Fractal Property for A Novel Function generated by Generalized Approximate 3x+1 Functions

WANG Jia
Experimental Instrument Center
Dalian Polytechnic University
Dalian, China, 116034
e-mail: wangjia@dlpu.edu.cn

LI Xin
Experimental Instrument Center
Dalian Polytechnic University
Dalian, China, 116034
e-mail: lixin@dlpu.edu.cn

Abstract—Today, study of generalized 3x+1 function becomes a highlight in fractal area. In this paper, since fractal property of generalized 3x+1 functions T(z) and C(z) is difficult to study, we study two approximate generalized 3x+1 functions B(z) and D(z), which are produced by T(z) and C(z), in the real axis. First, we provide a novel function A(z) from two functions B(z) and D(z), which are extended by T(z) and C(z). Then, we analyze fractal property of A(z), and point out that the fractal properties of A(z) contains the information of B(z) and D(z). Finally, we draw the fractal figures of A(z) by escape time algorithm. These fractal figures validate our conclusion of A(z).

Keywords- fractal; generalized 3x+1 function; generalized approximate 3x+1 function; convergence; escape time algorithm

I. INTRODUCTION

The 3n+1 conjecture, which was presented by Collatz, was born in year 1950. Soon it became a highlight in mathematical theory because of its simple definition and interesting conclusion. This conjecture is shown as below.

Definition 1. Collatz conjecture.

For a natural number n, to convert n with function F(n) in following Eq.1.

$$F(n) = \begin{cases} \dfrac{n}{2} & n \equiv 0 \bmod 2 \\ \dfrac{3n+1}{2} & n \equiv 1 \bmod 2 \end{cases} \quad (1)$$

When assuming $F^k(n)=F^{k-1}(F(n))$, Collatz conjecture can be described as follows.

There exists integer k⩾0 to make $F^k(n)=1$ for each n.

More than 60 years, many researchers have studied Collatz conjecture, which is called 3n+1/3x+1 problem/conjecture today.

Earlier, Garner researched in Collatz 3x+1 algorithm in 1981 [1]. His conclusion has been a base of followers. Then, in 1990th, more researchers began to study this strange problem and gained many conclusions.

In 1994, Bernstein finds a non-iterating 2-adic statement of the 3x+1 conjecture [2]. Then Belaga and his team extended 3x+1 conjecture to 3x+d context, and find a novel embedding method [3].

After year 2000, belong to the improvement of computer science, the study of 3x+1 conjecture becomes more rapid. Simons extended conclusions of Bernstein, and find the property of nonexistence of 2-cycles for the 3x+1 conjecture [4]. His team also found the theoretical and computational bound of m-cycles for the 3x+1 conjecture [5]. Meantime, Misiurewicz et al concluded the study of 3x+1 conjecture [6]. Soon, Monks proved the sufficiency of arithmetic progressions in the conjecture [7].

After Mandelbrot presented the thinking of fractal, the fractal soon used everywhere [8]. Since researchers tried many classic methods of mathematical theory to solve the conjecture, Pe J L generated a 3x+1 function and used computer to create its fractal [9]. On the other hand, Dumont and Reiter extended Eq.1 to a continuous function and generated a generalized 3x+1 function [10]. Then, Liu et al studied generalized 3x+1 functions. They found many fractal properties of generalized 3x+1 functions [11-12]. Then, they extend generalized 3x+1 functions to generalized approximate 3x+1 functions, and find fractal properties of these functions [13-14].

In fact, fractal figures of functions are same to its dynamical system. The convergence of fractal is its stable region. So after summarize studies heretofore, we also extend T(x) and C(x) to B(x) and D(x). We define attractive domain and series and find these characters of these two fractal functions. In fact attractive domain and series of T(x) and C(x) are similar to each other. Then we draw their fractal figures to validate the similarity.

The remainder of the paper is organized as follows. We present B(x), D(x) and our function with fractal properties in Section 2. Then, in order to validate our conclusion, we use escape time algorithm to create fractal of our function in Section 3. Finally, section 4 summarizes the main results of the paper.

