Non-Gromov hyperbolicity of asymptotic Teichmuller spaces

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***Abstract*- In this paper, we prove that the asymptotic Teichmuller space of Riemann surfaces of analytically infinite type with the asymptotic Teichmuller metricis not Gromov hyperbolic.**

Keywords-Riemannsurface, asymptotic Teichmuller space, Gromov hyperbolic, asymptotic extremal

I. INTRODUCTION

Let R be a hyperbolic Riemann surface, T(R) and AT(R) be the Teichmuller space and asymptotic Teichmuller space on R. The study of asymptotic Teichmuller space was initiated by Gardiner and Sullivan in [6] for the case that R is the unit disk. General results on asymptotic Teichmuller space AT(R) were obtained in the papers [1] [2] [4] [17] and the book [5]. You can refer to the papers [2][14][15][16][17] for recently progress onasymptotic Teichmuller. Masur and Wolf [11] proved that the Teichmuller space of compact Riemann surfaces of genus g>1 with the Teichmuller metric is not Gromov hyperbolic. This result had been proved by other mathematicians in the papers [7] [12] [13]. The goal of this paper is to prove that asymptotic Teichmuller space with the asymptotic Teichmuller metric is not Gromov hyperbolic. For AT(R) has interesting only when R is of analytically infinite type, otherwise, AT(R) consists of only one point. So, in what follows, we restrict R to be of analytically infinite type.

II. PRELIMINARIES

A. Asymptotic Teichmuller Space

The Teichmuller space T(R) is the set of equivalence classes of quasiconformal mappings on R. Two mappings f and g are equivalent if there is a conformal mapping c from f(R) onto g(R) and a homotopy through quasiconformal mappings h_t mappings R onto g(R) such that

$h_0 = c \circ f$, $h_1 = g$ and $h_t(p) = c? f(p) = g(p)$

for every p in the ideal boundary of R.

The asymptotic Teichmuller space AT(R) has the same definition with one exception. The mapping c is allowed to be asymptotically conformal. A mapping f is asymptotically conformal if, for every $\varepsilon > 0$, there is a compact subset E of R such that the dilatation of f outside of E is less than $1+\varepsilon$. Let [f] or $[\mu]$ denote the equivalence class of the quasiconformal mapping class of the quasiconformal mapping f in the asymptotic Teichmuller space (we call this equivalence class be asymptotic equivalence class), where μ is the Beltrami differential of f. Therefore, the asymptotic Teichmuller space AT(R) may be represented as the space of asymptotic equivalence classes of Beltrami differentials μ in the unit ball M(R) of the space L^{∞}(R) .

Let $\mu \in M(R)$, we define $h^*(\mu)$ and $h([\mu])$ as

$h^*(\mu) = \inf \left\{ \left\| \nu|_{R\backslash F} \right\|_{\infty} : F \right.$ is a compact subset of R$\}$,

$h([\mu]) = \inf \left\{ \left\| \nu|_{R\backslash F} \right\|_{\infty} : \nu \in [\mu], F \right.$ is a compact subset of R$\}$.

We say μ is asymptotically extremal if $h^*(\mu) = h([\mu])$.

Given two points $[\mu]$ and $[\nu]$ in AT(R), the asymptotic Teichmuller metric $d([\mu],[\nu])$ between $[\mu]$ and $[\nu]$ is defined as

$$d([\mu],[\nu]) = \frac{1}{2}\log\frac{1+h([f^{\mu}\circ (g^{\nu})^{-1}])}{1-h([f^{\mu}\circ (g^{\nu})^{-1}])}.$$

B. Degenerating sequence

We denote by A(R) the set of holomorphic quadratic differentials of R with norm

$$\|\phi\| = \int_R \phi dxdy < \infty .$$

By definition, a degenerating sequence in A(R) is a sequence of quadratic differentials $\{\phi_n\}$ such that $\|\phi_n\| = 1$ for all n and $\phi_n \to 0$ uniformly on compact subsets of R as $n \to \infty$. It is well known (see for instance Chapter 3 of [5]) that analytically infinite type of R implies A(R) contains degenerating sequences. It was proved in [2] that a Beltrami differential μ is asymptotically extremal if and only if there is a degenerating sequence $\{\phi_n\}$ such that

$$h^*(\mu) = \lim_{n\to\infty} \left| \int_R \mu\phi_n dxdy \right| .$$

There exists a special degenerating sequence $\{\phi_n\}$ which was constructed in the papers [8] [9] and [10] by Li Zhong and played an important role in papers [3] [8] [9] and [10]. The degenerating sequence is also important for our proof, we state it as

