Rasch Meta-Metres of Growth in Reading and Mathematics Attainment at Both Population and Individual Levels

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Abstract. Although hardly known, Georg Rasch had an approach to studying growth based on the principle of invariant comparisons, a principle for which he is well known for his models for measurement. The approach identifies a non-linear function of time, called a meta-metre, which governs the growth of all individuals of a population. Then within the meta-metre, each individual's rate of growth is linear and invariant. The approach, which focuses on the rate of growth, is analogous to that of constructing relationships in the natural sciences. This paper illustrates the approach with the educationally important variables of reading and mathematics attainment tests from two longitudinal studies showing that when the rate of growth has been identified for each individual, comparisons among subgroups of a population can be made using standard statistical procedures. Both meta-metres characterise early rapid, decelerating growth, which when related to the common grade scale, show that any grade difference between groups in the early years inevitably increases in later years. This increase has implications for interventions for groups at risk in their attainments.

Keywords: Rasch meta-metre, reading growth, mathematics growth.

1 Introduction

In the case that measurements of educational and psychological attainment variables have been placed on a scale with a common origin and unit across the relevant range of the variable, many researchers have quantified the relationship between growth on the variable and age or grade. This quantification is in the form of a regression equation (e.g. Shanley, 2016; Shin, et. al., 2013; Williamson, 2018, Lee, 2012). In particular, Williamson (2018) described growth satisfactorily with a quadratic function. Here, following Rasch (1977), we attempt to be more explanatory rather than descriptive, with a mathematical formalisation of a growth function, which focuses on the rate of growth. For illustrative purposes, we characterise the growth on two publicly available and educationally important longitudinal data sets, one in reading, the other in mathematics, attainment.

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Rasch's approach to studying growth on a physiological variable has the same principle of invariance of comparisons as his theory of measurement in the social sciences, and for which he is best known (Rasch, 1960; 1961; Wright, 1997; Andrich, 2018). This involves, either explicitly or implicitly, transforming the relevant variables over the relevant range such that their relationships are linear. The importance of linearity in characterising growth is that the rate of growth is then invariant over the relevant range. Invariant comparisons, as highlighted by Rasch, is a characteristic of a specific kind of objectivity and is the hallmark of relationships among measured variables constructed in the natural sciences.

Rasch's approach entails finding a function of time in which four principles hold. First, that if it can be identified, there is a common quantitative mode of growth, called a meta-metre of the time scale. This mode governs the growth of all individuals in the relevant class or population of individuals. Second, the common rate of growth is proportional to a meta-metre of time. Third, the rate of growth of every individual is linear in the meta-metre and therefore invariant over the range. Fourth, that although governed by a single equation for the population of individuals, there are individual differences in the rate of growth. In addition to Rasch's application to the approach to data from the increase in weight of calves (Olsen, 2003), Rao (1958) applied the approach to growth in weight of babies in their first year.

If the approach can be applied, it has several advantages in the study of the rate of growth. First, individuals and groups with varying relevant background characteristics can be compared on a single numerical value. Second, in formalising statistical comparisons among groups, standard procedures such as analysis of variance can be applied. Third, the characterisation is relatively simple, also a feature of laws in the physical sciences, where despite underlying processes being complex, the relationships among variables are almost invariably a simple multiplicative, or equivalent, functions of the variables (Ramsay, 1975). These features are shown in the characterisation of growth on Reading and Mathematics tests analysed.

A feature of examples involving physiological variables, and weight (mass) in particular, is that the measured variable has a natural origin of zero. This is relevant because Rasch's rationale led to taking the logarithms of weight. Measurements intellectual variables, such as those of intelligence and attainment tests, have an arbitrary origin. Provided that the measurements are in a constant unit across the relevant range of the variable, Andrich, Marais and Sappl (2023) showed that the arbitrary origin can be overcome readily. In this paper, the responses to the tests were analysed by methods of modern test theory, including Rasch models, thus providing measurements with a constant unit across the continuum, commonly referred as interval level measurement (Stevens, 1946).

The rest of the paper is structured as follows. Section 2 summarises the form of Rasch's a meta-metre for growth in weight; Section 3 applies this form for formalising analogous growth on social science variables; Section 4 provides estimation equations

for the relevant parameters at the population, subgroup, and individual levels; and Sections 5 and 6 show results of analyses of growth on a Reading and Mathematics test respectively. Section 7 is a summary of implications of decelerating growth on a quantitative scale when referenced to the common grade scale.

2 Formalisation of a Meta-Metre for Growth in Weight

From Rasch's interaction with physiological growth data, there were two pieces of information he wanted to accommodate in formalising growth. First, his rationale was that because growth in weight (mass) involves the splitting of cells, growth had to be a function of the current number of cells. Moreover, this rationale implies exponential growth (Olsen, 2003, p. 60). Therefore, for efficiency he considered growth in the logarithm of weight. To provide a linear function of growth, Rasch required an equation of the form

$$ln(Y_{nt}) = a_n + b_n\tau(t)\,, \tag{1}$$

where Y_{nt} is the measurement of weight of individual n measured at times t, b_n is the rate of growth of individual n, a_n is a constant which may be the measurement at time $t = 1$, and $\tau(t)$ is some *transformation of time*. Importantly, $\tau(t)$ is general – it governs, and is the mode of growth of all individuals in the relevant population within which there are individual differences. It is this function that Rasch referred to as a *meta-metre* of time. If it can be identified, it not only characterises linear growth, but the form of the meta-metre function is of interest in its own right.

Second, Rasch was aware that growth decelerated. Accommodating deceleration, led to specifying $\tau(t) = \ln(t + \lambda)$ resulting in

$$\ln(Y_{nt}) = a_n + b_n \ln(t + \lambda)\,. \tag{2}$$

where λ is a real valued constant to be estimated.

This paper does not detail Rasch's development of Eq. (2), which is provided in Olsen (2003), Rao (1958), Rasch (1977) and Andrich, et. al. (2023). Its purpose here is to provide a basis for an analogous approach to the study of growth in reading and mathematics attainment.

