A Survey on Cucker–Smale Model and Its Variant

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Abstract. Collective motion is a phenomenon widely observed in biological systems and engineering multi-agent networks, where a large number of individuals achieve coordinated movement through dispersed local interactions. Typical examples include flocks of birds, schools of fish, and swarms of robots. Although experience tells us that collective patterns can be generated in simulations, these cannot explain the mechanisms or conditions under which local interactions produce global coordination, so people began to seek rigorous mathematical models. Understanding how this global order emerges from simple interaction rules has become a central problem in applied mathematics, control theory, and complex systems.

The Cucker-Smale (CS) model provides a fundamental mathematical framework for this phenomenon, describing velocity alignment through distance-dependent pairwise interactions. Since its introduction in 2007, the CS model has inspired numerous theoretical developments, ranging from sophisticated swarm analysis techniques to extended models incorporating stochastic effects, finite sensing range, complex network structures, and geometric constraints.

This article presents a structured survey of the Cucker-Smale model and its major extensions. We first review the classical flocking theory and key analytical approaches, including Lyapunov methods and mean-field limits. We then discuss model refinements motivated by realism, such as noise, cut-off interactions, leader-follower networks, signed graphs, and normalized alignment. Finally, we highlight recent advances in adaptive dynamics and flocking on non-Euclidean manifolds. This article provides a reference for readers seeking an introduction to CS models. By organizing existing results within a unified framework, the article provides a solid foundation for beginners interested in the further study of collective motion.

Keywords: Cucker-Smale model, flocking dynamics, collective movement

1 Introduction

Collective movement is a ubiquitous phenomenon in both nature and technology. The formation flight of birds, the coordinated swimming of schools of fish, and the dense aggregation of insects all demonstrate remarkable patterns of self-organization. In all these examples, large numbers of individuals coordinate their movements without centralized control. Instead, order emerges spontaneously from simple local rules. Similar

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behaviors are widely applied in engineering systems, such as multi-robot formations, distributed sensor networks, and autonomous vehicle fleets. Therefore, understanding the mechanisms behind this phenomenon has become a central problem in applied mathematics, control theory, and complex systems research.

Early research has shown that real-world swarm patterns can be reproduced in simulations by prescribing suitable interaction rules. However, these rules are often empirical in nature and lack rigorous mathematical justification: they do not explain why or under what conditions global coordination emerges. This theoretical gap has motivated the search for mathematical frameworks capable of explaining how coherent collective behavior arises from decentralized, local interactions.

A key step toward such a framework is to formalize what it means for a group to flock. In mathematical terms, flocking refers to a dynamical regime in which agents both align their velocities and remain spatially cohesive. More precisely, consider Nagents with positions $x_i(t) \in \mathbb{R}^d$and velocities $v_i(t) \in \mathbb{R}^d$. The system is said to exhibit flocking if:

- the velocities asymptotically align;
- the spatial diameter of the group remains uniformly bounded in time;

which means the agents remain within a bounded region and do not disperse indefinitely. These two conditions capture the intuitive idea that a flock behaves as a single coherent unit, whos members move together, stay together, and maintain a stable formation over time.

A major breakthrough in the mathematical study of flocking came with the introduction of the Cucker–Smale (CS) model in 2007 [1]. This model provides an elegant system of differential equations that describes how agents adjust their velocities based on pairwise interactions influenced by spatial proximity.

We adopt the normalized classical Cucker–Smale model (with 1/N scaling) as our baseline flocking framework, where the dynamics of N agents are governed by

$$\dot{x}_i(t) = v_i(t), \text{ for } i = 1, \dots, N, \tag{1}$$

$$\dot{v}_i(t) = 1/N \sum_{j=1}^{N} \psi(| x_j(t) - x_i(t) |) \, (v_j(t) - v_i(t)), \tag{2}$$

where $x_i(t)$denotes the position of agent i, $v_i(t)$its velocity, and $\psi: [0, \infty] \to [0, \infty)$is a non-increasing communication weight, which is symmetric and unnormalized: closer agents exert stronger influence, and total influence grows with local density. We may employ the alternative notations (e.g., A_{ij}) in the following part. Normalization via $1/N$ ensures consistent scaling of the velocity update, and we treat unnormalized variants in the following parts.

