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Model Introduced SPRT for Structural Change Detection of Time Series (I) -- Formulation --

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Abstract

Previously, we have proposed a method applying Sequential Probability Ratio Test (SPRT) to the structural change detection problem of ongoing time series data. In this paper, we introduce a structural change model with Poisson process into a system that outputs a set of ongoing time series data, moment by moment. The model can be considered as a kind of Hidden Markov Model. According to the model, we formulate a method to find out the structural change, by defining a New Sequential Probability Ratio (NSPR), which can be calculated from the joint occurrence probability of the observing event with the event H_0 (the structural change is not occurred) and H_1 (the change is occurred). And also, we show the simple recurrence equation of the NSPR.

Keywords: Time series, Change detection, SPRT (Sequential Probability Ratio Test), Hidden Markov Model

1. Introduction

To make a prediction of ongoing time series data, we have three stages in general.[1, 2] First; we have to find a prediction model that adequately represents the characteristics of the early time series data. Second, we have to detect the structural change of the time series data, as quickly and correctly as possible, when the estimated prediction model does not meet the data any more as shown in Fig.1.[3, 4] Third, we have to reconstruct the next prediction model as soon as possible after the change detection.

For the second problem, we have already proposed an application of SPRT (sequential probability ratio test) that has been mainly used in the field of quality control.[5, 6] And we presented the experimental results in comparison with Chow Test that is well-known standard method for such structural change detection of time series data.[6, 7]

However, in the SPRT method that we previously proposed for the structural change detection problem, the probability of structural change occurrence is not known beforehand. In addition, in the method, only the ratio of the conditional probability is used. Then, the formulation of SPRT includes the notion that the structure may be able to recover even after the change occurrence. Then, there is a possibility that error happens in the change detection using the ratio of the conditional probability and so the threshold of the SPRT for the detection is decided based on the error probability.

In this paper, we introduce a Hidden Markov Model with Poisson process on the structural change one and propose a new probability ratio method for the change detection problem.

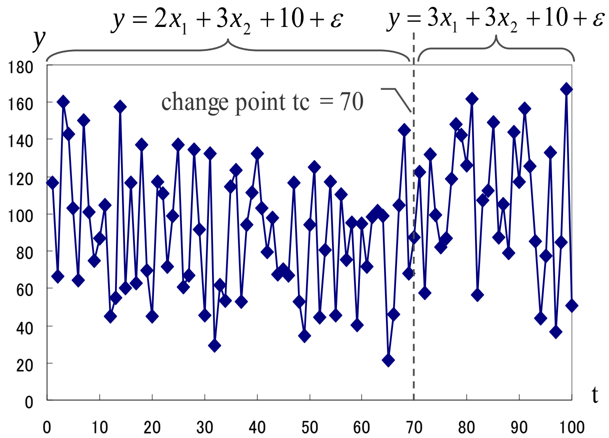


Fig. 1. Example of time series data where the true change point tc=70.

2. SPRT

2.1. *SPRT*[8]

The Sequential Probability Ratio Test (SPRT) is used for testing a null hypothesis H_0 (e.g. the quality is under pre-specified limit 1%) against hypothesis H_1 (e.g. the quality is over pre-specified limit 1%). And it is defined as follows:

Let Z_1, Z_2, …Z_i be respectively observed time series data at each stage of successive events, the probability ratio λ_i is computed as follows.

$$\lambda_i = \frac{P(Z_1 \mid H_1) \cdot P(Z_2 \mid H_1) \cdots P(Z_i \mid H_1)}{P(Z_1 \mid H_0) \cdot P(Z_2 \mid H_0) \cdots P(Z_i \mid H_0)} \quad (1)$$

where $P(Z \mid H_0)$ denotes the distribution of Z if H_0 is true, and similarly, $P(Z \mid H_1)$ denotes the distribution of Z if H_1 is true.

Two positive constants C_1 and C_2 ($C_1 < C_2$) are chosen. If $C_1 < \lambda_i < C_2$, the experiment is continued by taking an additional observation. If $C_2 < \lambda_i$, the process is terminated with the rejection of H_0 (acceptance of H_1). If λ_i < C1, then terminate this process with the acceptance of H_0.

$$C_1 = \frac{\beta}{1-\alpha}, \quad C_2 = \frac{1-\beta}{\alpha} \quad (2)$$

where α means type I error (reject a true null hypothesis), and β means type II error (accept a null hypothesis as true one when it is actually false).

2.2. *Procedure of SPRT*[8]

The concrete procedure of applying the SPRT method to the structural change detection problem is described using the notation of Fig. 2 as follows.

