Exponential Stability of Periodic Solutions for Cohen-Grossberg Neural Networks with Continuously Distributed Delays

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Abstract

A class of Cohen-Grossberg neural networks with distributed delays are considered. By using the coincidence degree theorem and differential inequality techniques, sufficient conditions for the existence and exponential stability of the periodic solutions are established, Without assuming the boundedness, monotonicity, and differentiability of activation functions and any symmetry of interconnections. The results of this paper are new.

Keywords: Cohen-Grossberg neural networks, Periodic solution, Exponential stability, Coincidence degree theory, Distributed delays

1. Introduction

Cohen and Grossberg neural networks (CGNNs) model, first introduced by Cohen and Grossberg in [1], have been widely investigated for the sake of theoretical interest as well as application considerations. Many good results have already been obtained by some authors in [2]-[15], and have been widely applied within various engineering and scientific fields such as neuro-biology, population biology, and computing technology. But to the best of our knowledge, the existing literature dealing with unbounded activation functions and distributed delays appears to be scarce. In fact, neural networks usually has a spatial extent due to the presence of an amount of parallel pathways with a variety of axon sizes and lengths. Thus, the delays in artificial neural networks are usually continuously distributed [14]-[20]. Moreover, in many applications, the properties of periodic oscillatory solutions and global exponential stability are of great interest. For example, the human brain is in a periodic oscillatory or chaos state, hence it is of prime importance to study periodic oscillation, global exponential stability and chaos phenomenon of neural networks. This motivates us consider the following CGNNs with distributed delays

$$\dot{x}_i(t) = -a_i(t, x_i(t))[b_i(t, x_i(t)) - \sum_{j=1}^{n} c_{ij}(t)f_j(x_j(t))$$

$$- \sum_{j=1}^{n} d_{ij}(t)f_j(\int_0^\infty K_{ij}(u)x_j(t-u)du) + I_i(t)],$$

$$i = 1, 2, \cdots, n, \tag{1.1}$$

where a_i and b_i are continuous functions on R^2, f_j, c_{ij}, d_{ij} and I_i are continuous functions on R; n corresponds to the number of units in a neural network; $x_i(t)$ denotes the potential (or voltage) of cell i at time t; a_i represents an amplification function; b_i is an appropriately behaved function; $c_{ij}(t)$ and $d_{ij}(t)$ denote the strengths of connectivity between cell i and j at time t respectively. The activation function $f_i(\cdot)$ shows how the ith neuron reacts to the input; $K_{ij}(t) \geq 0$ corresponds to the time delay required in processing and transmitting a signal from the jth cell to the ith cell at time t, and $I_i(t)$ denotes the ith component of an external input source introduced from outside the network to cell i at time t, $i, j = 1, 2, ..., n$.

Throughout this paper, it will be assumed that

(H_1) $c_{ij}, d_{ij}, I_i : R \to R$ are periodic functions with a common period $\omega(> 0)$, a_i and b_i are ω−periodic in the first variable, where $i, j = 1, 2, ..., n$.

(H_2) For $i = 1, 2, ..., n$, there exist positive constants $\underline{a_i}$ and $\overline{a_i}$ such that

$$\underline{a_i} < a_i(t, u) < \overline{a_i}, \quad \text{for all} \ \ t, u \in R.$$

(H_3) For $i = 1, 2, ..., n$, there exist positive constants $\underline{b_i}$ and $\bar{b}_i$ such that

$$0 \leq ub_i(t, u), \quad \underline{b_i} \mid u \mid \leq \mid b_i(t, u) \mid \leq \bar{b}_i \mid u \mid,$$

for all $\ t, u \in R$.

(H_4) For $i = 1, 2, ..., n$, there exist non-negative constants L_j such that

$$| f_j(u) - f_j(v) | \leq L_j | u - v | \quad \text{for all} \ \ u, v \in R.$$

(T_5) For each $i \in \{1, \ 2, \ \cdots, n\}, j \in \{1, \ 2, \ \cdots, n\}$, the delay kernels $K_{ij} : [0, \infty) \to R$ are continuous, integrable and there exit nonnegative constants k_{ij} such that

$$\int_0^\infty |K_{ij}(s)| du \leq k_{ij}.$$

(T_6) For each $i \in \{1, \ 2, \ \cdots, n\}, j \in \{1, \ 2, \ \cdots, n\}$, there exists a constant λ_0 such that

$$\int_0^\infty |K_{ij}(s)| \exp(\lambda_0 s) ds < +\infty.$$

For convenience, we introduce some notations. We can choose constants τ, $\overline{c_{ij}}$, $\overline{d_{ij}}$ and $\overline{I_i}$ such that

$$\overline{c_{ij}} = \max_{t \in [0, \ \omega]} |c_{ij}(t)|, \ \overline{d_{ij}} = \max_{t \in [0, \ \omega]} |d_{ij}(t)|,$$

$$\overline{I_i} = \max_{t \in [0, \ \omega]} |I_i(t)|, \quad (1.2)$$

We will use $x = (x_1, \ x_2, \ \cdots, x_n)^T \in R^n$ to denote a column vector, in which the symbol $(^T)$ denotes the transpose of a vector. For matrix $D = (d_{ij})_{n \times n}$, D^T denotes the transpose of D, and E_n denotes the identity matrix of size n. A matrix or vector $D \geq 0$ means that all entries of D are greater than or equal to zero. $D > 0$ can be defined similarly. For matrices or vectors D and E, $D \geq E$ (resp. $D > E$) means that $D - E \geq 0$ (resp. $D - E > 0$). For $V(t) \in C((a, \ +\infty), \ R)$, let

$$D^- V(t) = \limsup_{h \longrightarrow 0^-} \frac{V(t + h) - V(t)}{h},$$

$$D_- V(t) = \liminf_{h \longrightarrow 0^-} \frac{V(t + h) - V(t)}{h}, \forall t \in (a, \ +\infty).$$

As usual, we introduce the phase space $C((-\infty, \ 0]; \ R^n)$ as a Banach space of continuous mappings from $(-\infty, \ 0]$ to R^n equipped with the supremum norm defined by

$$\|\varphi\| = \max_{1 \leq i \leq n} \sup_{-\infty \leq t \leq 0} |\varphi_i(t)|$$

for all $\varphi = (\varphi_1(t), \ \varphi_2(t), \ \cdots, \varphi_n(t))^T \in C((-\infty, \ 0]; \ R^n)$.