Cucker and Smale showed that if ψdecays sufficiently slowly (e.g. algebraic decay with exponent $\leq 1/2$), then unconditional flocking occurs for all initial data. By formulating flocking as an ODE system given by distance-dependent alignment rules, the model simplified the complex collective behavior. Its simplicity and symmetry allow for rigorous analytical treatment, while its general structure allows a wide range of modification. As a result, the Cucker–Smale framework provides both a clear conceptual foundation and a flexible starting point for subsequent theory.

The aim of this article is to provide a structured survey of the CS model and its major extensions. We review analytical techniques developed to establish flocking, highlight key model refinements motivated by realistic factors, such as stochasticity, finite sensing, and complex networks, and discuss recent advances in adaptive and geometric settings. Rather than focusing on technical proofs, we emphasize conceptual ideas and mathematical structures that unify the theory across different scales.

2 Modified Cucker–Smale Model

Section 2 reviews major refinements and extensions of the Cucker–Smale model that address analytical simplicity, realism, and structural complexity. We organize the discussion according to three main directions: dissipative reformulations and mean-field limits, stochastic perturbations and finite-range interactions, and network-based extensions capturing asymmetric and heterogeneous influence structures.

2.1 Cucker–Smale Model Refinement

While mathematically groundbreaking, the original Cucker–Smale analysis relies on relatively complicated spectral estimates and treats finite-agent systems as the primary object of study. Ha and Liu (2009) [2] revisited this framework by uncovering the dissipative structure of the particle ODE system, using Lyapunov functionals to prove velocity alignment. Their key insight is that the high-dimensional particle system can be effectively reduced to a pair of scalar quantities measuring the overall spatial spread and velocity fluctuation of the group. By constructing Lyapunov functionals, they show that these quantities satisfy a system of dissipative differential inequalities, from which flocking behavior follows naturally. This approach not only yields a more transparent proof of velocity alignment, but also improves the treatment of the critical decay regime by removing certain conditional assumptions present in the original work. Beyond simplifying the finite-particle theory, [2] established a rigorous mean-field limit. They proved that, as the number of agents tends to infinity, the particle system converges to a Vlasov-type kinetic PDE with alignment dissipation.

In parallel, Zimper et al. (2024) [3] introduced a reduced PDE model formulated solely at the level of spatial densities. Unlike classical continuum descriptions of the Cucker–Smale system that arise from mean-field limits and depend on both position and velocity variables, their model incorporates dimensional reduction directly at the level of empirical densities, leading to a closed system in space alone. This formulation yields a natural and interpretable continuous notion of flocking that remains closely connected to the underlying particle dynamics. Through rigorous analytical results and supporting numerical simulations, the authors demonstrate that the reduced PDE captures velocity alignment and, in certain regimes, allows for a quantitative comparison between solutions of the PDE and the corresponding particle system, thereby contributing to a clearer multiscale understanding of alignment-based interaction models.

2.2 Cucker–Smale Model Considering Stochasticity and Cut-off

However, realistic biological systems seldom follow perfectly deterministic rules: individuals react with delays, make imperfect decisions, and are subject to environmental fluctuations. Moreover, sensing and communication are typically local, an animal or robot does not "feel" the entire group, but only those within a limited perceptual range. These two features: randomness and finite-range interactions—are therefore natural extensions of the classical ODE system.

To model uncertainty and randomness in collective dynamics, recent studies have investigated stochastic Cucker–Smale systems in which noise is incorporated directly into the velocity alignment mechanism. In particular, Han et al. (2024) [4] analyzed a stochastic Cucker–Smale model with multiplicative noise, where random perturbations represent environmental fluctuations and imperfect interactions among agents. The central theoretical question is whether velocity alignment can still be achieved when stochastic disturbances persist over time. Their results show that, under suitable sufficient conditions linking the noise intensity and the communication strength, the stochastic system preserves a dissipative structure and flocking behavior can be ensured in a probabilistic sense. Mathematically, this is established through the construction of Lyapunov functions for stochastic differential equations, which provide explicit probabilistic bounds on velocity fluctuations and guarantee finite-time convergence with high probability. Rather than identifying a sharp threshold beyond which alignment is necessarily lost or enhanced, the analysis characterizes admissible regimes of noise intensity under which coherent collective motion remains robust. These results clarify that stochasticity influences the analysis of flocking movement in a quantifiable way.

