Approximation Operators Based on L-Fuzzy Set-Valued Mappings

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Abstract

In this paper, the fuzziness of Gomolińska approximation space based on uncertainty mappings was introduced. It is proved that many approximation operators could be constructed by composition of basic approximation operators.

Keywords: Approximation space, Fuzzy rough set, Approximation operator

1. Introduction

A rough set is a set-theory-based technique to handle data with granular structures by using two sets called the rough lower approximation and the rough upper approximation to approximation object. By using this technique, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules. The classical definition of a rough set was introduced by Pawlak [1] with reference to an equivalence relation (a binary relation with reflexivity, symmetry and transitivity).

From both theoretical and practical viewpoint, the equivalence relation is a very stringent condition that may limit applications of rough sets. Various extensions of the Pawlak rough set were therefore developed from an equivalence relation to a more general mathematical concept, e.g. a similarity relation (a binary relation with reflexivity and symmetry), a covering [2], or a neighborhood system from the topological space [3].

Dubois and Prade studied first the fuzzification problem of rough sets [4], [5]. Morsi and Yakout [6] studied a set of axioms on approximation operators of fuzzy sets and defined a special family of approximation operators of fuzzy sets using the T-norms and the residuation implicators. Additionally, Radzikowska and Kerre [7] gave another general method for the fuzzification of rough sets. They defined a broad family of fuzzy rough sets, each of which is determined by a triangular norm and an implicator. But the discussions of fuzzy rough set in many of article are based on [0,1]-fuzzy set rather than L-fuzzy set. Later, Radzikowska and Kerre [8] generalized the model of fuzzy rough set to L-fuzzy rough set based on residuated lattice and discuss some basic properties of approximation operators of the L-fuzzy rough set. T. Deng [9] fuzzify the general rough approximation operators and present a new approach to fuzzy rough sets through the use of techniques provide by residuated lattice.

In those L-fuzzy rough set model, approximation operators have several types of definition. In this paper, we studies the relation of those approximation operators in L-fuzzy approximation space. It is proved that those approximation operators of L-fuzzy rough set are special cases of approximation operators based on L-fuzzy set-valued mappings.

2. Preliminaries

A monoid is a structure $(U, \otimes, \varepsilon)$, where U is a nonempty universe, $\otimes$ is a binary operator on U and ε is a unit element of $\otimes$, i.e. for every $x \in U$, $\varepsilon \otimes x = x \otimes \varepsilon = x$. A monoid $(U, \otimes, \varepsilon)$ is commutative if and only if $\otimes$ is commutative. Typical example of monoid is triangle norm on $[0, 1]$, where $\varepsilon = 1$.

Let $(L, \leq)$ be a poset and $\otimes$ a binary operation on L. The residuum of $\otimes$ is a binary operator $\rightarrow$ in L satisfying the following residuated condition: for all $a, b, c \in L$, $a \otimes b \leq c \Leftrightarrow a \leq b \rightarrow c$. The operator $\rightarrow$ is bounded if and only if for every $x \in L$, $1 \rightarrow x = x$.

`Definition 2.1` *[10] By a residuated lattice, we mean an algebra* $\boldsymbol{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$*, such that*

(1) $(L, \vee, \wedge, 0, 1)$ *is a bound lattice with the top element 1 and the bottom element 0.*
(2) $\otimes : L \times L \rightarrow L$ *is a binary operator and satisfies for all* $a, b, c \in L$,
   - $a \otimes b = b \otimes a$,

- $a \otimes (b \otimes c) = (a \otimes b) \otimes c$,
- $1 \otimes a = a$,
- $a \leq b \Rightarrow a \otimes c \leq b \otimes c$.