3 Formalisation of a Meta-Metre for Growth for a Social Measurement Variable

The hypothesis of a meta-metre for intellectual variables is by analogy with Rasch's rationale but with modifications to accommodate an arbitrary origin. Again, let Y_{nt} be the measurement of individual n measured at times t. Because Y is a variable with an arbitrary origin, it is not possible to assume that logarithms can be accommodated. Therefore, beginning with the measurement Y_{nt}, by analogy to Rasch's requirement for the growth in the logarithm of weight, we require that there exists a linear transformation of Y_{nt}, for example $Z_{nt} = g + hY_{nt}$ where $(g,\ h)$ are real valued constants, that gives

$$Z_{nt} = c_n + d_n\tau^*(t), \tag{3}$$

where $\tau^*(t)$ is a meta-metre of time with individual differences, c_n, d_n, in the intercept and the rate of growth respectively.

To identify an equation analogous to Eq. (2), two features need to be accommodated. First, measurements are in the observed variable Y. Second, by direct analogy to growth in weight, there is a wealth of evidence that growth on intellectual variables also decelerates (e.g. Lee, 2012; Shanley, 2016; Williamson, 2018, Rescorla & Rosenthal, 2004; Cameron, et. al. 2015, Bock, 1983) that needs to be accommodated. Appendix 1 summarises the derivation of the equation of growth which accommodates these two features resulting in a general growth equation

$$Y_{nt} = a_n + b_n\tau(t), \tag{4}$$

and the specific growth equation

$$Y_{nt} = a_n + b_n \ln(t + \lambda) \tag{5}$$

where $\tau(t) = (t + \lambda)$ and λ is a constant to be estimated. The parameter λ accommodates an arbitrary origin and reflects the rate of deceleration – the greater the value, the smaller the deceleration. Eq. (5) has exactly the form of Rasch's Eq. (2) except that the dependent variable is Y_{nt}, the measurement, not $\ln Y_{nt}$. It is stressed that the rationale leading to Eq. (5) is a hypothesis regarding growth that provides the basis for an empirical test that the meta-metre $\tau(t)$ is common for all members of the population.

4 Estimation of Parameters λ, b, a

To estimate the relevant parameters, we consider the estimation of the population parameter λ first. For this purpose, we take advantage of a distinctive property of Eq. (5), that of dynamic consistency (Keats, 1982).

4.1 Dynamic Consistency

Dynamic consistency ensures that the function for the growth of an individual is the same as the function for the growth in means of the population or any subgroup. Thus, taking means of measurements of individuals in Eq. (5) gives

$$\begin{aligned}\bar{Y}_{.t} = (\sum_{n=1}^{N} Y_{nt}) / N &= \sum_{n=1}^{N} [a_n + b_n \tau(t)] / N \\ &= \sum_{n=1}^{N} [a_n + b_n \ln(t+\lambda)] / N \\ &= \bar{a}_{.} + \bar{b}_{.} \ln(t+\lambda)\end{aligned} \tag{6}$$

where "." indicates the index over which the summation to form the mean, and $\bar{a}_{.}$, $\bar{b}_{.}$ are respective means of a_n, b_n, $n = 1,2,3,...,N$ in the population.

The growth in the means of any subgroup g is of the same form:

$$Y_{.gt} = \bar{a}_{.g} + \bar{b}_{.g} \ln(t+\lambda). \tag{7}$$

4.2 Least Squares Estimates of Population Parameters $(\lambda, \bar{b}_{.}, \bar{a}_{.})$

For completeness, and to ensure accurate implementation of the estimation equations, we elaborate the variable of time to distinguish between the occasion of measurement and the value of the time variable. Thus let θ_t be the values of the measurements at times $t = 1,2,3,...,T$. For example, five measurements might be taken at the students' ages of $5, 8, 10, 12, 15$, giving $\theta_t = 5, 8, 10, 12, 15,\ t = 1,2,3,.4,5.$

Following Rao (1958) we use least squares estimation. Andrich, Marais & Sappl (2023) also used the least squares method of estimation, but it is somewhat different and less efficient from the method summarised below.

In the elaborated notation, from Eq. (6), $\overline{Y}_{.t} = \overline{a}_{.} + \overline{b}_{.}\tau(\theta_t)$. Let $\overline{y}_{.t} = \overline{Y}_{.t} - \overline{Y}_{.t-1}$ be the differences in successive means. Then

$$\begin{aligned}\overline{y}_{.t} &= \overline{a}_{.} + \overline{b}_{.}\tau(\theta_t) - [\overline{a}_{.} + \overline{b}_{.}\tau(\theta_{t-1})],\ t = 2,3,...,T,\\ &= \overline{b}_{.}[\tau(\theta_t) - \tau(\theta_{t-1})].\\ &= \overline{b}_{.}\tau_{\theta t},\end{aligned} \tag{8}$$

where $\tau_{\theta t} = [\tau(\theta_t) - \tau(\theta_{t-1})]$. Note that the additive constant $\overline{a}_{.}$ is eliminated.

Let $(\hat{\lambda}, \hat{\overline{b}}_{.})$ be respective estimates of $(\lambda, \overline{b}_{.})$. Then

$$\hat{\tau}_{\theta t} = [\hat{\tau}(\theta_t) - \hat{\tau}(\theta_{t-1})] = [\ln(\theta_t + \hat{\lambda}) - \ln(\theta_{t-1} + \hat{\lambda})]. \tag{9}$$

For a chosen values $(\hat{\lambda}, \hat{\overline{b}}_{.})$, the predicted $\hat{\overline{y}}_{.t}$ difference is given by $\hat{\overline{y}}_{.t} = \hat{\overline{b}}_{.}\hat{\tau}_{\theta t}$ and the sum of squares, S, of the differences between the observed values $\hat{\overline{y}}_{.t}$ and predicted values $\hat{\overline{y}}_{.t}$ is given by

$$S = \sum_{t=2}^{T} (\overline{y}_{.t} - \hat{\overline{b}}_{.}\hat{\tau}_{\theta t})^2. \tag{10}$$

Derivations of the two least squares estimation equations are shown in Appendix 2. These are based on partially differentiating Eq. (10) with respect to each of $(\hat{\lambda}, \hat{\overline{b}}_{.})$ and setting each equation to 0. The resulting estimation equations are

$$\hat{\overline{b}}_{.} = \sum_{t=2}^{T} \overline{y}_{.t}\hat{\tau}_{\theta t} \Big/ \sum_{t=2}^{T} \hat{\tau}_{\theta t}^2; \tag{11}$$

$$\hat{\lambda}: \quad \sum_{t=2}^{T} (\overline{y}_{.t} - \overline{b}_{.}\hat{\tau}_{\theta t}) \left[\frac{1}{(\theta_t + \hat{\lambda})} - \frac{1}{(\theta_{t-1} + \hat{\lambda})} \right] = 0. \tag{12}$$