Complementing this direction, Jin (2018) [5] focused on a different but equally realistic constraint: limited sensing range. In many biological and engineered swarms, interactions are effectively cut off beyond a certain distance—agents only align with neighbors they can see, hear, or communicate with. Jin modeled this by allowing the communication function to have compact support (or a sharp cut-off), meaning that the interaction graph becomes state-dependent and potentially disconnected: if the group spreads out too much, distant subgroups may stop interacting altogether. This changes the nature of flocking from being primarily an "interaction strength" problem to being a connectivity and geometry problem. Jin's analysis emphasizes that long-term alignment now depends critically on whether the induced interaction network remains sufficiently connected over time. Intuitively, one can think of the swarm as a graph whose edges represent "who can influence whom"; with a cut-off, edges can appear or disappear as agents move. If this graph fragments into multiple components, each component may internally align but the whole population may fail to reach a single consensus velocity. In this way, Jin preserves the classical pairwise alignment of the CS model while introducing geometric and topological constraints.

2.3 Cucker–Smale Model Based on Complex Network

In many realistic collective systems, interactions are not fully symmetric: certain individuals exert a disproportionate influence on the motion of others. This interaction structure is commonly modeled as a leader–follower (or header–follower) network, in which a subset of agents, referred to as leaders (headers), guide the collective behavior, while the remaining followers primarily adjust their states in response to leaders rather than to all neighbors equally (as shown in Figure 1). Such asymmetric interaction patterns naturally arise in biological systems and engineered multi-agent systems. Unlike the classical Cucker–Smale model, which assumes reciprocal interactions, leader–follower networks introduce inherently asymmetric interaction relationships. From a mathematical perspective, these networks are typically represented by directed graphs, where edges encode influence. As a consequence, flocking behavior no longer depends solely on interaction strength or distance decay, but also on graph-theoretic properties such as directionality and connectivity.

For stochastic extensions, we focus on leader–follower settings with multiplicative noise, which align closely with prior work in finite-time flocking analysis. In particular, Shi et al. (2021) [6] establishes finite-time flocking guarantees in a leader–noise framework, where a fixed leader agent drives the system and follower agents are subject to multiplicative noise perturbations. The work derives explicit bounds on the convergence time to flocking, rather than presenting a generic robustness result for arbitrary multiplicative noise, as shown in Fig .1. Their analysis shows that classical consensus results no longer hold universally; instead, the long-time behavior depends on the structural balance of the underlying signed network. Balanced networks can still lead to consensus or clustered alignment, while unbalanced configurations may result in fragmentation or persistent disagreement. This work highlights how flocking dynamics are shaped not only by interaction strength and distance, but also by the sign structure and balance properties of the network, thereby linking flocking theory with modern developments in signed graph theory and social network analysis.