Our main contribution is to show the similarity and self-similarity of B(x) and D(x) by their attractive domains and sequences.

II. FRACTAL PROPERTY OF THE NOVEL FUNCTION

First, we present formulas of T(x), C(x), B(x) and D(x) in following Eqs.2-5.

$$T(x) = \frac{1}{2}[x3^{\sin^2(\frac{\pi x}{2})} + \sin^2(\frac{\pi x}{2})] \quad (2)$$

$$C(x) = x - \frac{x}{2}\cos \pi x + \frac{1 - \cos \pi x}{4} \quad (3)$$

$$B(x) = \frac{1}{2} x3^{\sin^2(\frac{\pi x}{2})} \quad (4)$$

$$D(x) = x - \frac{x}{2}\cos \pi x \quad (5)$$

Then, we present our function A(x)=B(x)-D(x) in Eq.6. Fractals of this function can show the similarity of fractal in B(x) and D(x).

$$A(x) = \frac{1}{2} x3^{\sin^2(\frac{\pi x}{2})} - x + \frac{x}{2}\cos \pi x \quad (6)$$

In this way, we can computer the fixed point of A(x) by Eq.7.

$$x = \frac{1}{2} x3^{\sin^2(\frac{\pi x}{2})} - x + \frac{x}{2}\cos \pi x \quad (7)$$

So we know that the solutions of Eq.7 is the fixed point of A(x), which is also the solutions of B(x)=D(x). Simplifying Eq.7, we have Eq.8 with change real number x to complex number z=a+bi.

$$3^{\sin^2(\frac{\pi(a+bi)}{2})} + \cos \pi(a + bi) = 4 \quad (8)$$

From complex analysis, we know Eqs.9-10.

$$\sin^2 \frac{\pi(a + bi)}{2} = \frac{1}{2}(1 - \cos(\pi a + \pi bi)) = \frac{1}{2} - \cos \pi a \frac{e^{\pi b} + e^{-\pi b}}{4} + i \sin \pi a \frac{e^{\pi b} - e^{-\pi b}}{4} = u + vi \quad (9)$$

$$\cos \pi(a + bi) = \frac{e^{\pi b} + e^{-\pi b}}{2}\cos \pi a - i\frac{e^{\pi b} - e^{-\pi b}}{2}\sin \pi a \quad (10)$$

So, Input Eqs.9-10 to Eq.8, we have Eq.11.

$$3^u(\cos v \ln 3 + i \sin v \ln 3) + (1 - 2u - 2vi) = 4 \quad (11)$$

In this way, we know the solutions of Eq.12 are just the solutions of Eq.8.

$$\begin{cases} 3^u \cos v \ln 3 = 2u + 3 \\ 3^u \sin v \ln 3 = 2v \end{cases} \quad (12)$$

Then, we know that z=a+bi is a fixed point of A(z) if there exist u(a, b) and v(a, b) suit to Eq.12.

It is admittedly that Eq.12 is difficult to solve. So we can use u(a, b) and v(a, b) to find if there exist u and v suit to Eq.13. The solutions are presented in Fig.1.

$$y = \log_3 \frac{2x}{\sin(x \ln 3)} \quad (13)$$

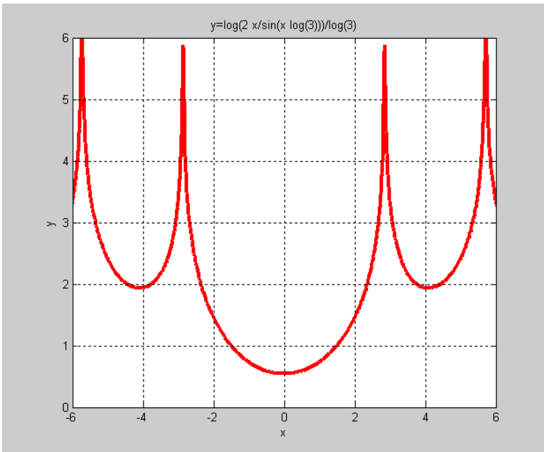

Figure 1. Solution line of Eq.13

To simplify it, when the discussing area from complex plane to real axis, we have b=0, z=a. Then, we know v=0. So the fixed points of A(z) is suit to Eq.14.

$$3u\text{-}2u\text{-}3=0 \quad (14)$$

The solutions of Eq.14 are presented in Fig.2. The two intersection points of red line and y=0 are the solutions. But we know u=1/2-cosπa. It means that -0.5≤u≤1.5. So these two solutions are all not fixed points. So we have theorem 1 to find the fixed points in real axis.