The initial conditions associated with system (1.1) are of the form

$$x_i(s) = \varphi_i(s), s \in (-\infty, \ 0], \ i = 1, 2, \cdots, n, \ \ (1.3)$$

where $\varphi = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t))^T \in C((-\infty, \ 0]; \ R^n)$.

Definition 1. Let $Z^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^T$ be an $\omega-$periodic solution of system (1.1) with initial value $\varphi^* = (\varphi_1^*(t), \ \varphi_2^*(t), \ \cdots, \varphi_n^*(t))^T \in C((-\infty, \ 0]; \ R^n)$. If there exist constants $\alpha > 0$ and $M > 1$ such that for every solution $Z(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ of system (1.1) with any initial value $\varphi \in C((-\infty, \ 0]; \ R^n)$, $|x_i(t) - x_i^*(t)| \leq M\|\varphi - \varphi^*\|e^{-\alpha t}, \forall t > 0, \quad i = 1, \ 2, \ \cdots, \ n$. Then $Z^*(t)$ is said to be global exponential stable.

Definition 2. A real $n \times n$ matrix $H = (h_{ij})$ is said to be an M-matrix if $h_{ij} \leq 0, \ i, j = 1, \ 2, \ \cdots, n, \ i \neq j,$ and $H^{-1} \geq 0$.

2. Preliminaries

First, consider an abstract equation in a Banach space X,

$$Lx = \lambda Nx, \ \ \lambda \in (0, \ 1), \qquad (2.1)$$

where $L : DomL \cap X \to X$ is a linear operator and λ is a parameter. Let P and Q denote two projectors, $P : DomL \cap X \to KerL$ and $Q : X \to X/ImL$. For convenience, we introduce a continuation theorem [23] as follows.

Lemma 2.1.[23] Let X be a Banach space. Suppose that $L : DomL \subset X \longrightarrow X$ is a Fredholm operator with index zero and $N : \overline{\Omega} \longrightarrow X$ is $L-$compact on $\overline{\Omega}$ with Ω open bounded in X. Moreover, assume that all the following conditions are satisfied.

(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap DomL, \lambda \in (0, \ 1)$;

(2) $QNx \neq 0, \forall x \in \partial\Omega \cap KerL$;

(3) $deg_B\{QN, \ \Omega \cap KerL, \ 0\} \neq 0, deg_B$ denotes the Brouwer degree. Then equation $Lx = Nx$ has at least one solution in $\overline{\Omega}$.

For ease of exposition, throughout this paper we will adopt the following notations:

$$|x_i|_\infty = \max_{t \in [0, \omega]} |x_i(t)|, u(t) = (x_1(t), x_2(t), \cdots,$$

$$x_n(t))^T, |x_i|_k = \left(\int_0^\omega |x_i(t)|^k dt \right)^{1/k}, i = 1, 2, \cdots, n.$$

We denote X as the set of all continuously $\omega-$periodic functions $u(t)$ defined on R, and denote $\|u\|_X = \max\{|x_1|_\infty, |x_2|_\infty, \cdots, |x_n|_\infty\}$. Then, X is a Banach space when it is endowed with the norm $\|u\|_X$. Let for $u(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in X$,

$$(Nu)_i(t) = -a_i(t, x_i(t))[b_i(t, x_i(t)) - \sum_{j=1}^n c_{ij}(t)$$

$$\times f_j(x_j(t)) - \sum_{j=1}^{n} d_{ij}(t) f_j \left(\int_0^\infty K_{ij}(u) x_j(t-u) du \right)$$

$$+ I_i(t)], \quad i = 1, 2, \cdots, n, \qquad (2.2)$$

$$(Lu)(t) = u'(t) = (x_1'(t), x_2'(t), \cdots, x_n'(t))^T,$$

$$DomL = \{u(t) : u(t) \in X, u'(t) \in X\}, \qquad (2.3)$$

$$Pu = Qu = \frac{1}{\omega} \int_0^\omega u(t) dt = \left(\frac{1}{\omega} \int_0^\omega x_1(t) dt, \right.$$

$$\left. \frac{1}{\omega} \int_0^\omega x_2(t) dt, \cdots, \frac{1}{\omega} \int_0^\omega x_n(t) dt \right)^T.$$