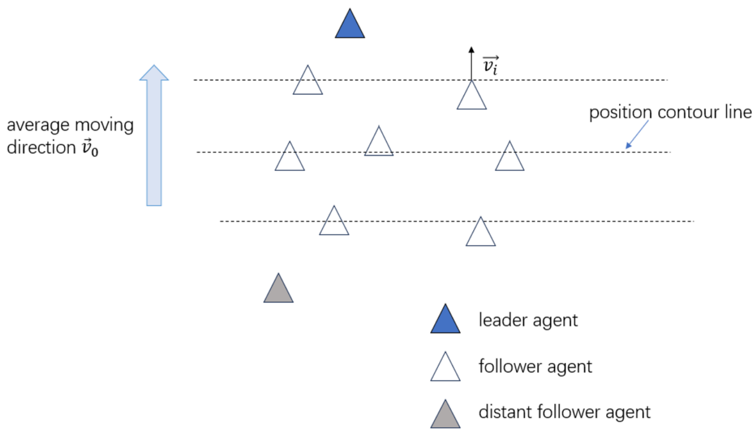

Fig. 1. leader-follower network

In parallel with these network-based developments, Motsch and Tadmor (2011) [7] identified a fundamental limitation of the classical CS model in non-uniform density environments. In the original formulation, interaction weights are symmetric and unnormalized, so agents located in densely populated regions exert disproportionately large influence on those in sparse regions. This can lead to unrealistic dynamics in which isolated agents are rapidly “dragged” by large clusters, even when their local information is limited. To address this issue, Motsch and Tadmor proposed a normalized alignment model, in which each agent averages the velocities of its neighbors after rescaling the interaction weights by the total influence it receives. This normalization ensures that alignment depends on relative rather than absolute interaction strength, thereby restoring a more local and biologically plausible response mechanism. From a mathematical standpoint, this modification alters the structure of the interaction operator, breaking symmetry while preserving dissipative. The authors proved enhanced flocking properties, including faster convergence rates and improved robustness in sparse or highly nonuniform configurations, establishing the normalized model as a direct refinement of the classical CS framework for heterogeneous populations.

Building on this normalized interaction paradigm, Liu, Li, and Xue (2025) [8] further incorporated pattern formation and velocity regulation through pinning control. Their multi-agent model combines three mechanisms: network coupling, spatial formation control, and sparse velocity pinning applied to a subset of agents. Using energy-function methods and gradient-based analysis, the authors show that the system can achieve both a desired spatial configuration and a common target velocity, even in large groups where controlling every agent is impractical. They additionally formulate an optimal control problem by adjusting coupling strengths and employ gradient descent to identify strategies that minimize system cost while accelerating flocking behavior. Numerical simulations confirm the effectiveness and robustness of the proposed sparse control strategies across different environmental and network settings. This work illustrates how CS-type alignment dynamics can be integrated with modern control techniques, bridging theoretical flocking models and practical coordination problems in multi-agent systems.

Beyond forward modeling, an important recent direction concerns the inverse problem of inferring interaction structures from observed trajectories. In this context, Cotil (2024) [9] investigated gregarious animal movement at the herd level with particular emphasis on asymmetric social interactions. Building on generalized Cucker–Smale dynamics, this work analyzes the long-time behavior of systems with weighted and possibly time-dependent interaction matrices, including Markov jump processes and matrix-valued coefficients. Using probabilistic interpretations together with Dobrushin-type inequalities, sufficient and necessary conditions for convergence to coherent collective motion are derived. Cotil further addresses statistical inference of interaction matrices from trajectory data by formulating a constrained least-squares problem with latent individual labels, reducing the inference task to clustering correlated time series. This approach establishes a direct link between flocking theory, network topology, and data-driven inference, demonstrating how CS-type models serve not only as descriptive tools but also as foundations for reconstructing underlying social interaction networks.

Taken together, these network-based extensions reveal that flocking behavior is fundamentally shaped by interaction topology. Beyond metric distance and interaction strength, directionality, sign structure, normalization, adaptivity, and controllability of the underlying graph play decisive roles in determining whether coherent collective motion emerges.

2.4 Cucker–Smale Model in Non-Euclidean Space

Classical Cucker–Smale models are formulated in Euclidean space, where agents evolve in $\mathbb{R}^d$ alignment dynamics rely on linear averaging, global coordinates, and translation-invariant distance measures. In this setting, notions such as distance, velocity comparison, and consensus are straightforward. When extending CS-type models beyond Euclidean space, however, these basic operations must be reconsidered. The absence of global linear structure, together with the presence of curvature and topological constraints, fundamentally changes how alignment, convergence, and collective behaviour are defined and analysed.

Beyond Euclidean space in the physical sense, recent research has explicitly extended flocking and consensus theory to curved configuration spaces, motivated by applications in orientation synchronization, attitude alignment of rigid bodies, and collective motion constrained by geometry. This line of work considers agents evolving on Riemannian manifolds (e.g., spheres, Lie groups) under metric-specific assumptions and other spaces.

