing and designing systems, particularly those with multiple inputs and outputs (MIMO systems) or systems defined by differential or difference equations. By expressing system dynamics in terms of state variables, inputs, and outputs, the state-space approach provides a powerful and flexible tool for system behavior, designing controllers, and analyzing stability and performance. Equations below represent the state space model for the Helicopter system.

$$\text{State vector } x = [\rho, \rho', \varepsilon, \varepsilon', \tau, \tau']^T, \text{ input } u = [u_1, u_2]^T \quad (10)$$

$$x_1' = x_2 \quad (11)$$

$$x_2' = \left\{ \frac{u_{(diff)} L_H - m_H g L_p \sin(x_1) \cos(x_3) - (k_{(d,\rho)} x_2 + k_{(s,\rho)} \text{sign}(x_2))}{\{l_p\}} \right\} \quad (12)$$

$$x_3' = x_4 \quad (13)$$

$$x_4' = \frac{\{u_{(sum)}L_M \cos\{x_1\} - M(x_3) - (I_r - I_p)x^2 \cos\{x_3\} \sin\{x_3\} - k_{(d,r)}x_6\}}{I_s} \quad (14)$$

$$x_5' = x_6 \quad (15)$$

$$x_6' = \left\{ \frac{-u_{(sum)}L_M \cos\{x_3\} \sin\{x_1\} - k_{(d,r)}x_6}{I_{\tau(x_3)}} \right\} \quad (16)$$

Upon Simplification, we get:

$$x_1' = x_2 \quad (17)$$

$$x_2' = -\beta_p x_2 + \alpha_p u_2 \quad (18)$$

$$x_3' = x_4 \quad (19)$$

$$x_4' = -k_\varepsilon x_3 - \beta_e x_4 + \alpha_e u_1 \quad (20)$$

$$x_5' = x_6 \quad (21)$$

$$x_6' = \alpha_t x_1 - \beta_t x_6 \quad (22)$$

Numeric Coefficient Evaluation

$$\beta_p = 0.053 / 0.094 = 0.56383 \text{ s}^{-1} \quad (23)$$

$$\beta_e = 1.78 / 6.78 = 0.26254 \text{ s}^{-1} \quad (24)$$

$$\beta_t = 0.5 / 6.78 = 0.07375 \text{ s}^{-1} \quad (25)$$

$$m_h l_c = 0.22995, m_w l_w = 3.35442 \quad (26)$$

$$\text{Difference} = -3.12447 \times g = -30.65105 \text{ N} \cdot \text{m} \quad (27)$$

$$k_s = -30.65105 / 6.78 = -4.52080 \text{ s}^{-2} \quad (28)$$

$$k_{lin} = 4.52080 \text{ s}^{-2} \quad (29)$$

$$\alpha_p = 2.13830 \text{ K}_t \quad (30)$$

$$\alpha_e = 0.17257 \text{ K}_t \quad (31)$$

$$\alpha_t = 0.17257 \text{ K}_t u_{10} \quad (32)$$

3.1 PID Design

The Ziegler–Nichols method was used to obtain optimal PID coefficients. The ultimate gain K_u is obtained by increasing proportional gain, till constant oscillations are observed. T_u is measured from this, and other coefficients are calculated as below.

Measured (averaged) values used in this project:

- Pitch: $K_u=0.50, T_u=2.00\text{s}$
- Elevation: $K_u=0.17, T_u=3.00\text{s}$
- Travel: $K_u=0.167, T_u=2.857$

Compute Ziegler–Nichols initial gains

Applied Ziegler–Nichols PID formulas (parallel form):

Ziegler–Nichols:

$$K_p = 0.6K_u \quad (33)$$

$$K_i = 2K_p/T_u \quad (34)$$

$$K_d = K_p T_u / 8 \quad (35)$$

This yields the initial controller set and output of Ziegler–Nichols is shown in Fig. 1 and PID parameters are found in Table 2.

3.2 MATLAB/Simulink Implementation

In Fig. 2, the 3 DOF Helicopter system modelled in MATLAB and Simulink is shown with multi-input and multi-output control system. Fig. 3 shows the MATLAB and Simulink System Equations Modelling block for 3 DOF Helicopter system.