In view of (2.2) and (2.3), the operator equation

$$Lx = \lambda Nx$$

is equivalent to the following equations

$$x_i'(t) = -\lambda a_i(t, x_i(t))[b_i(t, x_i(t)) - \sum_{j=1}^{n} c_{ij}(t) f_j(x_j(t))$$

$$- \sum_{j=1}^{n} d_{ij}(t) f_j \left(\int_0^\infty K_{ij}(u) x_j(t-u) du \right) + I_i(t)],$$

$$i = 1, 2, \cdots, n, \lambda \in (0, 1). \qquad (2.4)_\lambda$$

Again from (2.2) and (2.3), it is not difficult to show that $KerL = R^n$, $ImL = \{u(t) : u(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in X, \int_0^\omega x_1(t) dt = \int_0^\omega x_2(t) dt = \cdots = \int_0^\omega x_n(t) dt = 0\}$ is closed in X, $\dim KerL = n = \text{codim Im } L$, and P, Q are continuous projectors such that

$$ImP = KerL \quad \text{and} \quad KerQ = ImL.$$

It follows that the operator L is a Fredholm operator with index zero. Furthermore, the generalized inverse (of L) $K_p : ImL \longrightarrow DomL \cap KerP$ reads as

$$(K_p u)_i(t) = \int_0^t x_i(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t x_i(s) ds dt,$$

$$u(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in ImL, \qquad (2.5)$$

for $i = 1, 2, \cdots, n$. Therefore, from (2.2) and (2.5), we have that N is $L-$compact on $\overline{\Omega}$, where Ω is an open bounded set in X.

The following lemmas will be useful to prove our main results in Section $3 - 4$.

Lemma 2.2.[24]-[25]. Let $H = (h_{ij})_{n \times n}$ with $h_{ij} \leq 0$, $i, j = 1, 2, \cdots, n$, $i \neq j$. Then the following statements are equivalent.

(1) H is an M-matrix.

(2) There exists a vector $\eta = (\eta_1, \eta_2, \cdots, \eta_n) > (0, 0, \cdots, 0)$ such that $\eta H > 0$.

(3) There exists a vector $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^T > (0, 0, \cdots, 0)^T$ such that $H\xi > 0$.

Lemma 2.3 ([24]-[25]). Let $A \geq 0$ be an $n \times n$ matrix and $\rho(A) < 1$, then $(E_n - A)^{-1} \geq 0$, where E_n denotes the identity matrix of size n.

3. Existence of a periodic solution

Theorem 3.1. Let $(H_1) - (H_6)$ hold. Assume that the following condition is satisfied .

(H_7) there exists a vector $\eta = (\eta_1, \eta_2, \cdots, \eta_n) > (0, 0, \cdots, 0)$ such that

$$\bar{\eta} = (\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_n) = \eta(E_n - \Lambda) > (0, 0, \cdots, 0),$$

where $\Lambda = (q_{ij})_{n \times n}$, $q_{ij} = \underline{b}_i^{-1}(\overline{c_{ij}} L_j + \overline{d_{ij}} L_j k_{ij})(i, j = 1, 2, \cdots, n)$. Then system (1.1) has at least one $\omega-$periodic solution.

Proof. We shall seek to apply Lemma 2.1. To do this, it suffices to prove that the set of all possible ω-periodic solutions of Eq. $(2.4)_\lambda$ is bounded.

Let $u(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ be an arbitrary $\omega-$ periodic solution of Eq. $(2.4)_\lambda$. Then, for any $i = 1, 2, \cdots, n$, $x_i(t)$, as the components of $u(t)$, are all continuously differentiable. Thus, there exist $t_i \in [0, \omega]$ such that $|x_i(t_i)| = \max_{t \in [0, \omega]} |x_i(t)|$. Hence, $x_i'(t_i) = 0, i = 1, 2, \cdots, n$. This implies that

$$b_i(t_i, x_i(t_i)) = \sum_{j=1}^{n} c_{ij}(t_i) f_j(x_j(t_i)) + \sum_{j=1}^{n} d_{ij}(t_i)$$

$$\times f_j \left(\int_0^\infty K_{ij}(u) x_j(t_i - u) du \right) - I_i(t_i), \qquad (3.1)$$

In view of $(H_1) - (H_6)$, for $i = 1, 2, \cdots, n$, we have

$$\underline{b}_i |x_i(t_i)| \leq |b_i(t_i, x_i(t_i))|$$

$$\leq \sum_{j=1}^{n} (\overline{c_{ij}} L_j + \overline{d_{ij}} L_j k_{ij}) |x_j(t_j)|$$

$$+ \sum_{j=1}^{n} (\overline{c_{ij}} + \overline{d_{ij}}) |f_j(0)| + |I_i(t_i)|, \qquad (3.2)$$

Set $F_i = \sum_{j=1}^{n} \underline{b}_i^{-1}(\overline{c_{ij}} + \overline{d_{ij}}) |f_j(0)| + \underline{b}_i^{-1} \overline{I_i}$, $i = 1, 2, \cdots, n$. Clearly, (3.2) implies that

$$|x_i(t_i)| \leq \sum_{j=1}^{n} \underline{b}_i^{-1}(\overline{c_{ij}} L_j + \overline{d_{ij}} L_j k_{ij}) |x_j(t_j)|$$

$$+ \sum_{j=1}^{n} \underline{b}_i^{-1}(\overline{c_{ij}} + \overline{d_{ij}}) |f_j(0)| + \underline{b}_i^{-1} \overline{I_i}$$

$$= \sum_{j=1}^{n} q_{ij} |x_j(t_j)| + F_i, \quad i = 1, 2, \cdots, n.$$

Thus,

$$(E_n - \Lambda)(|x_1(t_1)|, |x_2(t_2)|, \cdots, |x_n(t_n)|)^T \leq$$

$$(F_1, F_2, \cdots, F_n)^T := F, \qquad (3.4)$$

which, together with (H_7), implies that

$$\min\{\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_n\}(|x_1(t_1)|+|x_2(t_2)|+\cdots+|x_n(t_n)|)$$

$$
\begin{aligned}
&\leq \quad \bar{\eta}_1|x_1(t_1)| + \bar{\eta}_2|x_2(t_2)| + \cdots + \bar{\eta}_n|x_n(t_n)| \\
&= \quad \eta(E_n - \Lambda)(|x_1(t_1)|, |x_2(t_2)|, \cdots, |x_n(t_n)|)^T \\
&\leq \quad \eta(F_1, F_2, \cdots, F_n)^T \\
&= \quad \eta_1 F_1 + \eta_2 F_2 +, \cdots, + \eta_n F_n. \qquad (3.5)
\end{aligned}
$$