(3) $\rightarrow$: $L \times L$ *is a residuum of* $\otimes$, *i.e.* $\rightarrow$ *satisfies for all* $a, b, c \in L$

$$a \otimes b \leq c \Leftrightarrow a \leq b \rightarrow c.$$

A residuated lattice $\mathbf{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is called complete iff the underlying lattice $(L, \vee, \wedge, 0, 1)$ is complete. Given a residuated lattice $\mathbf{L}$, we define the precomplement operator $\sim$ as following, for every $a \in \mathbf{L}$, $\sim a = a \rightarrow 0$.

Proposition 2.2 *[10]-[12], Suppose* $\boldsymbol{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ *is a residuated lattice, and* $\sim$ *is the precomplement operator on* $\boldsymbol{L}$. *Then for all* $a, b, c \in \boldsymbol{L}$,

(1) $a \otimes b \leq a \wedge b$, $a \rightarrow b \geq b$.
(2) $a \rightarrow (b \rightarrow c) = (a \otimes b) \rightarrow c$.
(3) *If* $a \leq b$, *then* $c \rightarrow a \leq c \rightarrow b$ *and* $b \rightarrow c \leq a \rightarrow c$.
(4) $a \leq b \Leftrightarrow a \rightarrow b = 1$.
(5) $a \leq b \Rightarrow \sim a \geq \sim b$.
(6) $a \leq \sim\sim a$.
(7) $a \rightarrow b \leq \sim (a \otimes \sim b)$,
$a \otimes b \leq \sim (a \rightarrow \sim b)$.
(8) *If* L *is a complete lattice, then*

$$(\vee_{i \in I} a_i) \otimes b = \vee_{i \in I}(a_i \otimes b),$$
$$a \rightarrow \wedge_{i \in I} b_i = \wedge_{i \in I}(a \rightarrow b_i),$$
$$\vee_{i \in I} a_i \rightarrow b = \wedge_{i \in I}(a_i \rightarrow b),$$
$$a \rightarrow \vee_{i \in I} b_i \geq \vee_{i \in I}(a \rightarrow b_i),$$
$$\wedge_{i \in I} a_i \rightarrow b \geq \vee_{i \in I}(a_i \rightarrow b).$$

Definition 2.3 *[9] Let* $\otimes$ *be a conjunction on a complete residuated lattice* L, *A* L*-fuzzy relation* R_L *on* U *is a* L*-fuzzy set* $R_L \in \boldsymbol{F}_L(U) \times \boldsymbol{F}_L(U))$ *is called:*

- *reflexive if* $R(x, x) = 1$, *for every* $x \in U$;
- *symmetric if* $R(x, y) = R(y, x)$, *for every* $x, y \in U$;
- $\otimes$*-transitive if* $R(x, z) \otimes R(z, y) \leq R(x, y)$, *for every* $x, y, z \in U$.

If the R *is reflexive, symmetric and* $\otimes$*-transitive, then* R *is* $\otimes$*-similarity.*

Proposition 2.4 *[7] Let* $\boldsymbol{L}$ *be a residuated lattice. If the* $R \in \boldsymbol{F}_L(U) \times \boldsymbol{F}_L(U)$ *is a reflexive and* $\otimes$*-transitive fuzzy relation, then for every* $x, y \in U$

$$R(x, y) = \bigvee_{y \in U} (R(x, z) \otimes R(z, y)) = \bigwedge_{y \in U} (R(x, z) \rightarrow R(z, y)).$$

3. Approximation operators in L-fuzzy approximation space

Definition 3.1 *[2] By an approximation space, we mean a triple* $A = (U, I, k)$, *where* U *is a non-empty set called the universe,* $I : U \rightarrow \boldsymbol{P}(U)$ *is an uncertainty mapping, and* $k : \boldsymbol{P}(U) \times \boldsymbol{P}(U) \rightarrow [0, 1]$ *is a rough inclusion function and satisfies: for every* $X, Y \in \boldsymbol{P}(U)$

- $k(X, Y) = 1$ *if and only if* $X \subseteq Y$;
- $k(X, Y) > 0$ *if and only if* $X \cap Y \neq \emptyset$.