Theorem A. *There exist a compact subset sequence* $\{E_n\}$ *of R and a degenerating sequence* $\{\phi_n\}$ *which satisfy the following conditions (1)-(7).*

$$E_n \subset E_{n+1}, n=1,2,...; R=\cup_{n=1}^{\infty} E_n \quad (1)$$

$$|\phi_n(z)| < 2^{-n} \text{ for all } z \in E_{n-1}, \quad (2)$$

$$\int_{E_n \setminus E_{n-1}} |\phi_n| > 1-2^{-n}, \quad (3)$$

$$\int_{R \setminus E_n} |\phi_m| < 2^{-n} \text{ if } 1 \le m \le n, \quad (4)$$

$$\int_{E_{n-1}} |\phi_n| < O(2^{-n}) \text{ as } n \to \infty, \quad (5)$$

$$\int_{E_n \setminus E_{n-1}} |\phi - \phi_n| < O(n2^{-n}) \text{ as } n \to \infty, \quad (6)$$

$$\int_{E_n \setminus E_{n-1}} |\phi| < 1 + O(n2^{-n}) \text{ as } n \to \infty, \quad (7)$$

where $\phi = \sum_{n=1}^{\infty} \phi_n$.

You can refer to the papers [8][9] and [10] for the proof of Theorem A.

C. Gromov hyperbolicity

The following definition of Gromov hyperbolicity comes from [10]. Let X be a geodesic metric space, that is, a metric space (X, d_X) where every pair of $A,B \in X$ can be connected by the isometric image of the segment $[0, d_X(A,B)]$. In such a space,we can define the notion of a triangle with vertices A, B and C in X to be the union of geodesic segments [AB], [BC] and [AC] which connecting A and B, B and C, and A and C , respectively.

Definition 1. *The geodesic metric space X is Gromov hyperbolic if there is a number* $\delta > 0$ *so that for every triangle* $\Delta = [AB] \cup [BC] \cup [AC]$ *and every* $D \in [AB]$, *we have* $d_X(D, [BC] \cup [AC]) < \delta$

By the result of Chapter 15 of [5], we know that there is at least one geodesic segment connecting any two given points, so the asymptotic Teichmuller space with the asymptotic Teichmuller metric is a metric space.

III. MAIN THEOREM AND PROOF

The goal of this section is to prove the following theorem.

Theorem1: *Asymptotic Teichmuller space of Riemann surfaces of analytically infinite type with the asymptotic Teichmuller metric is not Gromov hyperbolic.*

The main idea to prove Theorem 1 is to constructing a sequence of triangles so that the condition in Definition 1 dose not hold. To prove the theorem, we need the following lemmas.

Lemma 1. *Let* $\{E_n\}$, $\{\phi_n\}$ *and* ϕ *be constructed in Theorem A, and*

$$\mu(z) = \begin{cases} k_1 \dfrac{\overline{\phi(z)}}{|\phi(z)|}, z \in E_{2n+1} \setminus E_{2n}, \\ k_2 \dfrac{\overline{\phi(z)}}{|\phi(z)|}, z \in E_{2n} \setminus E_{2n-1}, \end{cases}$$

where $k_1, k_2 \in (-1,1)$ *are real constant. Then the Beltrami differential* $\mu(z)$ *is asymptotically extremal.*

Proof. If $k_1 > k_2$, then $h^*(\mu) = k_1$, we claim that $\lim_{n\to\infty} \left|\int_R \mu\phi_{2n+1} dxdy\right| = h^*(\mu)$, which implies the Lemma. For

$$\int_R \mu\phi_{2n+1} = \int_{R\setminus E_{2n+1}} \mu\phi_{2n+1} + \int_{E_{2n+1}\setminus E_{2n}} k_1 \frac{\bar\phi}{|\phi|}\phi_{2n+1} + \int_{E_{2n}} \mu\phi_{2n+1} . \quad (8)$$