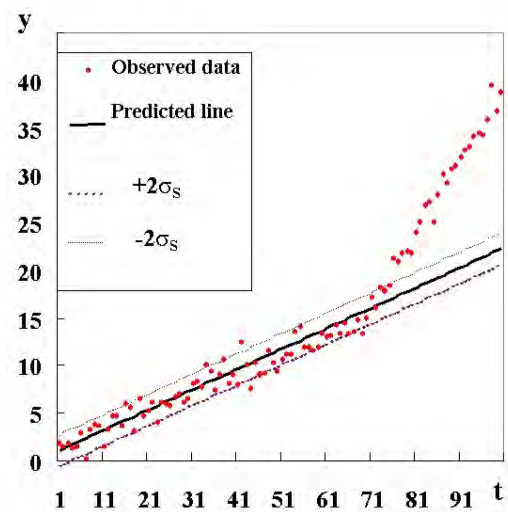


Fig. 2. Example of time series data and single regression line, where the true change point tc =70.[6]

Step 1: Make a prediction expression and set the tolerance band (***a***) (e.g. ***a***=$2\sigma_s$) that means permissible error margin between the predicted data and the observed one. (σ_s denotes a standard deviation in learning sample data at early stage.)

Step 2: Set up the null hypothesis H_0 and alternative hypothesis H_1.

H_0 : Change has not occurred yet.
H_1 : Change has occurred.

Set the values α, β and compute C_1 and C_2, according to Eq. (2). Initialize i = 0, $\lambda_0 = 1$.

Step 3: Incrementing i (i = i+1), observe the following data y_i. Evaluate the error | ε_i | between the data y_i and the predicted value from the aforementioned prediction expression.

Step 4: Judge as to whether the data y_i goes in the tolerance band or not, i.e., the ε_i is less than (or equal to) the permissible error margin or not. If it is Yes, then set λ_i = 1 and return to Step3. Otherwise, advance to Step5.

Step 5: Calculate the probability ratio λ_i, using the following Eq. (3) that is equivalent to Eq. (1).

$$\lambda_i = \lambda_{i-1} \frac{P(\varepsilon_i \mid H_1)}{P(\varepsilon_i \mid H_0)} \qquad (3)$$

where, if the data y_i goes out the tolerance band,
($P(\varepsilon_i | H_0)$, $P(\varepsilon_i | H_1)$)= (θ_0, θ_1), otherwise,
($P(\varepsilon_i | H_0)$, $P(\varepsilon_i | H_1)$)= ($(1-\theta_0)$, $(1-\theta_1)$).

Step 6: Execution of testing.

(i) If the ratio λ_i is greater than C_2 (= $(1-\beta)/\alpha$), dismiss the null hypothesis H_0, and adopt the alternative hypothesis H_1, and then End.

(ii) Otherwise, if the ratio λ_i is less than C_1 (= $\beta/(1-\alpha)$), adopt the null hypothesis H_0, and dismiss the alternative hypothesis H_1, and then set λ_i = 1 and return to Step3.

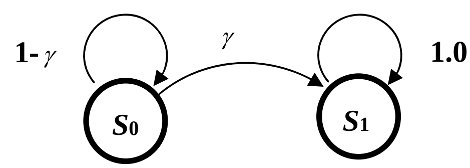
(iii) Otherwise (in the case where $C_1 \le \lambda_i \le C_2$), advance to Step7.

Step 7: Observe the following data y_i incrementing i. Evaluate the error | ε_i | and judge whether the data y_i goes in the tolerance band, or not. Then, return to Step5 (calculation of the ratio λ_i).

3. Structural Change Model Introduced

3.1. *Hidden Markov Model*

We present a Hidden Markov Model where the structural change is Poisson occurrence of average γ. In this model, once the change has occurred during the observing period, the structure does not go back to the previous one. The reason why we set such a model is that we focus on the detection of the first structural change in the sequential processing. The concept of the structural change model is shown in Fig.3. Then, we introduce the detailed model. Let ***R*** be the probability of the prediction failure that the data value goes out the tolerance zone when the structure is unchanged. Let ***Rc*** be the probability of the failure when the structure change occurred. We can consider that ***Rc*** is greater than ***R***, i.e., ***Rc*** > ***R***. The detailed model is illustrated as a probabilistic finite state automaton in Fig.4.



S$_0$ **:** State that the structure is unchanged.
S$_1$ **:** State that the structural change has occurred.
γ **:** Probability of the structural change occurrence. (Poisson Process.)