Eqs. (11) and (12) are solved iteratively. Specifically, an initial value $\hat{\lambda}^{(0)}$ is chosen, inserted in Eq. (11) to obtain $\hat{\overline{b}}_{.}^{(0)}$. Both are inserted in Eq. (12) to check its deviation from 0. If the value is different from 0, as it is almost certain to be with the initial value, an updated value of $\hat{\lambda}^{(1)}$ is obtained followed by an updated value of $\hat{\overline{b}}_{.}^{(1)}$. The process is repeated till Eq. (12) is satisfied to any chosen number of decimal places. For estimates of the Reading and Mathematics tests for this paper, Eq. (12) was satisfied to 4 decimal places.

Given $(\hat{\lambda}, \hat{\bar{b}}_{\cdot})$, $\hat{\bar{a}}_{\cdot}$ can be obtained as a standard regression estimate, given by

$$\hat{\bar{a}}_{\cdot} = \bar{Y}_{\cdot\cdot} - \hat{\bar{b}}_{\cdot}\bar{\tau}(\theta_{\cdot}) \tag{13}$$

where $\bar{\tau}(\theta_{\cdot}) = [\sum_{t=1}^{T} \tau(\theta_t)]/T$ is the mean of the meta-metre values at times t, $t = 1,2,...,T$.

4.3 Estimation of Subgroup Parameters $(\bar{b}_{\cdot g}, \bar{a}_{\cdot g})$

Given the population estimate $\hat{\lambda}$, estimates of the mean rate of growth $\bar{b}_{\cdot g}$ of some subgroup, g is given by the same form as Eq. (11), that is

$$\hat{\bar{b}}_{\cdot} = \sum_{t=2}^{T} \bar{y}_{\cdot gt}\hat{\tau}_{\theta t} / \sum_{gt=2}^{T} \hat{\tau}^2_{\theta t} \tag{14}$$

where $\bar{y}_{\cdot gt} = \bar{Y}_{\cdot gt} - \bar{Y}_{\cdot gt-1}$, $t = 2,3,...,T$ are successive differences of the means. Then

$$\hat{\bar{a}}_{\cdot g} = \bar{Y}_{\cdot g\cdot} - \hat{\bar{b}}_{\cdot g}\bar{\tau}(\theta_{\cdot}). \tag{15}$$

4.4 Estimation of Individual Parameters (b_n, a_n)

Given the population estimate $\hat{\lambda}$, estimates of the rate of growth of individual b_n is also given by the form of Eq. (11), that is

$$\hat{b}_n = \sum_{t=2}^{T} y_{nt}\hat{\tau}_{\theta t} / \sum_{gt=2}^{T} \hat{\tau}^2_{\theta t} \tag{16}$$

where $y_{nt} = Y_{nt} - Y_{nt-1}$, $t = 2,3,...,T.$

Then

$$\bar{a}_n = \bar{Y}_{n\cdot} - \hat{\bar{b}}_{n\cdot}\bar{\tau}(\theta_{\cdot}). \tag{17}$$

For the purposes of this paper, the rate of growth and intercept means of subgroups of males and females are obtained by finding the means from Eqs. (16) and (17). However, if individual measurements are not available, but only subgroup means, then Eqs. (14) and (15) can be used.

4.5 Quality of the Estimates

Given that the growth is expected to be linear in a meta-metre, the quality of the estimates can be estimated with the standard regression correlation. Thus, for the population, given the estimates $(\hat{\lambda}, \hat{\bar{b}}_{\cdot}, \hat{\bar{a}}_{\cdot})$ predicted measurements $\hat{\bar{Y}}_{\cdot t} = \hat{\bar{a}}_{\cdot} + \hat{\bar{b}}_{\cdot}(\theta_t)$ can be compared with the observed ones to form

$$R^2 = 1 - \sum_{t=1}^{T} (\bar{Y}_{\cdot t} - \hat{\bar{Y}}_{\cdot t})^2 / \sum_{t=1}^{T} (\bar{Y}_{\cdot t} - \hat{\bar{Y}}_{\cdot \cdot})^2 , \tag{18}$$

which is the standard form for the proportion of variance that is accounted for by the linear equation. Analogous values can be obtained for each subgroup g from $\hat{\bar{Y}}_{\cdot gt} = \hat{\bar{a}}_{g\cdot} + \hat{\bar{b}}_{g\cdot}(\theta_t)$ and for each individual from $\hat{Y}_{nt} = \hat{a}_n + \hat{b}_n(\theta_t)$.

5 Analysis of Growth on a Reading Test

The reading test data analysed in this paper are publicly available, and are from the National Longitudinal Survey of Youth – Children and Young Adults (NLSY) in which the measurements were taken in the Spring every two years between 1996 and 2006 in Grades 1 to 12 (Center for Human Resource Research, 2004). The measurements analysed are from vertically equated tests across grades 1,3, 5, 7 and 9. The sample contained considerable missing data resulting in only 234 students who had all five measurements, though not in the same years. Of these, 133 were female and 101 were male. This is clearly a small sample, but it is used illustratively to show that even with the small sample, the method is successful. The second example has a much larger sample. Although the sample contains identification of three groups according to race, we compare the rate of growth only for two subgroups based on gender.

However, of particular interest is the possibility of comparing the rate of growth of students who were present for all five measurements with that of students who missed at least one measurement. Therefore, in addition to the means and standard deviations of students who had completed data, Table 1 also shows the means and standard deviations of those students with at least one measurement missing.

For the variable of time, we take simply the grades in which the students were measured, that is, $\theta_t = 1, 3, 5, 7, 9; t = 1,2,3,4,5.$ Table 1 shows the descriptive statistics for the sample with complete data, disaggregated by gender.

Table 2 shows the estimated value of λ for the total sample together with the means and standard deviations of the parameters b, a disaggregated by gender. It also shows the value of R^2 for the means of the total group and the mean and standard deviation of this value for individuals.