Definition 3.2 *By a L-fuzzy approximation space, we mean a triple* $\boldsymbol{A} = (U, F_L, K)$, *where* U *is a non-empty set called the universe,* $F_L : U \rightarrow \boldsymbol{F}_L(U)$ *is an uncertainty mapping, and* $k : \boldsymbol{F}_L(U) \times \boldsymbol{F}_L(U) \rightarrow [0, 1]$ *is a rough inclusion function and satisfies: for every* $X, Y \in \boldsymbol{F}_L(U)$

- $K(X, Y) = 1$ *if and only if* $\wedge_{x \in U}[X(x) \rightarrow Y(x)] = 1$;
- $K(X, Y) > 0$ *if and only if* $\vee_{x \in U}[X(x) \otimes Y(x)] > 0$.

The mapping F may be viewed as a granulation function which assigned each $u \in U$ to a fuzzy set of $\mathbf{F}(U)$, i.e. an elementary granule of information. In this way an indexed family of fuzzy sets which is elementary granules of information from our perspective,

$$F_L^{\rightarrow}(U)(x) = \bigvee_{u \in U} [F_L(u)](x)$$

is obtained. Then $F_L^{\rightarrow}(U)$ is a fuzzy semi-partition of U, i.e.

$$\bigwedge_{x \in U} F_L^{\rightarrow}(U)(x) < 1,$$

or a fuzzy partition of U, i.e.

$$\bigwedge_{x \in U} F_L^{\rightarrow}(U)(x) = 1.$$

In the approximation space $\mathbf{A}$, we determine what fundamental properties any reasonable rough approximation mapping $f : \mathbf{F}_L(U) \rightarrow \mathbf{F}_L(U)$ should possibly posses. We distinguish two kinds of approximation mappings: lower and upper approximation mappings (in short low- and upp-mappings). Such "rationality" postulates for low- and upp-mapping could have the following forms:

- Every low-mapping f is decreasing. (i.e. for each $X \subseteq U, x \in U$, $f(X)(x) \leq X(x)$);

- Every upp-mapping f is increasing. (i.e. for each $X \subseteq U, x \in U$, $X(x) \leq f(X)(x)$);
- If f is a low-mapping, then for every $X \subseteq U, f(X)(x) = 1$, $K(f(x), X(x)) = 1$;
- If f is an upp-mapping, then for every $X \subseteq U, f(X)(x) = 1$, $K(f(x), X(x)) > 0$;
- For each $X \subseteq U$, $f(X)$ is definable in $\mathbf{A}$;
- For each $X \subseteq U$ is definable in $\mathbf{A}$, $f(X) = X$.

In the L-fuzzy approximation space $\mathbf{A} = (U, F_L, K)$, The mapping F_L generates a L-fuzzy binary relation $R_L \in U \times U$, such that for every $x, y \in U$

$$[F_L(x)](y) = R_L(x, y).$$

Then

- R_L is reflexive if and only if for every $x, y \in U$, $[F_L(x)](x) = 1$;
- R_L is symmetry if and only if for every $x, y \in U$, $[F_L(x)](y) = [F_L(y)](x)$;
- R_L is $\otimes$-transitive if and only if for every $x, y, z \in U$, $[F_L(x)](z) \otimes [F_L(z)](y) \leq [F_L(x)](y)$;
- R_L is $\otimes$-similarity if and only if for every $x, y, z \in U$, $[F_L(x)](z) \otimes [F_L(z)](y) = [F_L(x)](y)$.

$X \subseteq \mathbf{F}(U)$ is *definable* in an approximation space $\mathbf{A}$ if and only if there is a set $Y \subseteq U$ such that

$$X(x) = \bigvee_{y \in U} ((F_L(x))(y) \otimes Y(y)).$$

Let $\mathbf{C} = \{f(X) | f(x) \textit{ is definable}\}$, then $\mathbf{C}$ is a L-fuzzy partition of the universe U.