It following from (4) that

$$\lim_{n\to\infty} \left|\int_{R\setminus E_{2n+1}} \mu\phi_{2n+1}\right| \le |k_1| \lim_{n\to\infty} \int_{R\setminus E_{2n+1}} |\phi_{2n+1}| = 0 ,$$

so

$$\lim_{n\to\infty} \int_{R\setminus E_{2n+1}} \mu\phi_{2n+1} = 0 . \quad (9)$$

It following from (5) that

$$\lim_{n\to\infty} \left|\int_{E_{2n+1}} \mu\phi_{2n+1}\right| \le |k_1| \lim_{n\to\infty} \int_{E_{2n+1}} |\phi_{2n+1}| = 0 ,$$

so

$$\lim_{n\to\infty} \int_{E_{2n}} \mu\phi_{2n+1} = 0 . \quad (10)$$

It following from (6) and (7) that

$$\int_{E_{2n+1}\setminus E_{2n}} k_1 \frac{\bar\phi}{|\phi|}\phi_{2n+1} = \int_{E_{2n+1}\setminus E_{2n}} k_1 \frac{\bar\phi}{|\phi|}\phi + \int_{E_{2n+1}\setminus E_{2n}} k_1 \frac{\bar\phi}{|\phi|}(\phi_{2n+1} - \phi) \to k_1 (n \to \infty) . \quad (11)$$

From (8)-(11), we conclude that

$$\lim_{n\to\infty} \left|\int_R \mu\phi_{2n+1}\right| = |k_1| = h^*(\mu)$$

If $k_2 \ge k_1$, by similar argument, we have

$$\lim_{n\to\infty} \left|\int_R \mu\phi_{2n}\right| = |k_2| = h^*(\mu) .$$

Which completes the proof.

Lemma 2. *Let* $f(t) = \dfrac{e^{2t}-1}{e^{2t}+1}, g(t) = \dfrac{t+k}{1+tk}$, *where* $k \in (-1,1)$ *is a real constant. Then* $g \circ f(t)$ *is an isometry from* $([0, \log\frac{1+k}{1-k}], d_E)$ *to* $([-k,k], \rho_\Delta)$, *where* d_E *and* ρ_Δ *denote the Euclidean metric on* $[0, \log\frac{1+k}{1-k}]$ *and the Poincare metric* ρ_Δ *on unit disk* respectively.

Proof. For g is a conformal mapping which preserve the unit disk , then g is an isometry $([0,\frac{2k}{1+k^2}],\rho_\Delta)$ to $([-k,k],\rho_\Delta)$. By some computation, we know that $f(t)$ is an isometry from $([0,\log\frac{1+k}{1-k}],d_E)$ to $([0,\frac{2k}{1+k^2}],\rho_\Delta)$, so $g\circ f(t)$ is isometric. Which completes the proof.

Let $\{E_n\}$ and ϕ be constructed in Theorem A, we define two Beltrami differentials on R as

$$\nu(z)=k\frac{\bar{\phi}}{|\phi|} \tag{12}$$

and

$$\eta(z)=\begin{cases} k\dfrac{\overline{\phi(z)}}{|\phi(z)|}, z\in E_{2n+1}\setminus E_{2n}, n=0,1,\ldots \\ -k\dfrac{\overline{\phi(z)}}{|\phi(z)|}, z\in E_{2n}\setminus E_{2n-1}, n=1,2,\ldots \end{cases} \tag{13}$$

Denote $A=[\nu]$, $B=[-\nu]$, $c=[\eta]$ and $\alpha=\frac{1}{2}\log\frac{1+k}{1-k}$, from the definition of asymptotic Teichmuller metric and Lemma 1, we know that $d(A,B)=d(B,C)=d(A,C)=2\alpha$. To prove Theorem 1, we will give an explicit formula for geodesic segments $[AB],[BC],[AC]$. By Lemma 2, we know that these geodesic segments are the isometric images of $([-k,k],\rho_\Delta)$.

Lemma 3. *Let* $\Gamma_{AB}(t)=[t\frac{\bar{\phi}}{|\phi|}]$, *then* Γ_{AB} *is an isometric image of* $([-k,k],\rho_\Delta)$ *which connects points A and B.*

Proof. It is easy to check that $\Gamma_{AB}(k)=A$ and $\Gamma_{AB}(-k)=B$. For any $t_1,t_2\in[-k,k]$,

$$\frac{\Gamma_{AB}(t_1)-\Gamma_{AB}(t_2)}{1-\Gamma_{AB}(t_1)\overline{\Gamma_{AB}(t_2)}}=\frac{t_1-t_2}{1-t_1t_2}\frac{\bar{\phi}}{|\phi|}.$$

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By Lemma 1, we have

$$d(\Gamma_{AB}(t_1),\Gamma_{AB}(t_2))=\frac{1}{2}\log\frac{1+h([\frac{t_1-t_2}{1-t_1t_2}\frac{\bar{\phi}}{|\phi|}])}{1-h([\frac{t_1-t_2}{1-t_1t_2}\frac{\bar{\phi}}{|\phi|}])}$$

$$=\frac{1}{2}\log\frac{1+\left|\frac{t_1-t_2}{1-t_1t_2}\right|}{1-\left|\frac{t_1-t_2}{1-t_1t_2}\right|}=\rho_\Delta(t_1,t_2).$$

Which completes the proof.