Figure 1 shows graphically the decelerating relationship of the observed means $\overline{Y}_{.t}$ as a function of $\theta_t + \hat{\lambda}$ (Panel 1), and the linear relationship as a function of the meta-metre $\ln(\theta_t + \hat{\lambda})$. Both the statistical information in Table 2 and the graphical depiction in Figure 1 indicate that the meta-metre is extremely well estimated for the total group.

Figure 2, Panel 1, shows analogous graphical information to that of Figure 1 Panel 2, disaggregated by gender. The statistics in Table 2 and the graphs in Figure 2 confirm that the meta-metre provides an accurate depiction of the respective growth for both males and females. In addition, it is evident without formal statistical analyses, that the rate of growth is equivalent for the two genders, and that the intercept is not very different. However, for completeness, Table 3 shows a one-way analysis of variance (ANOVA) showing no significant difference at the 5% level for the difference in means in the rate of growth $\overline{b}_{.}$, with a marginally significant difference in favour of the females for the intercept $\overline{a}_{.}$.

It is possible to estimate the rate of growth, and intercept, of those with missing responses by using only the means of the groups, that is Eqs. (14) and 15. Given the estimate of the meta-metre from the means of students with all five measurements, Figure 2 (Panel 2) shows the rate of growth of this group relative to that of the group with complete data. Table 4 reproduces the estimates for the rate of growth and the intercept from these students with at least one measurement missing. It is evident that the meta-metre is more than satisfactory to characterise the growth of this latter group of students ($R^2 = 0.9971$). Assuming the rate of growth and intercept of the large group with missing measurements as the population estimate, and using the statistics from the sample with complete data, Table 4 also shows the t *test* values for a significant difference between their respective means. It is evident that the group with complete data, and at a 5% level of significance, has a marginally significantly better intercept, and almost marginally significantly better rate of growth. This difference suggests that missing at least one measurement, but maybe more, is a symptom reflecting negatively on overall achievement and rate of growth.

In addition to graphical depiction of the growth in means, individual growth curves can be instructive. Figure 3 shows graphically the rate of growth of selected individuals. Although a linear growth in the meta-metre for these individuals is excellent as shown by their R^2 values, because of much larger individual measurement error compared to that of a group, as well as idiosyncrasies of individuals growth between times of measurement, the observed measurements are not as close to the regression line as for the means of the groups.

Table 1. NLSY reading mean (standard deviation) for students with complete data disaggregated by gender and those with at least one measurement missing.

Mean (Standard Deviation)		θ_t : Grade					$\bar{Y}_{..}$
Sample	N	1	3	5	7	9	
Total	234	19.12	36.91	47.18	55.92	60.72	43.97
		(4.52)	(9.25)	(9.93)	10.37	(10.56)	
Male	101	18.23	35.98	46.37	55.98	60.72	43.46
		(4.30)	(9.47)	(9.96)	(10.16)	(10.44)	
Female	133	19.79	37.62	47.80	55.88	60.72	44.36
		(4.58)	(9.05)	(9.90)	(10.56)	(10.69	
Missing*	N	3584	3599	3498	3380	2245	
		17.48	33.58	45.06	53.05	59.06	40.05
		(5.67)	(9.67)	(10.52)	(12.24)	(11.58)	

*At least one measurement missing

Table 2. NLSY reading estimate of λ and the means and standard deviations of the rate of growth b and the intercept a disaggregated by gender for students with all measurements and estimates of the rate of growth and intercept for the means of students with at least one measurement missing.

Meta-metre from the total group means complete data: $\hat{\lambda} = 1.0852$

$$\hat{\bar{b}}_{\cdot} = 26.4658, \hat{\bar{a}}_{\cdot} = -0.2234, R^2 = 0.9992$$

	Mean (Standard Deviation) from individual estimates		
Sample	$\hat{\bar{b}}_{\cdot}$	$\hat{\bar{a}}_{\cdot}$	R^2
Total	26.4658	-0.2234	0.9294
	(6.8493)	(6.8835)	(0.0685)
Male	26.8046	-1.3027	0.9286
	(7.0088)	(6.9845)	(0.0671)
Female	26.2086	0.5962	0.9299
	(6.8038)	(6.7069)	(0.0695)
Incomplete*	25.6518	-1.1891	0.9971

*At least one measurement missing

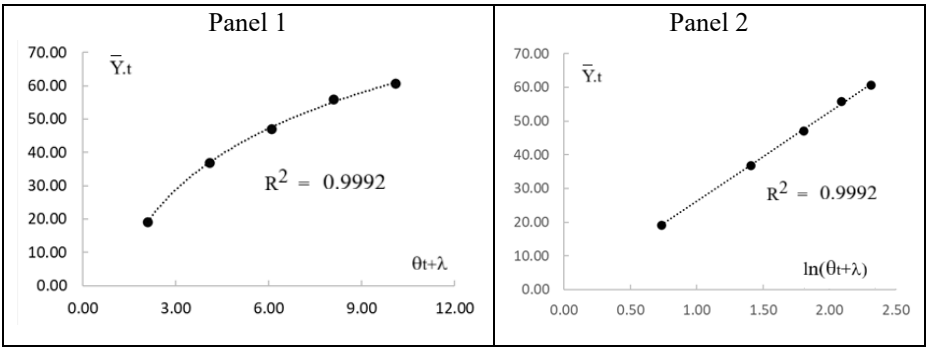


Fig 1. Means as a function of $\theta_t + \hat{\lambda}$ (Panel 1) and meta-metre $\ln(\theta_t + \hat{\lambda}),\ \hat{\lambda} = 1.0852$ (Panel 2).

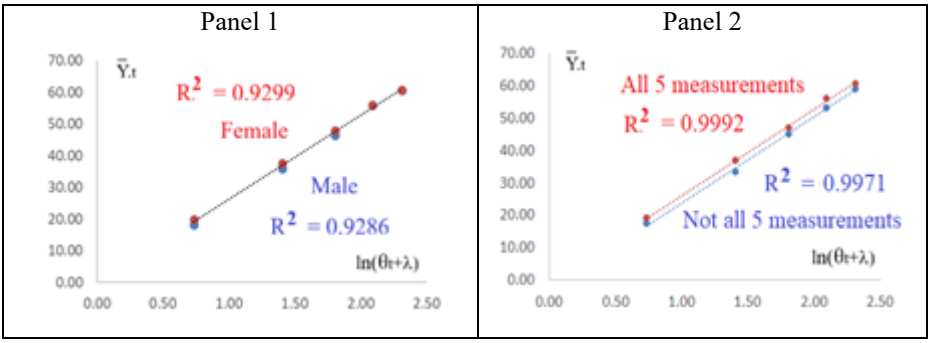


Fig 2. Means disaggregated by gender as a function of meta-metre $\ln(\theta_t + \hat{\lambda}),\ \hat{\lambda} = 1.0852$ (Panel 1); means of students with all 5, and at least one measurement missing (Panel 2)

Table 3. ANOVA analysis for mean differences between genders for the rate of growth and the intercept.