In many such systems, agents do not evolve in $\mathbb{R}^d$, but rather on manifolds such as spheres, rotation groups, or more general Riemannian spaces. In these settings, linear averaging and global coordinate representations are no longer available. As a result, notions such as distance, velocity comparison, and alignment must be defined intrinsically through the geometry of the underlying space, typically using geodesic distance and differential geometric tools.

Representative examples along this line include relativistic and second-order extensions on Riemannian manifolds, where curvature enters the dynamics explicitly through differential operators. Ahn, Ha, Kang, and Shim (2021) [10] extended the Cucker–Smale model to a relativistic setting on curved spaces, introducing the manifold RCS model. In this geometric context, the evolution of velocities is defined via covariant derivatives and parallel transport along geodesics, capturing how curvature affects interactions. The manifold RCS model demonstrates that topology and curvature impose fundamental constraints on alignment: antipodal or distant configurations on compact manifolds cannot align continuously without accounting for geodesic structure. Using a Lyapunov functional and LaSalle's invariance principle, Ahn et al. proved the emergence of weak velocity alignment on compact manifolds and analysed explicit cases on the unit sphere and hyperbolic space, highlighting how curvature shapes both asymptotic spatial patterns and alignment dynamics. Their work provides a rigorous framework showing that, even in relativistic or highly curved settings, flocking and consensus phenomena persist, though modified by geometric effects.

These issues are particularly visible on compact manifolds such as S^2, where global notions of direction and averaging can fall into a genuinely topological way. Choi, Kwon, and Seo (2025) [11] study a Cucker–Smale-type flocking model where agents move on the 2-sphere rather than in ordinary Euclidean space. This change of domain matters: on a sphere, agents' velocities lie in different tangent spaces, so there is no single global "straight-line" direction for everyone, and common Euclidean tools (like momentum conservation) generally do not apply. Their model also includes an interparticle bonding force, so agents not only align but are also pulled toward preferred distances. Using an energy dissipation argument and a reformulation that tracks the diameters of positions and velocities, the authors derive a sufficient condition guaranteeing that agent converges and that the spread of positions decays exponentially over time, leading to time-asymptotic flocking.

2.5 Adaptive Cucker–Smale networks

A recent and mathematically refined advance in this direction is due to Kuehn and Yoon (2025) [12], who studied adaptive Cucker–Smale networks from the perspective of time-varying Laplacian dynamics arising in a singular limit. Motivated by adaptive dynamical networks in neuroscience and social opinion dynamics, they proposed a co-evolutionary framework in which interaction strengths evolve in response to agents' states, combining ideas from the Cucker–Smale model with normalization and bounded-confidence mechanisms inspired by the Hegselmann–Krause model. In the regime of fast adaptation, the fully adaptive system reduces to a linear consensus dynamics governed by a time-dependent Laplacian matrix on a temporal graph, allowing the authors to rigorously analyse the asymptotic behaviour of the original nonlinear model.

3 Conclusion and Further Studies

In summary, this review has surveyed the development of alignment-based flocking models centered around the Cucker–Smale framework, which has emerged as a unifying mathematical paradigm for the study of collective motion. Beginning with the original formulation of the Cucker–Smale model and its rigorous flocking theory, we have reviewed subsequent extensions addressing stochastic perturbations, finite sensing ranges, normalization mechanisms, complex interaction networks, adaptive dynamics, and geometric constraints. Together, these works demonstrate how increasingly realistic features can be incorporated into a common modeling structure while preserving mathematical tractability.

Despite significant progress, many unresolved issues remain. Future research directions include cluster theory under multiple interaction perturbations, rigorous analysis of adaptive and data-driven interaction laws, and in-depth exploration of topological and geometric effects in high-dimensional or constrained environments. CS model can also provide structures that can be embedded into neural network architectures used to model collective behavior. From a broader perspective, the Cucker-Smale paradigm remains a fruitful bridge connecting dynamical systems, graph theory, geometry, and applications in biology and engineering.

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