Theorem 1. A(x) has not fixed point in real axis.

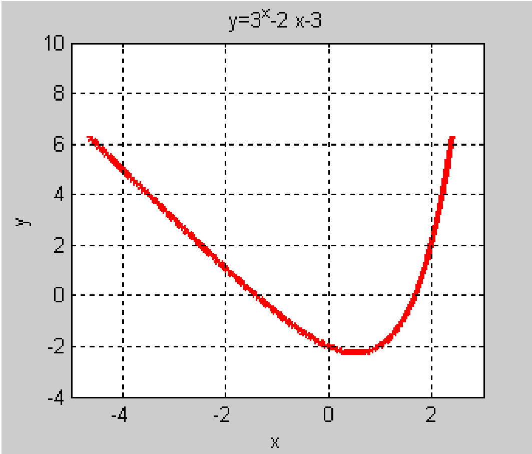

Figure 2. Solution line of $y=3^u-2u-3$

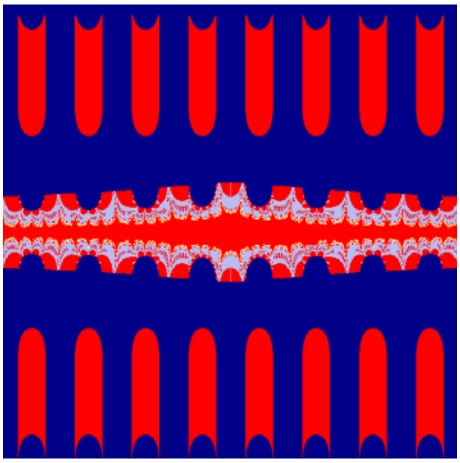

Figure 3. Fractals of A(z), (-8,8)×(-8,8)

Then, it is admittedly that the solutions of Eq.13 contain solutions of Eq.14. Meantime, solutions of Eq.13 changes to Eq.14 when $v=2k\pi/\ln 3$. So we know the distribution for fixed point of A(z) at complex plane.

III. Fractals of the Novel Function

We create fractals of A(z) by escape time algorithm [15], which is already extended to improved escape time algorithm [16]. Today, this algorithm is also extended to distributional and cloud environment [17-18].

Algorithm ETA (Escape Time Algorithm)

Step1. Let N_t be escape threshold. Let I_{Max} be the number of maximum iteration times. Let $m_c=0$ be the initial number of iteration times of any point c. Set $N_c=c$ as initial function value $f^0(c)$ of point c.

Step2. For each point c in computation area,

While $m_c \le I_{Max}$ and $|N_c|<N_t$ Repeat Let $m_c=m_c+1$, and $N_c=f(N_c)$.

Color point c with color value m_c.

Then, If $m_c> I_{Max}$, then point c is convergent; otherwise, it is divergent.

First, fig.3 presents the main fractal image of A(z). The color is changed from red to blue, where convergence colors red and divergence colors blue. The escape threshold is 1000, max iterating time is 100.

Then, because the outer convergence are usually taken as computational error, we present the middle of fig.3 in figs.4-5. They show the details of A(z)'s fractal.

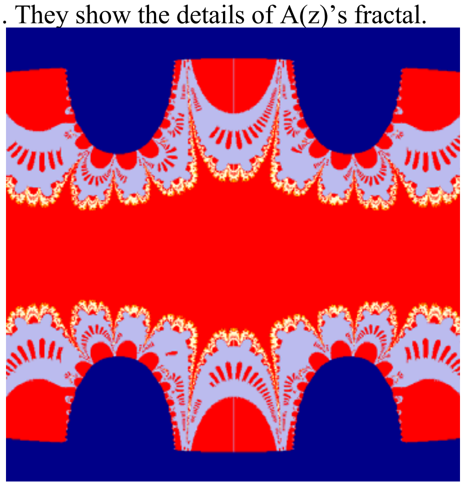

Figure 5. Fractals of A(z), (-2,2)×(-2,2)

Then, we can see the fragment of this fractal in detail. They are presented in figs.6-8. In these figures, we can validate the similar of different fragment in this fractal. Especially, fig.8 is also a fragment of fig.7. So it also validate the property of self-similar in A(z)'s fractals.