Fig. 3. Structural change model.

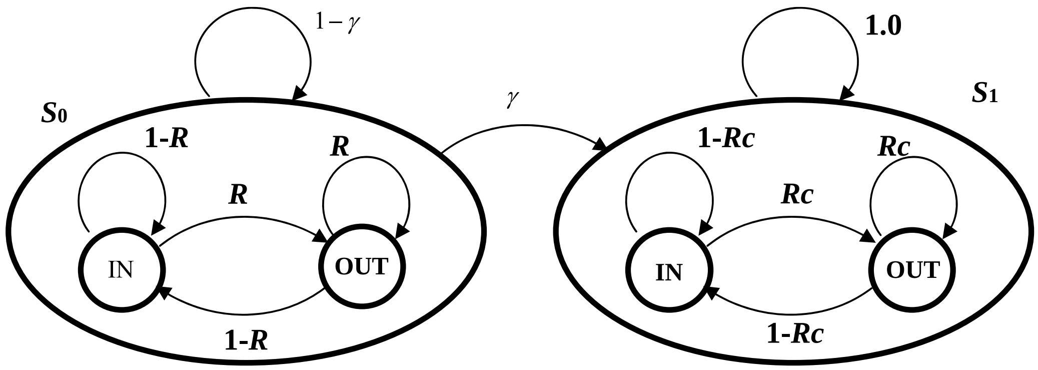
3.2. *New sequential probability ratio*

Let $a_1 a_2,\ldots,a_i,\ldots a_n \quad a_i \in \{\mathrm{IN}, \mathrm{OUT}\}$ be a string (or symbol sequence) obtained from the observed data.

Let θ_i and $\tilde{\theta}_i$ be the conditional probability that outputs the observed data (or above symbol sequence, $C_n = a_1 a_2 \ldots a_n$ in the state ***S*₀** and ***S*₁**, respectively. That is, it means that $\theta_i \in \{R, 1-R\}$ and $\tilde{\theta}_i \in \{R_c, 1-R_c\}$, respectively.

And let $P(a_1 \ldots a_n, H_0)$ and $P(a_1 \ldots a_n, H_1)$ be the joint probability of the symbol sequence C_n that happen with the event H_0 (the structural change is not occurred) and H_1 (the change is occurred), respectively.

Then, the following equations hold.



R : Probability of prediction failure (**OUT**) when the structure is unchanged.
Rc : Probability of prediction failure (**OUT**) when the structural change has been occurred.
IN : The state that observed data goes into the tolerance zone.
OUT: The state that observed data goes out of the tolerance zone.

Fig.4. Proposed Hidden Markov Model.

$$P(a_1 \ldots a_n, \mathrm{H}_0) \equiv P(C_n, \mathrm{H}_0) = (1-\gamma)^n \theta_1 \ldots \theta_n = (1-\gamma)^n \prod_{i=1}^{n} \theta_i \quad (4)$$

$$\begin{aligned} P(a_1 \ldots a_n, \mathrm{H}_1) &\equiv P(C_n, \mathrm{H}_1) \\ &= \gamma \prod_{i=1}^{n} \tilde{\theta}_i + ((1-\gamma)\theta_1)(\gamma \prod_{i=2}^{n} \tilde{\theta}_i) \\ &\quad + ((1-\gamma)^2 \theta_1 \theta_2)(\gamma \prod_{i=3}^{n} \tilde{\theta}_i) + \ldots \\ &= \sum_{k=1}^{n} ((1-\gamma)^{k-1} \cdot \prod_{j=0}^{k-1} \theta_j)(\gamma \prod_{i=k}^{n} \tilde{\theta}_i) \end{aligned} \quad (5)$$

$$\begin{aligned} P(\mathrm{H}_0 \mid a_1 \ldots a_n) &\equiv P(\mathrm{H}_0 \mid C_n) = \frac{P(\mathrm{H}_0 \mid C_n) \cdot P(C_n)}{P(C_n)} \\ &= \frac{P(C_n, \mathrm{H}_0)}{P(C_n)} = \frac{P(C_n, \mathrm{H}_0)}{P(C_n, \mathrm{H}_0) + P(C_n, \mathrm{H}_1)} \end{aligned} \quad (6)$$

$$\begin{aligned} P(\mathrm{H}_1 \mid a_1 \ldots a_n) &\equiv P(\mathrm{H}_1 \mid C_n) = \frac{P(\mathrm{H}_1 \mid C_n) \cdot P(C_n)}{P(C_n)} \\ &= \frac{P(C_n, \mathrm{H}_1)}{P(C_n)} = \frac{P(C_n, \mathrm{H}_1)}{P(C_n, \mathrm{H}_0) + P(C_n, \mathrm{H}_1)} \end{aligned} \quad (7)$$