	$MS_{between}$	MS_{within}	df	F	p
$\bar{b}_.$	20.3905	47.0270	1, 232	0.4336	0.5109
$\bar{a}_.$	206.9976	46.6948	1, 232	4.4330	0.0363*

*Marginally significant at the 5% level

Table 4. $t\ test$ comparisons for the rate of growth and intercept between those with complete data and those with at least one measurement missing.

Data	$\hat{\bar{b}}_{\cdot}$	$\hat{\bar{a}}_{\cdot}$
Complete*	26.4658	-0.2234
Missing**	25.6518	-1.1891
$t\ test$	1.7717	2.0390
p	0.0777	0.0426(1)

*All five measurements present; **At least one of five measurements missing; (1)Marginally significant at the 5% level

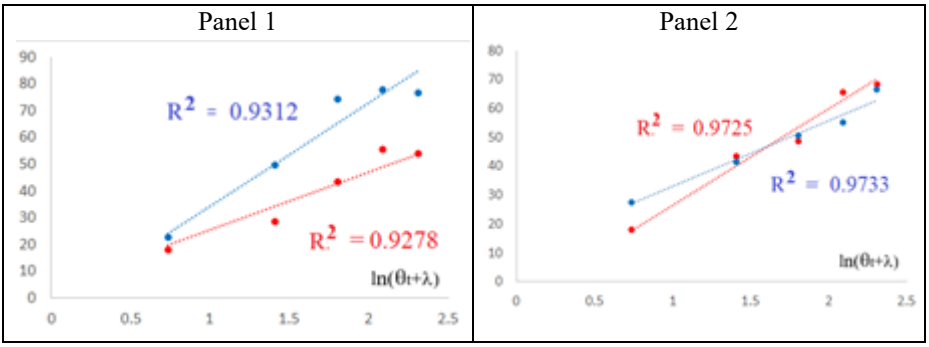


Fig 3. Two individuals with similar initial measurements but different rates of growth $(b = 21.7600, 38.2606)$ (Panel 1); two individuals with different initial measurements and different rates of growth ($(b = 22.7118, 33.2606)$ (Panel 2).

6 Analysis of Growth on a Mathematics Test

The data for the study of growth in attainment in mathematics come from the Early Childhood Longitudinal Study - Kindergarten Class [ECLS-K] (Tourangeau, Nord, Le, Pollack, & Atkins-Burnett, 2006), which are publicly available. The sample is a nationally representative consisting of 21260 children who entered kindergarten in 1998-1999. The assessments were administered in the Spring of Kindergarten, and then in Grades 1, 3, 5, and 8 and contain considerable amounts of missing data. However, 8339 students have scores on all five grades, which in contrast to that from the reading test, is a very large sample. These are the measurements, constructed by vertical scaling using modern test theory, analysed in this paper. Also, unlike the reading test, the mathematics assessments were not in successively equidistant grades with the first measurement in Kindergarten and then Grades 1, 3, 5, and 8. Although the data contain groups identified by race, this paper considers again only two gender subgroups.

For the variable of time, we again take the grades as time measurements, but we begin with setting Kindergarten as time 0, that is,

$\theta_t = 0(K), 1, 3, 5, 8; t = 1,2,3,4,5$ for the analysis. The initial time could have been $K = 1$ giving $\theta_t = 1(K), 2, 4, 6, 0; t = 1,2,3,4,5$, in which case the estimate of λ would be less by 1.

Table 5 shows the descriptive statistics for the ECLS-K sample with complete data, disaggregated by gender. It also shows these descriptive statistics for the group which has at least one measurement missing.

Table 5. Descriptive statistics for the ECLS-K sample with complete data, disaggregated by gender.

Mean (Standard Deviation)		θ_t : Grade					$\bar{Y}_{..}$
Sample	N	0	1	3	5	8	
	8339	-0.61	0.12	0.77	1.16	1.49	0.49
		(0.44)	(0.39)	(0.38)	(0.40)	(0.44)	
Male	4157	-0.60	0.14	0.81	1.21	1.52	0.62
		(0.47)	(0.41)	(0.39)	(0.40)	(0.45)	
Female	4182	-0.62	0.10	0.73	1.12	1.46	0.56
		(0.42)	(0.37)	(0.36)	(0.38)	(0.42)	
Missing*	N	11,310	8302	6035	2935	946	
		-0.74	0.01	0.65	1.01	1.36	0.00
		(0.46)	(0.44)	(0.39	(0.41)	(0.46)	

*At least one measurement missing

Table 6 shows the estimated value of λ from the ECLS-K sample with all five measurements, together with the means and standard deviations of the parameters b, a disaggregated by gender. It also shows the R^2 value for the means of the total group and the mean and standard deviation of this value for individuals. Figure 4 shows graphically the decelerating relationship of the observed means $\bar{Y}_{.t}$ as a function of $\theta_t + \hat{\lambda}$ (Panel 1), and the linear relationship as a function of the meta-metre $\ln(\theta_t + \hat{\lambda})$. Both the statistical information in Table 6 and the graphical depiction in Figure 4 indicate that, as with the Reading test, the meta-metre is extremely well estimated for the total group.

Table 7 shows the ANOVA for gender differences respectively for the rates of growth and intercepts (b, a), and primarily because of the large sample size, they are both significant, slightly favouring the males.

Applying *a t test*, as for the Reading data, Table 8 also shows a marginally significant difference in the rate of growth in favour of the group with a measurement missing compared to those with all five measurements. However, that group has a significantly worse intercept, suggesting again, that at least one measurement missing reflects a disadvantage in achievement. This may result for many reasons, including a symptom of missing more schooling and therefore missing out on teaching, or some other compounding factors such as health which might also affect school attendance.