For every $X \in \mathbf{F}_L(U)$, $x \in U$, let

$$\begin{aligned}
&[D^+(x)](y) = R_L(y, x), \\
&[D^-(x)](y) = R_L(x, y); \\
&[D^+(X)](y) = \bigvee_{x \in X} R_L(y, x), \\
&[D^-(X)](y) = \bigvee_{x \in X} R_L(x, y).
\end{aligned}$$

represent *dominating set* and *dominated set* with respect to X, respectively.

In this section, we consider some mappings $f_1, f_2, f_3, f_4 : \mathbf{F}_L(U) \to \mathbf{F}_L(U)$, where for every $X \in \mathbf{F}_L(U)$, $x \in U$.

$$\begin{aligned}
f_1(X)(x) &= (D^+(X) \otimes X)(x) \\
&= \bigvee_{y \in U} (R_L(y, x) \otimes X(y)), \\
f_2(X)(x) &= (D^-(X) \otimes X)(x) \\
&= \bigvee_{y \in U} (R_L(x, y) \otimes X(y)); \\
f_3(X)(x) &= D^+(X)(x) \otimes X(x) \\
&= \bigvee_{y \in U} (R_L(y, x) \otimes X(x)), \\
f_4(X)(x) &= D^-(X)(x) \otimes X(x) \\
&= \bigvee_{y \in U} (R_L(x, y) \otimes X(x)).
\end{aligned}$$

and their respective "dual" mappings $f_1^d, f_2^d, f_3^d, f_4^d$

$$\begin{aligned}
f_1^d(X)(x) &= (D^+(X) \to X)(x) \\
&= \bigwedge_{y \in U} (R_L(y, x) \to X(y)), \\
f_2^d(X)(x) &= (D^-(X) \to X)(x) \\
&= \bigwedge_{y \in U} (R_L(x, y) \to X(y)); \\
f_3^d(X)(x) &= D^+(X)(x) \to X(x) \\
&= \bigwedge_{y \in U} (R_L(y, x) \to X(x)), \\
f_4^d(X)(x) &= D^-(X)(x) \to X(x) \\
&= \bigwedge_{y \in U} (R_L(x, y) \to X(x)).
\end{aligned}$$

If a set $X \in \mathbf{F}_L(U)$ is definable in an approximation space $\mathbf{A}$ if and only if there is $Y \in \mathbf{F}_L(U)$ such that for every $x \in U$, $X(x) = f_0(Y)(x)$.

Proposition 3.3 *Consider any $f : \boldsymbol{F}_L(U) \to \boldsymbol{F}_L(U)$. $f(X)$ is definable for any $X \subseteq U$ if and only if there is a mapping $g : \boldsymbol{F}_L(U) \to \boldsymbol{F}_L(U)$ such that $f = f_0 \circ g$.*

Proposition 3.4 *In the L-fuzzy approximation space $\boldsymbol{A} = (U, F_L, K)$ based on residuated lattice $\boldsymbol{L}$, it holds for every $X \in \boldsymbol{F}_L(U)$, $x \in U$,*

(1) If R_L is symmetric, then $f_1(X)(x) = f_2(X)(x)$ and $f_3(X)(x) = f_4(X)(x)$;
(2) If R_L is reflexive, then $f_1(X)(x) = f_3(X)(x)$ and $f_2(X)(x) = f_4(X)(x)$;
(3) If R_L is $\otimes$-similarity, then $f_1(X)(x) = f_2(X)(x) = f_3(X)(x) = f_4(X)(x)$ and $f_i(X)(x) = f_i(X)(x) \circ f_i(X)(x)$, $i = 1, 2, 3, 4$;

From the Proposition 3.3 and 3.4, we draw a conclusion

Corollary 3.5 *Suppose that the L-fuzzy binary relation is $\otimes$-similarity. For any approximation operator f in L-fuzzy approximation space $\boldsymbol{A}$, there exists a mapping $g : \boldsymbol{F}_L(U) \to \boldsymbol{F}_L(U)$ such that $f = f_0 \circ g$.*

From the definition of L-fuzzy approximation operators $f_i, i = 1,2,3,4$, the L-fuzzy approximation operators can represent other approximation operator by the compound of $f_i, i = 1,2,3,4$. Such as the approximation operator of L-fuzzy rough set based on residuated lattice in [8], f_2, f_2^d are upper and lower approximation operators respectively. In the approximation operator in [9], the upper and lower operators are represented as $f_2^d \circ f_2$ and $f_2 \circ f_2^d$, respectively.