Using these equations, the New Sequential Probability Ratio (NSPR) that we propose can be represented as follows.

$$\begin{aligned} \mathrm{NSPR} &= \frac{P(\mathrm{H}_1 \mid a_1 \ldots a_n)}{P(\mathrm{H}_0 \mid a_1 \ldots a_n)} \equiv \frac{P(\mathrm{H}_1 \mid C_n)}{P(\mathrm{H}_0 \mid C_n)} = \frac{P(C_n, \mathrm{H}_1)}{P(C_n, \mathrm{H}_0)} \\ &= \frac{\sum_{k=1}^{n} ((1-\gamma)^{k-1} \cdot \prod_{j=0}^{k-1} \theta_j)(\gamma \prod_{i=k}^{n} \tilde{\theta}_i)}{(1-\gamma)^n \prod_{i=1}^{n} \theta_i} \end{aligned} \quad (8)$$

When the NSPR is greater than 1.0, we can regard that the structural change has been occurred before the present time.
In the similar way to the case of SPRT (see Eq.(3)), the definition of NSPR can also be described in a recursion formula.

Let Λ_n be the value of NSPR defined by Eq.(8).

$$\Lambda_n = \frac{\sum_{k=1}^{n} ((1-\gamma)^{k-1} \cdot \prod_{j=0}^{k-1} \theta_j)(\gamma \prod_{i=k}^{n} \tilde{\theta}_i)}{(1-\gamma)^n \prod_{i=1}^{n} \theta_i}$$

where $\theta_0 = 1, \tilde{\theta}_0 = 1$ (9)

Accordingly, we have the followings.

$$\Lambda_1 = \frac{\gamma \tilde{\theta}_1}{(1-\gamma)\theta_1} \quad (10)$$

$$\begin{aligned}\Lambda_2 &= \frac{\gamma \tilde{\theta}_1\tilde{\theta}_2 + (1-\gamma)\theta_1\gamma \tilde{\theta}_2}{(1-\gamma)^2 \theta_1\theta_2} \\ &= \frac{1}{1-\gamma}\left(\frac{\gamma \tilde{\theta}_1 + (1-\gamma)\theta_1\gamma}{(1-\gamma)\theta_1}\right)\left(\frac{\tilde{\theta}_2}{\theta_2}\right) \\ &= \frac{1}{1-\gamma}\left(\frac{\gamma \tilde{\theta}_1}{(1-\gamma)\theta_1} + \frac{(1-\gamma)\theta_1\gamma}{(1-\gamma)\theta_1}\right)\left(\frac{\tilde{\theta}_2}{\theta_2}\right) \\ &= \frac{1}{1-\gamma}(\Lambda_1 + \gamma)\left(\frac{\tilde{\theta}_2}{\theta_2}\right)\end{aligned}$$

(11)

Similarly, we have

$$\begin{aligned}\Lambda_3 &= \frac{\gamma \tilde{\theta}_1\tilde{\theta}_2\tilde{\theta}_3 + (1-\gamma)\theta_1\gamma \tilde{\theta}_2\tilde{\theta}_3 + (1-\gamma)^2\theta_1\theta_2\gamma \tilde{\theta}_3}{(1-\gamma)^3\theta_1\theta_2\theta_3} \\ &= \frac{1}{1-\gamma}(\Lambda_2 + \gamma)\left(\frac{\tilde{\theta}_3}{\theta_3}\right)\end{aligned}$$

(12)

Therefore, by inductive inference, we have the following simple recurrence equation.

$$\Lambda_n = \frac{1}{1-\gamma}(\Lambda_{n-1} + \gamma)\left(\frac{\tilde{\theta}_n}{\theta_n}\right) \quad (13)$$

where $\Lambda_0 = 0$.

4. Conclusion

We have presented a Hidden Markov Model for the structural change detection problem, and have formulated a New Sequential Probability Ratio (NSPR) for solving the problem of the detection, using the occurrence probability based on the proposed model.
Moreover, we have shown that the definition of NSPR can also be described in a simple recurrence equation.
We consider that this formulation based on our proposed model will be more promising than the SPRT one that we previously applied it to the change detection problem.
At another opportunity, we will describe the nature of NSPR in the details and analyze the difference between NSPR and the conventional SPRT, and also show the effectiveness of the NSPR by experimentation.

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