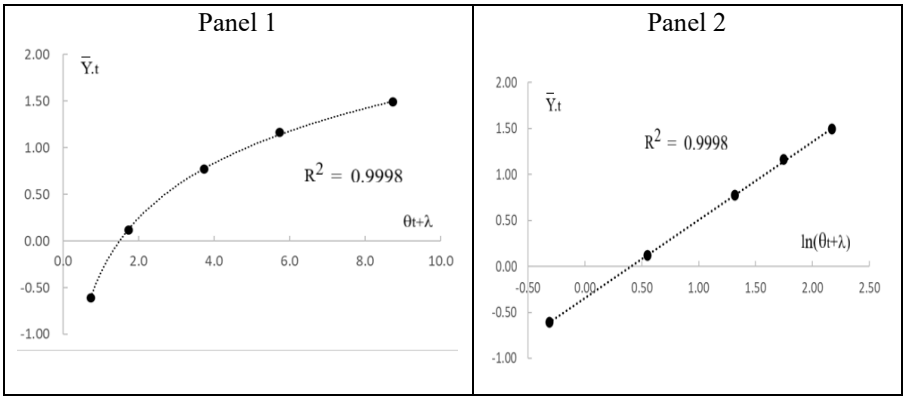
Table 6. ECLS-K mathematics estimate of λ and the means and standard deviations of the rate of growth b and the intercept a disaggregated by gender for students with all measurements and estimates from means of the rate of growth and intercept for students with at least one measurement missing.

Meta-metre from the ECLS-K total group means complete data: $\hat{\lambda} = 0.7325$

$$\hat{\bar{b}}_{\cdot} = 0.8481, \hat{\bar{a}}_{\cdot} = -0.3412, R^2 = 0.9998$$

	Mean (Standard Deviation) from individual estimates		
Sample	$\hat{\bar{b}}_{\cdot}$	$\hat{\bar{a}}_{\cdot}$	R^2
Total	0.8481	-0.3412	0.9862
	(0.1497)	(0.4122)	(0.0171)
Male	0.8587	-0.3242	0.9862
	(0.1542)	(0.4314)	(0.0178)
Female	0.8375	-0.3580	0.9862
	(0.1444)	(0.3814)	(0.0165)
Incomplete*	0.8519	-0.4758	0.9699

*At least one measurement missing



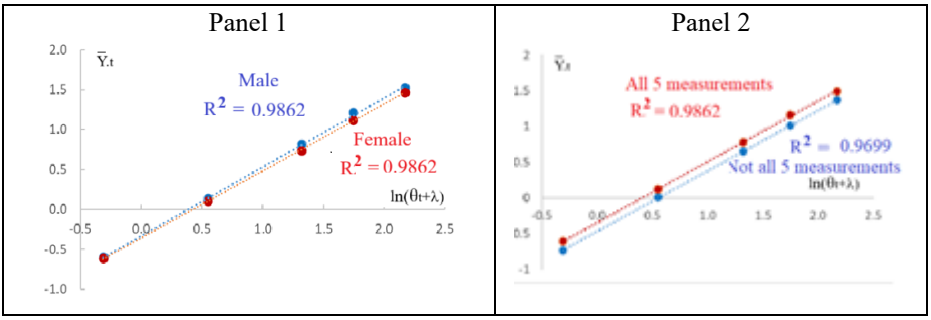


Fig 5. Means disaggregated by gender as a function of meta-metre $\ln(\theta_t + \hat{\lambda}),\ \hat{\lambda} = 0.7325$ (Panel 1); means of students with all 5, and at least one measurement missing (Panel 2).

Table 7. ANOVA analysis for mean differences between genders for the rate of growth and the intercept.

	$MS_{between}$	MS_{within}	df	F	p
$\bar{b}_.$	0.9342	0.0223	1,8337	41.8686	0.000*
$\bar{a}_.$	2.3925	0.1696	1,8337	14.1061	0.000*

*Significant at the 5% level

Table 8. $t\ test$ comparisons for the rate of growth and intercept between those with complete data and those with at least one measurement missing.

Data	$\hat{\bar{b}}_.$	$\hat{\bar{a}}_.$
Complete*	0.8481	-0.3412
Missing**	0.8519	-0.4758
$t\ test$	-2.3180	29.8191
p	0.0205(1)	0.000(2)

*All five measurements present; **At least one of five measurements missing; (1)Marginally significant at the 5% level; (2) Very significant at the 5% level

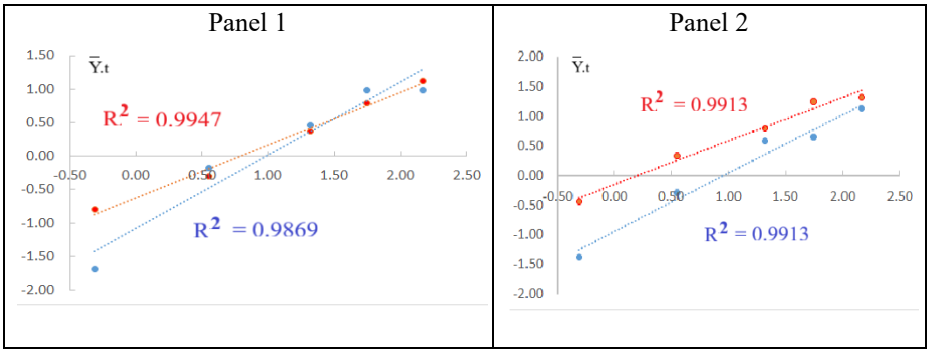


Fig 6. Two individuals different initial measurements, different rates of growth $(b = 0.7471, 1.1095)$ (Panel 1); two individuals with different initial measurements and less different rates of growth $(b = 0.7390, 1.0843)$ (Panel 2) with both pairs closer at the last measurement.

7 Implications of Decelerating Growth

The approach taken in this paper differs from other approaches in modelling growth, primarily because it separates the mode of growth, the meta-metre in Rasch's terminology which governs the growth of the population, from the rate of growth of individuals and subgroups. Both the form of the meta-metre and the individual and subgroup growth within the meta-metre, are important in their own right.

In the two examples of data analysed, the meta-metre showed deceleration. Researchers who have modelled growth in these, and related data, have noted a form of deceleration, but have not made it an object of study. Because it is tempting to assume that growth on both quantitative and the grade scale should be more or less linear, the decelerating nature of growth on the quantitative scale is worth further examination.