Therefore,

$$|x_i|_\infty = \max_{t\in[0,\ \omega]} |x_i(t)| = |x_i(t_i)| \leq$$

$$\frac{\eta_1 F_1 + \eta_2 F_2 + \cdots + \eta_n F_n}{\min\{\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_n\}} := \delta^*, \ i = 1,\ 2,\ \cdots, n.$$
$$(3.6)$$

Again from (H_7), it follows from Lemma 2.2 that $E_n - \Lambda$ is an $M-$matrix and there exist a vector $\zeta = (\zeta_1,\ \zeta_2, \cdots,\ \zeta_2)^T > (0,\ 0, \cdots,\ 0)^T$ such that $(E_n - \Lambda)\zeta > (0,\ 0, \cdots,\ 0)^T$, which implies that we can choose a constant $d > 1$ such that $\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2, \cdots, \bar{\xi}_n)^T = (d\zeta_1, d\zeta_2, \cdots, d\zeta_n)^T = d\zeta$ and $\xi_i = d\zeta_i > \delta^*,\ i = 1,\ 2,\ \cdots, n$, and

$$(E_n - \Lambda)\bar{\xi} = d(E_n - \Lambda)\zeta > F. \qquad (3.7)$$

We take

$$\Omega = \{u(t) \in X, -\bar{\xi} < u(t) < \bar{\xi}, \forall t \in R\}, \qquad (3.8)$$

which satisfies Condition (1) of Lemma 2.1. If $u(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in \partial\Omega \cap KerL$, then $u(t)$ is a constant vector in R^n, and there exists some $i \in \{1, 2, \cdots, n\}$ such that $|x_i| = \bar{\xi}_i$. It follows that

$$(QNu)_i = -\frac{1}{\omega}\int_0^\omega a_i(t, x_i)[b_i(t, x_i)-\sum_{j=1}^n c_{ij}(t)f_j(x_j)$$

$$-\sum_{j=1}^n d_{ij}(t)f_j(\int_0^\infty K_{ij}(u)x_j(t-u)du)+I_i(t)]dt \quad (3.9)$$

We claim that

$$|(QNu)_i| > 0. \qquad (3.10)$$

By way of contradiction, suppose that $|(QNu)_i| = 0$, i.e.,

$$-\frac{1}{\omega}\int_0^\omega a_i(t, x_i)[b_i(t, x_i) - \sum_{j=1}^n c_{ij}(t)f_j(x_j)$$

$$-\sum_{j=1}^n d_{ij}(t)f_j(\int_0^\infty K_{ij}(u)x_j(t-u)du)+I_i(t)]dt = 0.$$

Then, there exists some $t^* \in [0,\ \omega]$ such that

$$a_i(t^*, x_i)[b_i(t^*, x_i) - \sum_{j=1}^n c_{ij}(t^*)f_j(x_j)$$

$$-\sum_{j=1}^n d_{ij}(t^*)f_j(\int_0^\infty K_{ij}(u)x_j(t^*-u)du)+I_i(t^*)] = 0,$$

which implies that

$$\underline{b}_i|x_i| \leq |b_i(t^*, x_i)| = |\sum_{j=1}^n c_{ij}(t^*)f_j(x_j)$$

$$+\sum_{j=1}^n d_{ij}(t^*)f_j(\int_0^\infty K_{ij}(u)x_j(t^* - u)du) - I_i(t^*)|.$$

Thus,

$$
\begin{aligned}
\bar{\xi}_i &= |x_i| \\
&\leq \sum_{j=1}^n \underline{b}_i^{-1}[|c_{ij}(t^*)|(|f_j(x_j) - f_j(0)|) \\
&\quad + |d_{ij}(t^*)|(|f_j(\int_0^\infty K_{ij}(u)x_j(t^* - u)du) - f_j(0)|) \\
&\quad + (|c_{ij}(t^*)| + |d_{ij}(t^*)|)|f_j(0)|] + \underline{b}_i^{-1}|I_i(t^*)| \\
&\leq \sum_{j=1}^n q_{ij}\bar{\xi}_j + F_i.
\end{aligned}
$$

This implies that $((E_n - \Lambda)\bar{\xi})_i \leq F_i$, which contradicts $(E_n - \Lambda)\bar{\xi} > F$. Therefore, (3.10) holds, and hence, Condition (2) of Lemma 2.1 is satisfied.

Furthermore, we define a continuous function $\Psi : \Omega \bigcap KerL \times [0, 1] \longrightarrow X$ by $\Psi(u, \mu) = \mu\text{diag}(-\bar{a}_1\bar{b}_1, -\bar{a}_2\bar{b}_2, \cdots, -\bar{a}_n\bar{b}_n)u+(1-\mu)QNu$, for all $u = (x_1, x_2, \cdots, x_n)^T \in \Omega\bigcap KerL = \Omega\bigcap R^n$ and $\mu \in [0, 1]$.