4. Properties of L-fuzzy approximation operators

Proposition 4.1 *For any sets $X, Y \subseteq \boldsymbol{F}_L(U)$, object $x, y \in U$, and $i = 1,2,3,4$, it holds that:*

(1) $f_i^d \le id \le f_i^d$;
(2) $f_1(x)$ *is definable;*
(3) $f_i(F^{\to}(U)) = f_i^d(F^{\to}(U)) = F^{\to}(U)$;
(4) $[f_1(x)](y) = 1, K([F_L(y)](x), X(x)) > 0$;
(5) $[f_1^d(x)](y) = 1, K([F_L(y)](x), X(x)) = 1$;
(6) $f_i(\emptyset) = \emptyset$, $f_i(U) = U$;
(7) f_i *and* f_i^d *are monotone;*
(8) $f_i(X \cup Y) = f_i(X) \cup f_i(Y)$;
(9) $f_i(X \cap Y) \subseteq f_i(X) \cap f_i(Y)$;
(10) $f_i^d(X \cup Y) \supseteq f_i(X) \cup f_i(Y)$;
(11) $f_i^d(X \cap Y) = f_i^d(X) \cap f_i^d(Y)$.

IMTL-algebra is the special case of residuated lattice [11],[12] . The upper and lower approximation operators of L-fuzzy rough set based on IMTL-algebra are dual [13] . Similarity, in the L-fuzzy approximation space based on IMTL-algebra, the L-fuzzy approximation operator $f_i, i = 1,2,3,4$ can find its dual operator.

Proposition 4.2 *If the L-fuzzy approximation space $\boldsymbol{A} = (U, F_L, K)$ based on IMTL-algebra, the approximation operator $f_i, i = 1,2,3,4$ have the their dual operator $f_i^d, i = 1,2,3,4$, respectively. that is say for every $X \in \boldsymbol{F}_L(U)$, $x \in U$,*

$$f_i(X)(x) = \sim f_i^d(\sim X)(x).$$

5. Conclusions

In approximation space A, the L-fuzzy approximation operators $f_i, i = 1,2,3,4$ degenerate the classical approximation operators of generalized rough set as following:

(1) $f_1'(X) = \bigcup\{R_s(x) | R_s(x) \cap X \neq \emptyset\}$,
$f_1'^d(X) = \bigcap\{R_s(x) | R_s(x) \subseteq X\}$;
(2) $f_2'(X) = \bigcup\{R_p(x) | R_p(x) \cap X \neq \emptyset\}$,
$f_2'^d(X) = \bigcap\{R_p(x) | R_p(x) \subseteq X\}$;
(3) $f_3'(X) = \{x | R_s(x) \cap X \neq \emptyset\}$,
$f_3'^d(X) = \{x | R_s(x) \subseteq X\}$;
(4) $f_4'(X) = \{x | R_p(x) \cap X \neq \emptyset\}$,
$f_4'^d(X) = \{x | R_p(x) \subseteq X\}$;

where $R_s(x)$ and $R_p(x)$ are dominating set $R_s(x) = \{y | y \in I(x)\}$ and dominated set $R_p(x) = \{y | x \in I(y)\}$ with respect to X, respectively.

Following the methods [2] , we can define the $f_i, i = 5,6,\cdots$. But the properties of the approximation operators in L-fuzzy approximation space isn't thoroughly extended from the approximation in generalized approximation space in [2] , such as, f_2 is L-fuzziness of $f_0' \circ f_1'^d$, and those properties needs more researchers to study.

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