To illustrate two points regarding deceleration, the meta-metre of the ECLS-K data is taken and interpolated for all grades from Kindergarten to Grade 9, that is $Y_t = -0.3412 + 0.84811\text{n}(\theta_t + \lambda),\ \lambda = 0.7325;\ t = 0,1,2,...,8,9.$ To show explicitly the deceleration, Panel 1 of Figure 7 shows the growth as a function of $(\theta_t + \lambda)$.

7.1 Comparisons of Growth in the Earlier and Later Years

First, on the vertical axis of Panel 1 in Figure 7 is highlighted the attainment of Kindergarten, Grade 2 and Grade 9. The growth from Kindergarten to Grade 2 is,

1.1165, and that between Grade 2 to Grade 9 is 1.0773, which are almost equal with the former growth slightly greater. That is, the growth over two years from Kindergarten is the same as the growth over seven years from Grade 2! This can readily give the impression that not much growth is present in the later years. However, to understand the growth qualitatively, the syllabus differences between those of Grades 2 and 9 need to be examined, and these shows substantive, as distinct from quantitative growth.

7.2 Comparisons on the Grade Scale - Same Rate of Growth

Second, Panel 2 of Figure 7 reproduces the meta-metre of Panel 1 together with a second meta-metre where the rate of growth, $b = 0.8481$, is the same, but where the intercept is lower, $a = -0.6141$ instead of $a = -0.3412$. The Panel shows that in Grade 1, the attainment for the lower-intercept group is about half a year behind the students in the higher-intercept group, that is, the same attainment is reached by the higher-intercept group half a year earlier. However, by Grade 9, the same Panel shows that the higher-intercept group attains the same as that attained by the lower-intercept group three years earlier, that is Grade 6. This is a direct result of deceleration, that for the same rate of growth, a group with a slightly lower attainment at Kindergarten, will have increasingly lower attainment on the Grade scale. The substantive implication is that it is crucial to intervene early for any students at risk of attainment, and that it is important to set as high a rate of growth as early as possible.

7.3 Factors Governing Growth

In the above, growth is taken simply as a function of time. Clearly it is not time as such which is relevant, but what happens in time. Moreover, the simple formulation and its success imply that it is as if growth is governed by a *single factor* (Rao, 1958, P.15). In the two examples, the dominant effect is taken as schooling, but clearly maturation and home environment are also important factors. Therefore, it is *as if* the very many factors that govern growth are woven together into a single higher order, factor.

The measurements for means in both examples, which resulted in excellently modelled meta-metres, were taken at the same time of the year. In particular, the ECLS-K measurements of attainment in mathematics were taken in Spring, that is towards the end of the school year. The ECLS-K example also had a measurement in the Fall of the Kindergarten year, and in Grade 1 which is at the beginning of the school year. Figure 8, Panel 2, includes the measurement in the Fall of Grade 1, which shows that it is below the line of the meta-metre. This attainment below the line is attributed to the summer vacation period before the schooling begins again in the Fall, generally September. However, together with the school learning in the first few weeks of the year, there is growth following the attainment in the Spring of Kindergarten, but not at the same rate as during the school year.

Finally, it is reiterated that it seems that missing at least one measurement in the five measurements considered, is a symptom of features that lead to lower overall attainment in both the Reading and the Mathematics tests analysed.

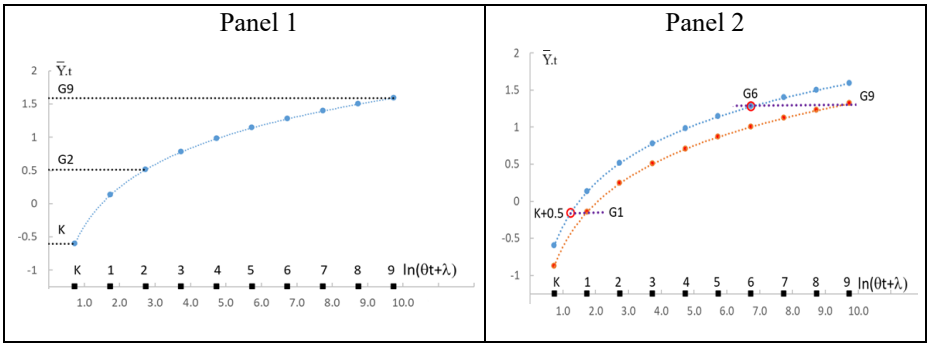


Fig 7. Comparison of the growth between Kindergarten and Grade 2, and Grade 2 and Grade 9(Panel 1); two identical rates of growth, $b = 0.8481$, but two different intercepts $a = -0.3412$, $a = -0.6141$ (Panel 2)

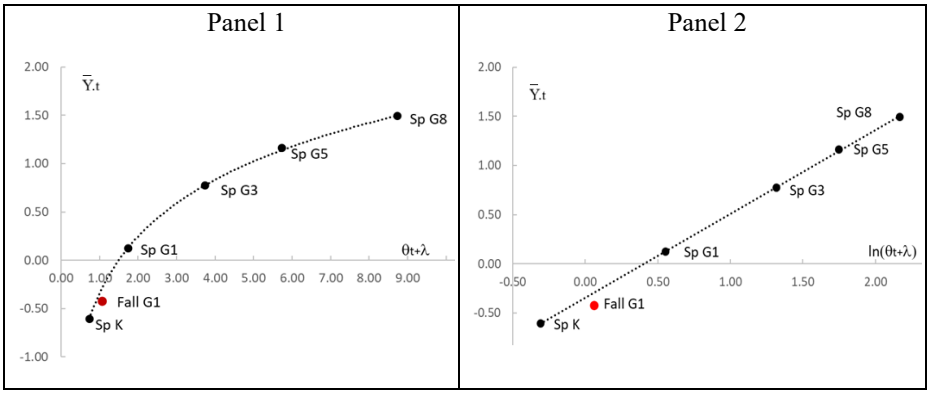


Fig 8. Fall measurement in Grade 1 relative to the meta-metre from the Spring measurements for the ECLS-K mathematics attainment as a function of $\theta_t + \hat{\lambda}$ (Panel 1) and $\ln(\theta_t + \hat{\lambda})$, $\hat{\lambda} = 0.7325$ (Panel 2).