If $u(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in \partial\Omega \cap KerL$, then $u(t)$ is a constant vector in R^n, and there exists some $i \in \{1, 2, \cdots, n\}$ such that $|x_i| = \bar{\xi}_i$. It follows that

$$(\Psi(u, \mu))_i = -\mu\bar{a}_i\bar{b}_ix_i + (1-\mu)\{-\frac{1}{\omega}\int_0^\omega a_i(t, x_i)$$

$$[b_i(t, x_i)-\sum_{j=1}^n c_{ij}(t)f_j(x_j)-\sum_{j=1}^n d_{ij}(t)f_j(\int_0^\infty K_{ij}(u)$$

$$x_j(t - u)du) + I_i(t)]dt\}. \qquad (3.11)$$

We claim that

$$|(\Psi(u, \mu))_i| > 0. \qquad (3.12)$$

If this is not true, then $|(\Psi(u, \mu))_i| = 0$, i.e.,

$$-\mu \bar{a}_i \bar{b}_i x_i + (1 - \mu)\{-\frac{1}{\omega} \int_0^\omega a_i(t, x_i)[b_i(t, x_i)$$

$$-\sum_{j=1}^n c_{ij}(t) f_j(x_j)$$

$$-\sum_{j=1}^n d_{ij}(t) f_j(\int_0^\infty K_{ij}(u) x_j(t-u)du) + I_i(t)]dt\} = 0.$$

Hence, there exists some $t^{**} \in [0, \omega]$ such that

$$-\mu \bar{a}_i \bar{b}_i x_i + (1 - \mu)\{-a_i(t^{**}, x_i)[b_i(t^{**}, x_i)$$

$$-\sum_{j=1}^n c_{ij}(t^{**}) f_j(x_j) - \sum_{j=1}^n d_{ij}(t^{**}) f_j(\int_0^\infty K_{ij}(u)$$

$$x_j(t^{**} - u)du) + I_i(t^{**})]\} = 0. \qquad (3.13)$$

Now, we shall consider the following two cases.

Case (i). If $x_i > 0$, in view of (H_2) and (H_3), we get

$$a_i(t^{**}, x_i) b_i(t^{**}, x_i) - \bar{a}_i \bar{b}_i x_i \leq 0.$$

Thus, from (3.13), we obtain

$$-a_i(t^{**}, x_i) b_i(t^{**}, x_i) + (1 - \mu)\{-a_i(t^{**}, x_i)$$

$$[-\sum_{j=1}^n (c_{ij}(t^{**}) f_j(x_j) + d_{ij}(t^{**}) f_j(\int_0^\infty K_{ij}(u)$$

$$\times x_j(t^{**} - u)du)) + I_i(t^{**})]\}$$

$$\geq \mu[a_i(t^{**}, x_i) b_i(t^{**}, x_i) - \bar{a}_i \bar{b}_i x_i] - a_i(t^{**}, x_i) b_i(t^{**}, x_i)$$

$$+ (1 - \mu)\{-a_i(t^{**}, x_i)[-\sum_{j=1}^n (c_{ij}(t^{**}) f_j(x_j) + d_{ij}(t^{**}) \times$$

$$f_j(\int_0^\infty K_{ij}(u) x_j(t^{**} - u)du)) + I_i(t^{**})]\} = 0,$$
$$(3.14)$$

which implies that

$$a_i(t^{**}, x_i) b_i(t^{**}, x_i) \leq (1 - \mu)\{-a_i(t^{**}, x_i)$$

$$\times \quad [-\sum_{j=1}^n (c_{ij}(t^{**}) f_j(x_j) + I_i(t^{**})$$

$$+ \quad d_{ij}(t^{**}) f_j(\int_0^\infty K_{ij}(u) x_j(t^{**} - u)du))]\}$$

$$\leq \quad a_i(t^{**}, x_i)|\sum_{j=1}^n (c_{ij}(t^{**}) f_j(x_j) - I_i(t^{**})$$

$$+ \quad d_{ij}(t^{**}) f_j(\int_0^\infty K_{ij}(u) x_j(t^{**} - u)du))|. (3.15)$$

Noting that $x_i > 0$ and $b_i(t^{**}, x_i) \geq 0$, from (3.15), we have

$$\underline{b}_i x_i \leq b_i(t^{**}, x_i) \leq |\sum_{j=1}^n (c_{ij}(t^{**}) f_j(x_j) +$$

$$d_{ij}(t^{**}) f_j(\int_0^\infty K_{ij}(u) x_j(t^{**} - u)du)) - I_i(t^{**})|.$$

Therefore,

$$\bar{\xi}_i = x_i \leq \sum_{j=1}^n q_{ij} \bar{\xi}_j + F_i.$$

This implies that $((E_n - \Lambda)\bar{\xi})_i \leq F_i$, which contradicts $(E_n - \Lambda)\bar{\xi} > F$. Therefore, (3.12) holds.

Case (ii). If $x_i < 0$, using the methods similar to those used in Case (i), we can show that (3.12) holds. It follows that $\Psi(x_1, x_2, \cdots, x_n, \mu) \neq (0, 0, \cdots, 0)^T$, $\forall (x_1, x_2, \cdots, x_n)^T \in \partial \Omega \cap KerL, \mu \in [0, 1]$.

Hence, using the homotopy invariance theorem, we obtain $deg_B\{QN, \Omega \cap KerL, (0, 0, \cdots, 0)^T\} = deg_B\{(-\bar{a}_1 \bar{b}_1 x_1, -\bar{a}_2 \bar{b}_2 x_2, \cdots, -\bar{a}_n \bar{b}_n x_n)^T, \Omega \cap KerL, (0, 0, \cdots, 0)^T\} \neq 0$. To summarize, we have proved that Ω satisfies all the conditions of Lemma 2.1. This completes the proof.

4. Uniqueness and exponential stability of the periodic solution

In this section, we establish some results for the uniqueness and exponential stability of the $\omega-$periodic solution of (1.1).

Theorem 4.1. Suppose that $(H_1)-(H_7)$ hold. Assume also that the following conditions are satisfied.