This model implements a closed-loop control system for a Three-Degree-of-Freedom (3DOF) helicopter. It is structured into three main sections: reference input modules, PID controller units, and a nonlinear model that simulates the helicopter's dynamics. This configuration supports testing and analysing the impact on the pitch, elevation, and travel behaviour of the system.

The system receives two separate input setpoints, each representing the target value for pitch, and travel angles while the roll PID is interconnected to the travel PID. These inputs are compared with the actual measured outputs to generate error signals. Each error is then processed by a PID controller responsible for one specific DOF. The controllers continuously modify the motor inputs to reduce the errors and achieve the desired orientation.

The signals from the PID controllers are passed through two intermediary blocks that convert the separate control actions into two combined motor commands: the total thrust ($v_1 + v_2$) and the thrust differential ($v_1 - v_2$). Since the system is underactuated, these two values serve as the only available control inputs for operating the helicopter's actuators.

The motor inputs are fed into a nonlinear plant model that replicates the helicopter's dynamics based on the previously derived equations of motion as represented in Fig. 2 and Fig 3. The model accepts the total thrust ($v_1 + v_2$) and thrust difference ($v_1 - v_2$) as inputs and outputs the pitch, elevation (and travel angles. The plant outputs are looped back into the system to create a closed-loop feedback structure, allowing continuous comparison between the actual states and the target values. Additionally, these outputs are sent to display or scope blocks for real-time observation of the system's behaviour during simulation.

4 Results

A thorough analysis, modelling, and control design framework for a 3-DOF helicopter system is provided by this research. It includes every step of the procedure, from

developing and validating sophisticated control strategies to deriving nonlinear dynamic equations. The study starts by formulating nonlinear equations of motion using free-body diagram analysis and first principles. These are then linearized to produce a simplified state-space representation that can be used in controller design. Building on this model, MATLAB/Simulink is used to design, simulate, and assess a number of control schemes, such as Proportional-Integral-Derivative (PID). The results of this research are depicted below.

The above given pitch angle response to a 51.1° reference shows a rapid rise with a peak overshoot of about 15%, settling to the target within roughly 15 seconds over the 50 second simulation as shown in Fig.4. The system reaches steady state with negligible errors, indicating effective integral action, while the overshoot and smooth decay demonstrate adequate damping from the derivative term.

The travel rate response to a 5400 input shows a very fast rise to the target, reaching close to the desired value within about 3 seconds, with minimal overshoot and smooth convergence as shown in Fig. 5. The curve maintains stability throughout the 50-second simulation and holds steady with negligible error, indicating well-tuned PID parameters that ensure both rapid response and strong steady-state accuracy for the travel control loop.

The roll angle response, indirectly influenced by the travel rate PID, rises quickly to a peak in the low- 40° range with an overshoot of about 10–15% before gradually settling to the high- 30° range within roughly 15 seconds as shown in Fig. 6. This behaviour highlights the coupled dynamics between travel and roll, with the roll control loop providing sufficient damping to maintain stability and eliminate steady-state error.

5 Conclusions

This paper on the modelling, simulation, and control of a 3-DOF helicopter successfully combined theoretical formulation with experimental tuning to achieve a stable, accurate, and responsive control system for pitch, roll, and travel. Beginning with a full derivation of the system's motion equations, a simplified linearised model was developed to capture the key dynamics of the system. This model is used to derive state space model of 3 DOF Helicopter system which is used for controller design. PID gains were determined through measured ultimate gain and oscillation periods, applying the Ziegler–Nichols approach for an initial estimate, and then refined using stability analysis and nonlinear simulations.

The tuned controllers provided quick rise times, moderate overshoot within design limits, and minimal steady-state error, even under axis coupling and nonlinear effects such as actuator constraints and friction. The pitch loop demonstrated precise setpoint tracking, the travel loop ensured smooth rate control, and the roll loop maintained a stable coupled response.

Overall, this work demonstrates that blending experimental measurements with model-based validation can produce reliable controllers for interconnected dynamic systems.

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