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Appendix 1

Derivation of the meta-metre for a social science variable with an arbitrary origin

The measurement of individual n is given by the variable Y_{nt}, which has an arbitrary origin and unit. We begin with hypothesis is that there is a linear transformation $Z_{nt} = g + hY_{nt}$ whose rate of change with respect to the time variable is directly proportional to the rate of change of the meta-metre, that is

$$\frac{dZ_{nt}}{dt} = k_n \frac{d(\tau(t))}{dt}, \tag{A1.1}$$

where k_n is a constant of proportionality pertaining individual n.

Invoking the principle of parsimony, the simplest function of time is considered first, that is a linear function, say $\tau(t) = r + st$ where $(r,\ s)$ are constants. However, as indicated in the body of the paper, growth on intellectual variables, like growth in weight, decelerates. Accommodating deceleration, the hypothesis taken is that the rate of growth is inversely proportional to a function of time, that is $d\tau(t)/dt = 1/(r+st)$ giving

$$\frac{dZ_{nt}}{dt} = k_n \frac{1}{r+st}. \tag{A1.2}$$

Upon integration

$$Z_{nt} = (k_n / s)\ln(r+st) + l_n \tag{A1.2}$$

where l_n is an additive constant. However, the measurements are in the variable Y_{nt}. Therefore

$$Z_{nt} = g + hY_{nt} = (k_n / s)\ln(r+st) + l_n. \tag{A1.4}$$

Rearranging (A1.4) gives,

$$Y_{nt} = \{-g/h + l_n/h\} + ((k_n/s))/h(\ln(r+st)), \tag{A1.5}$$

and consolidating terms gives

$$Y_{nt} = c_{n+} b_n \ln(r+st) \tag{A1.6}$$

where $c_n = \{-g/h + l_n/h\}$, $b_n = ((k_n/s))/h$.

Expanding Eq. (A1.6) gives

$$\begin{aligned} Y_{nt} &= c_n + b_n \ln(s(t + r/s)), \\ &= c_n + b_n \{\ln s + \ln(t + r/s)\}, \\ &= c_n + b_n \ln s + b_n \ln(t + r/s). \end{aligned} \tag{A1.7}$$

That is,

$$Y_{nt} = a_n + b_n \ln(t + \lambda), \tag{A1.8}$$

where $a_n = c_n + b_n \ln s$, $\lambda = r/s$. Eq. (A1.8) is of the required form.

Appendix 2

Estimation of $(\hat{\lambda}, \hat{\bar{b}}_{\cdot})$

Obtaining least squares estimates involves finding the values of $(\hat{\lambda}, \hat{\bar{b}}_{\cdot})$ which minimises the sum of squares of Eq. (10): $S = \sum_{t=2}^{T} (\bar{y}_{\cdot t} - \hat{\bar{b}}_{\cdot}\hat{\tau}_{\theta t})^2$. This involves differentiating Eq. (10) with respect to each of $(\hat{\lambda}, \hat{\bar{b}}_{\cdot})$ and setting the resulting equations to 0.

First, differentiating with respect $\bar{b}_{\cdot}$ and setting it equal to 0 gives

$$\frac{\partial S}{\partial \bar{b}_{\cdot}} = 2\sum_{t=2}^{T} (\bar{y}_{\cdot t} - \hat{\bar{b}}_{\cdot}\hat{\tau}_{\theta t})\hat{\tau}_{\theta t} = 0 \qquad \text{(A2.1)}$$

from which $\sum_{t=2}^{T} (\bar{y}_{\cdot t}\hat{\tau}_{\theta t} - \hat{\bar{b}}_{\cdot}\hat{\tau}_{\theta t}^2) = 0$ and $\sum_{t=2}^{T} \bar{y}_{\cdot t}\hat{\tau}_{\theta t} - \hat{\bar{b}}_{\cdot}\sum_{t=2}^{T} \hat{\tau}_{\theta t}^2 = 0$. Further simplification and rearrangement results in

$$\hat{\bar{b}}_{\cdot} = \frac{\sum_{t=2}^{T} \bar{y}_{\cdot t}\hat{\tau}_{\theta t}}{\sum_{t=2}^{T} \hat{\tau}_{\theta t}^2}. \qquad \text{(A2.2)}$$

Equation (A2.2) is an explicit equation for estimating $\bar{b}_{\cdot}$ given $\hat{\tau}_{\theta t}$, which requires an estimate of $\hat{\lambda}$

Second, differentiating S, in expanded form, with respect $\hat{\lambda}$ and setting it equal to 0 gives

$$\frac{\partial S}{\partial \lambda} = \frac{\partial S}{\partial \lambda} \sum_{t=2}^{T} (\bar{y}_{\cdot t} - \hat{\bar{b}}_{\cdot}[\ln(\theta_t + \hat{\lambda}) - \ln(\theta_{t-1} + \hat{\lambda})])^2 = 0. \qquad \text{(A2.3)}$$

That is,

$$\sum_{t=2}^{T} 2(\bar{y}_{\cdot t} - \sum_{t=2}^{T} (\bar{y}_{\cdot t} - \hat{\bar{b}}_{\cdot}[\ln(\theta_t + \hat{\lambda}) - \ln(\theta_{t-1} + \hat{\lambda})])\hat{\bar{b}}_{\cdot}[\frac{1}{\theta_t + \hat{\lambda}} - \frac{1}{\theta_{t-1} + \hat{\lambda}}] = 0. \qquad \text{(A2.4)}$$

Given $2\bar{b}_{\cdot}$ is a constant in Eq. (A2.4), and applying $\hat{\tau}_{\theta t} = [\hat{\tau}(\theta_t) - \hat{\tau}(\theta_{t-1})] = [(\theta_t + \hat{\lambda}) - (\theta_{t-1} + \hat{\lambda}))$ simplifies Eq. (A4) to

$$\lambda: \quad \sum_{t=2}^{T}(\bar{y}_{.t} - \bar{b}_{.}\hat{\tau}_{\theta t})\left[\frac{1}{\theta_t + \hat{\lambda}} - \frac{1}{\theta_{t-1} + \hat{\lambda}}\right] = 0. \qquad \text{(A2.5)}$$

Equation (A2.5) is an implicit equation for estimating λ given $\bar{b}_{.}$ and is solved iteratively.

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