(H_8) for $i = 1, 2, ..., n$, there exists a positive constant L_i^a such that

$$|a_i(t, u) - a_i(t, v)| < L_i^a|u - v|, \quad \text{for all } t, u, v \in R;$$

(H_9) for $i = 1, 2, ..., n$, there exists a positive constant L_i^{ab} such that

$$[a_i(t, u) b_i(t, u) - a_i(t, v) b_i(t, v)](u - v) \geq 0,$$

$$|a_i(t, u) b_i(t, u) - a_i(t, v) b_i(t, v)| \geq L_i^{ab}|u - v|,$$

for all $t, u, v \in R$;

(H_{10}) for $i = 1, 2, ..., n$, $|f_j(u)| \leq L_j |u|$ for all $u \in R$;

(H_{11}) for $i, j = 1, 2, \cdots, n$, set $\bar{F}_i = \underline{b}_i^{-1} \overline{I}_i$, $\delta = \frac{\eta_1 \bar{F}_1 + \eta_2 \bar{F}_2 + \cdots + \eta_n \bar{F}_n}{\min\{\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_n\}}$, and assume that $E_n - H$ is an M-matrix, where $H = (h_{ij})_{n \times n}$ and

$$h_{ij} = [L_i^{ab} - L_i^a \sum_{j=1}^n (\overline{c_{ij}} L_j \delta + \overline{d_{ij}} L_j \delta k_{ij}) - L_i^a \overline{I}_i]^{-1}$$

$$\times \bar{a}_i (\overline{c_{ij}} L_j + \overline{d_{ij}} L_j k_{ij}),$$

$$L_i^{ab} - L_i^a \sum_{j=1}^n (\overline{c_{ij}} L_j \delta + \overline{d_{ij}} L_j \delta k_{ij}) - L_i^a \overline{I}_i > 0.$$

Then system (1.1) has exactly one ω-periodic solution, and the ω-periodic solution of system (1.1) is globally exponentially stable.

Proof. Since $(H_1) - (H_7)$ hold, it follows from Theorem 3.1 that system (1.1) has at least one ω-periodic solution $Z^*(t) = (x_1^*(t), x_2^*(t), \cdots, x_n^*(t))^T$ with initial value $\varphi^* = (\varphi_1^*(t), \varphi_2^*(t), \cdots, \varphi_n^*(t))^T$. from (H_{10}), we have

$$| f_j(0) | = 0, \quad \text{for} \quad j = 1, 2, ..., n.$$

Thus, by using a similar proof in $(3.2) - (3.6)$, we obtain

$$|x_i^*|_\infty = \max_{t \in [0, \omega]} |x_i^*(t)| \le \frac{\eta_1 F_1 + \eta_2 F_2 + \cdots + \eta_n F_n}{\min\{\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_n\}}$$

$$= \frac{\eta_1 \bar{F}_1 + \eta_2 \bar{F}_2 + \cdots + \eta_n \bar{F}_n}{\min\{\bar{\eta}_1, \bar{\eta}_2, \cdots, \bar{\eta}_n\}} = \delta, \qquad (4.1)$$

where $i = 1, 2, \cdots, n$.

Let $Z(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ be an arbitrary solution of system (1.1) with initial value $\varphi = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t))^T$ and define $y(t) = Z(t) - Z^*(t)$. Then, for $i = 1, 2, \cdots, n$, set

$$\alpha_i(t, y_i(t)) = a_i(t, y_i(t) + x_i^*(t)) b_i(t, y_i(t) + x_i^*(t))$$

$$- a_i(t, x_i^*(t)) b_i(t, x_i^*(t)),$$

$$\beta_i(t, y_i(t)) = a_i(t, y_i(t) + x_i^*(t)) \sum_{j=1}^n c_{ij}(t)[f_j(y_j(t)$$

$$+ x_j^*(t)) - f_j(x_j^*(t))],$$

$$\bar{\beta}_i(t, y_i(t)) = a_i(t, y_i(t) + x_i^*(t)) \sum_{j=1}^n d_{ij}(t)$$

$$\times [f_j(\int_0^\infty K_{ij}(u)(y_j(t-u) + x_j^*(t-u)du)$$

$$- f_j(\int_0^\infty K_{ij}(u)(x_j^*(t-u)du)],$$

$$\gamma_i(t, y_i(t)) = [a_i(t, y_i(t) + x_i^*(t))$$

$$- a_i(t, x_i^*(t))] \sum_{j=1}^n c_{ij}(t) f_j(x_j^*(t)),$$

$$\bar{\gamma}_i(t, y_i(t)) = [a_i(t, y_i(t) + x_i^*(t)) - a_i(t, x_i^*(t))]$$

$$\times \sum_{j=1}^n d_{ij}(t) f_j(\int_0^\infty K_{ij}(u) x_j^*(t-u) du),$$

$$\theta_i(t, y_i(t)) = [a_i(t, y_i(t) + x_i^*(t)) - a_i(t, x_i^*(t))] I_i(t).$$

Then, we get

$$y_i'(t) = -\alpha_i(t, y_i(t)) + \beta_i(t, y_i(t)) + \bar{\beta}_i(t, y_i(t))$$

$$+ \gamma_i(t, y_i(t)) + \bar{\gamma}_i(t, y_i(t)) - \theta_i(t, y_i(t)), \qquad (4.3)$$