Lemma 4. *Let*

$$\Gamma_{AC}(t)=\begin{cases} k\dfrac{\overline{\phi(z)}}{|\phi(z)|}, z\in E_{2n+1}\setminus E_{2n}, n=0,1,\ldots \\ -t\dfrac{\overline{\phi(z)}}{|\phi(z)|}, z\in E_{2n}\setminus E_{2n-1}, n=1,2,\ldots \end{cases}$$

then Γ_{AC} *is an isometric image of* $([-k,k],\rho_\Delta)$ *which connects points A and C.*

Proof. It is easy to check that $\Gamma_{AC}(k)=A$ and $\Gamma_{AC}(-k)=C$. For any $t_1,t_2\in[-k,k]$,

$$\frac{\Gamma_{AC}(t_1)-\Gamma_{AC}(t_2)}{1-\Gamma_{AC}(t_1)\overline{\Gamma_{AC}(t_2)}}=\begin{cases} 0, z\in E_{2n+1}\setminus E_{2n}, n=0,1,\ldots \\ \dfrac{t_1-t_2}{1-t_1t_2}\dfrac{\overline{\phi(z)}}{|\phi(z)|}, z\in E_{2n}\setminus E_{2n-1}, n=1,2,\ldots \end{cases}$$

By Lemma 1, we have

$$d(\Gamma_{AC}(t_1),\Gamma_{AC}(t_2))=\frac{1}{2}\log\frac{1+\left|\frac{t_1-t_2}{1-t_1t_2}\right|}{1-\left|\frac{t_1-t_2}{1-t_1t_2}\right|}=\rho_\Delta(t_1,t_2).$$

Which completes the proof.

By the same argument, we have the similar following lemma.

Lemma 5. *Let*

$$\Gamma_{BC}(t)=\begin{cases} t\dfrac{\overline{\phi(z)}}{|\phi(z)|}, z\in E_{2n+1}\setminus E_{2n}, n=0,1,\ldots \\ -k\dfrac{\overline{\phi(z)}}{|\phi(z)|}, z\in E_{2n}\setminus E_{2n-1}, n=1,2,\ldots \end{cases},$$

then Γ_{BC} *is an isometric image of* $([-k,k],\rho_\Delta)$ *which connects points B and C.*

From Lemma3-Lemma5, we know that $\Gamma_{AB}(t)$, $\Gamma_{BC}(t)$ and $\Gamma_{AC}(t)$ can be regarded as an explicit formula for the geodesic segment [AB], [BC] and [AC] respectively.

Proof of Theorem 1.

Let $\Delta=[AB]\cup[BC]\cup[AC]$ be a triangle in the asymptotic Teichmuller space AT(R), where [AB], [AC] and [BC] be the same as them in Lemma 3-Lemma 5 respectively. Let $D=[0]=\Gamma_{AB}(0)\in[AB]$, by Lemma 4,5 and Lemma 1, then

$$d(D,[BC]\cup[AC])=\frac{1}{2}\log\frac{1+k}{1-k}.$$

As $k\to 1$, we see $d(D,[BC]\cup[AC])\to\infty$. Which completes the proof.

From the proof of Theorem 1, the following corollary is obvious.

Corollary 1. *Any ball in an asymptotic Teichmuller space of Riemann surfaces of analytically infinite type is not strictly convex with respect to geodesics.*

Proof.

For the Ball $B([0],r)=\{p\in AT(R),d([0],p)\le r\}$, where $r=\frac{1}{2}\log\frac{1+k}{1-k}$, $\Gamma_{AC}(t)$ and $\Gamma_{BC}(t)$ in Lemma 4 and 5 are geodesics on the

$$S([0],r)=\{p\in AT(R),d([0],p)=r\},$$

which implies the result of the corollary.

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