Thus, for $i = 1, 2, \cdots, n$, from $(H_2), (H_4), (H_5)$, $(H_8), (H_9)$ and (H_{10}), we have

$$D^- \quad |y_i(t)| = D^- \{\text{sgn}(x_i(t) - x_i^*)(x_i(t) - x_i^*)\}$$

$$\le \quad -L_i^{ab} |y_i(t)| + |\beta_i(t, y_i(t))| + |\bar{\beta}_i(t, y_i(t))|$$

$$+ |\gamma_i(t, y_i(t))| + |\bar{\gamma}_i(t, y_i(t))| + |\theta_i(t, y_i(t))|$$

$$\le \quad -[L_i^{ab} - L_i^a \sum_{j=1}^n (\overline{c_{ij}} L_j \delta + \overline{d_{ij}} L_j k_{ij} \delta) -$$

$$L_i^a \overline{I}_i] |y_i(t)| + \bar{a}_i \sum_{j=1}^n (\overline{c_{ij}} L_j + \overline{d_{ij}} L_j k_{ij}) \bar{y}_j(t),$$

where $\bar{y}_j(t) = \sup_{-\infty < s \le t} |y_j(s)|$. From (H_9) and Lemma 2.2, we obtain that there exist a constant $\sigma > 0$ and a vector $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^T > (0, 0, \cdots, 0)^T$ such that

$$(E_n - H) \xi > (\sigma, \sigma, \cdots, \sigma)^T.$$

Therefore, for $i = 1, 2, \cdots, n$,

$$\xi_i - \sum_{j=1}^n k_{ij} \xi_j = \xi_i - \sum_{j=1}^n [L_i^{ab} - L_i^a \sum_{j=1}^n (\overline{c_{ij}} L_j \delta$$

$$+ \overline{d_{ij}} L_j \delta k_{ij}) - L_i^a \overline{I}_i]^{-1} \bar{a}_i (\overline{c_{ij}} L_j + \overline{d_{ij}} L_j k_{ij}) \xi_j > \sigma,$$

which implies that

$$-[L_i^{ab} - L_i^a \sum_{j=1}^n (\overline{c_{ij}} L_j \delta + \overline{d_{ij}} L_j \delta k_{ij}) - L_i^a \overline{I}_i] \xi_i$$

$$+ \bar{a}_i \sum_{j=1}^n (\overline{c_{ij}} L_j + \overline{d_{ij}} L_j k_{ij}) \xi_j < 0, \qquad (4.5)$$

where $i = 1, 2, \cdots, n$. We can choose a constant $\alpha < 1$ such that

$$\alpha \xi_i + \{-[L_i^{ab} - L_i^a \sum_{j=1}^n (\overline{c_{ij}} L_j \delta + \overline{d_{ij}} L_j \delta k_{ij}) - L_i^a \overline{I}_i] \xi_i$$

$$+\bar{a}_i\sum_{j=1}^n(\overline{c_{ij}}L_j+\overline{d_{ij}}L_jk_{ij})\xi_je^{\alpha\tau}\}<0, \qquad (4.6)$$

We can also choose a constant $\beta>1$ such that

$$\beta\xi_ie^{-\alpha t}>1,\ \ \text{for all}\ \ t\in(-\infty,0],\ i=1,2,\cdots,n. \qquad (4.7)$$

For $\forall\varepsilon>0$, let

$$\Upsilon_i(t)=\beta\xi_i[\sum_{j=1}^n\bar{y}_j(0)+\varepsilon]e^{-\alpha t},\ i=1,2,\cdots,n. \qquad (4.8)$$

From (4.6) and (4.8), we obtain

$$D_-\Upsilon_i(t)=-\alpha\beta\xi_i[\sum_{j=1}^n\bar{y}_j(0)+\varepsilon]e^{-\alpha t}$$

$$>\ -[L_i^{ab}-2L_i^a\sum_{j=1}^n(\overline{c_{ij}}L_j\delta+\overline{d_{ij}}L_j\delta k_{ij})-L_i^a\overline{I_i}]$$

$$\Upsilon_i(t)+\bar{a}_i\sum_{j=1}^n(\overline{c_{ij}}L_j+\overline{d_{ij}}L_jk_{ij})\bar{\Upsilon}_j(t), \quad (4.9)$$

where $\bar{\Upsilon}_j(t)=\sup_{-\infty<s\le0}\Upsilon_j(s)$. In view of (4.7) and (4.8), for $i=1,2,\cdots,n$, we have

$$\Upsilon_i(t)=\beta\xi_i[\sum_{j=1}^n\bar{y}_j(0)+\varepsilon]e^{-\alpha t}>\sum_{j=1}^n\bar{y}_j(0)$$

$$+\varepsilon>|y_i(t)|,\ \ \text{for all}\ \ t\in(-\infty,0]. \qquad (4.10)$$

We claim that

$$|y_i(t)|<\Upsilon_i(t),\ \ \text{for all}\ \ t>0, \qquad (4.11)$$

Contrarily, there must exist $i\in\{1,2,\cdots,n\}$ and $t_i>0$ such that

$$|y_i(t_i)|=\Upsilon_i(t_i)\ \ \text{and}\ \ |y_j(t)|<\Upsilon_j(t),$$

$$\text{for all}\ \ t\in[-\tau,\ t_i),\ j=1,2,\cdots,n, \qquad (4.12)$$

which implies that

$$|y_i(t_i)|-\Upsilon_i(t_i)=0\ \ \text{and}\ \ |y_j(t)|-\Upsilon_j(t)<0,$$

$$\text{for all}\ \ t\in[-\tau,\ t_i),\ j=1,2,\cdots,n. \qquad (4.13)$$

It follows that

$$0\ \le\ D^-(|y_i(t_i)|-\Upsilon_i(t_i))$$

$$\le\ \limsup_{h\to0^-}\frac{|y_i(t_i+h)|-|y_i(t_i)|}{h}$$

$$-\liminf_{h\to0^-}\frac{\Upsilon_i(t_i+h)-\Upsilon_i(t_i)}{h}$$

$$=\ D^-|y_i(t_i)|-D_-\Upsilon_i(t_i). \qquad (4.14)$$

From (4.4), (4.9) and (4.12), we obtain

$$D^-|y_i(t_i)|$$

$$\le\ -[L_i^{ab}-L_i^a\sum_{j=1}^n(\overline{c_{ij}}L_j\delta+\overline{d_{ij}}L_j\delta k_{ij})-L_i^a\overline{I_i}]$$

$$|y_i(t_i)|+\bar{a}_i\sum_{j=1}^n(\overline{c_{ij}}L_j+\overline{d_{ij}}L_jk_{ij})\bar{y}_j(t_i)$$

$$\le\ -[L_i^{ab}-L_i^a\sum_{j=1}^n(\overline{c_{ij}}L_j\delta+\overline{d_{ij}}L_j\delta k_{ij})-L_i^a\overline{I_i}]$$

$$\Upsilon_i(t_i)+\bar{a}_i\sum_{j=1}^n(\overline{c_{ij}}L_j+\overline{d_{ij}}L_jk_{ij})\bar{\Upsilon}_j(t_i)$$

$$<\ D_-\Upsilon_i(t_i), \qquad (4.15)$$

which contradicts (4.14). Hence, (4.11) holds. Letting $\varepsilon\longrightarrow0^+$ and $M=n\max_{1\le i\le n}\{\beta\xi_i+1\}$, we have from (4.8) and (4.11) that

$$|x_i(t)-x_i^*(t)|=|y_i(t)|\le\beta\xi_i\sum_{j=1}^n\bar{y}_j(0)e^{-\alpha t}$$

$$\le\beta\xi_in\|\varphi-\varphi^*\|e^{-\alpha t}\le M\|\varphi-\varphi^*\|e^{-\alpha t},$$

for all $t>0$. This completes the proof.

5. An illustrative example

Example 5.1. Consider the following GCNNs

$$\begin{cases}x_1'(t)=-(2+\frac{1}{10\pi}\arctan x_1(t))[x_1(t)\\ \quad-\frac{1}{24}(\sin t)|x_1(t)|-\frac{1}{24}(\cos t)|x_2(t)|\\ \quad-\frac{1}{24}(\sin t)|\int_0^\infty(\sin u)e^{-u}x_1(t-u)du|\\ \quad-\frac{1}{24}(\cos t)|\int_0^\infty(\sin u)e^{-u}x_2(t-u)du|\\ \quad+\frac{5}{12}\sin^2 t],\\ x_2'(t)=-(2+\frac{1}{10\pi}\arctan x_2(t))[x_2(t)-\\ \quad\frac{1}{24}(\sin 2t)|x_1(t)|-\frac{1}{24}(\cos 4t)|x_2(t)|-\\ \quad\frac{1}{24}(\sin 2t)|\int_0^\infty(\sin u)e^{-u}x_1(t-u)du|-\\ \quad\frac{1}{24}(\cos 4t)|\int_0^\infty(\sin u)e^{-u}x_2(t-u)du|\\ \quad+\frac{5}{12}\cos t.\end{cases}$$
$$(5.1)$$

Notice that $f_1(x)=f_2(x)=|x|,K_{ij}(u)=(\sin u)e^{-u},a_1(t,x)=a_2(t,x)=2+\frac{1}{10\pi}\arctan x$, $b_1(t,x)=b_2(t,x)=x$, it is easy to show that we can choose the constants in the conditions $(H_1)-(H_{10})$ and (H_{11}^*) as follows.
$\underline{a}_1=\underline{a}_2=1,\bar{a}_1=\bar{a}_2=3,\bar{b}_1=\bar{b}_2=\underline{b}_1=\underline{b}_2=1,L_1=L_2=1,L_1^a=L_2^a=\frac{1}{30},L_1^{ab}=L_2^{ab}=1,\bar{I}_1=\bar{I}_2=\frac{5}{12},\overline{c_{11}}=\overline{d_{11}}=\overline{c_{12}}=\overline{c_{12}}=\overline{c_{21}}=\overline{d_{21}}=\overline{c_{22}}=\overline{d_{22}}=\frac{1}{24},\eta=(\eta_1,\ \eta_2)=(1,\ 1),k_{ij}=1,i,j=1,2$. Then, we obtain

$$\bar{\eta}=(\bar{\eta}_1,\ \bar{\eta}_2)=\eta(E_n-\Lambda)=(\frac{5}{6},\ \frac{5}{6}),\ \ \delta=1$$

$$L_i^{ab} - L_i^a \sum_{j=1}^{2} (\overline{c_{ij}} L_j \delta + \overline{d_{ij}} L_j \delta k_{ij}) - L_i^a \overline{I_i} = \frac{353}{360},$$

i=1,2, and

$$H = (h_{ij})_{2 \times 2} = \begin{pmatrix} \frac{90}{353} & \frac{90}{353} \\ \frac{90}{353} & \frac{90}{353} \end{pmatrix},$$

which implies that

$$\rho(H) \leq \|H\|_\infty = \frac{90}{353} + \frac{90}{353} = \frac{180}{353} < 1,$$

where $\|H\|_\infty$ the row norm of matrix H. It is straight forward to check that all the conditions needed in Corollary 4.1 are satisfied. Therefore, by Corollary 4.1, system (5.1) has exactly one 2π-periodic solution, and the 2π-periodic solution of system (5.1) is globally exponentially stable.

6. Conclusion

In this paper, Cohen-Grossberg neural networks with continuously distributed delays have been studied. Some sufficient conditions for the existence and exponential stability of the periodic solutions have been established. These obtained results are new. Moreover, an example is given to illustrate the effectiveness of the new results.

Acknowledgements

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