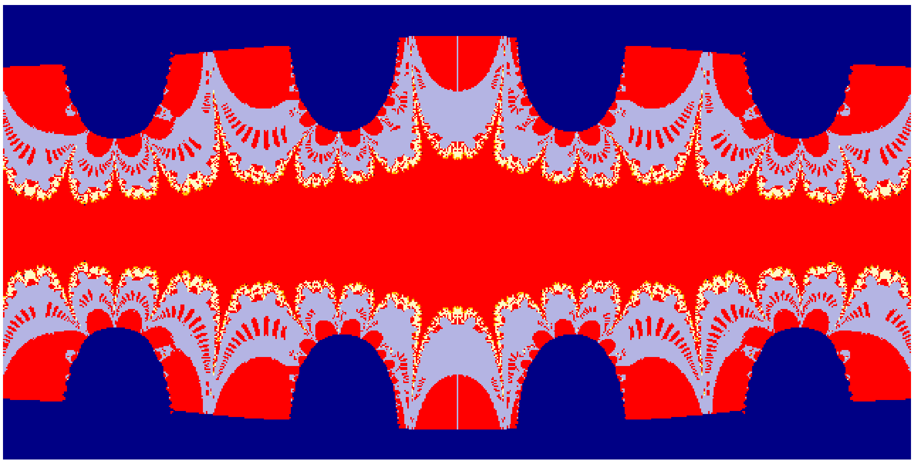

Figure 4. Fractals of A(z), (-4,4)×(-2,2)

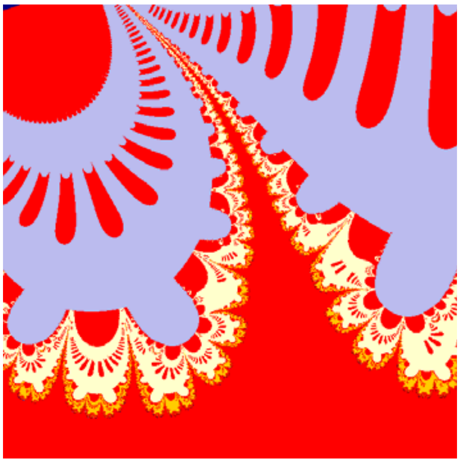

Figure 6. Fractals of A(z) in detail, (1.125,1.7)×(0.3375,0.9125)

Figure 7. Fractals of A(z) in detail, (1.775,1.9125)×(0.425,0.5625)

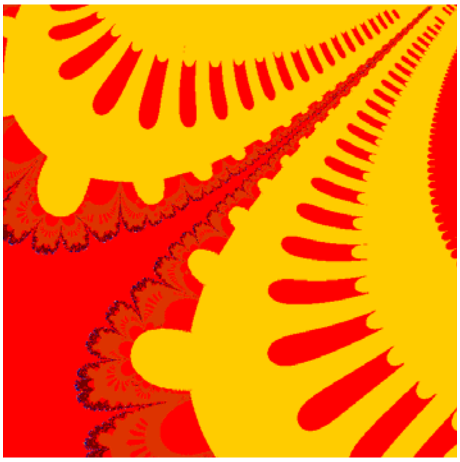

Figure 8. Fractals of A(z) in detail, (1.8410,1.8543)×(0.4714,0.4847)

IV. CONCLUSION AND FUTURE WORK

In this paper, we have produced a novel function A(z) from two generalized approximate 3x+1 functions B(z) and D(z). We analyzed the fractal properties of A(z) and created its fractal images to validate our conclusions.

.In next step, we will research in the function A(z) deeper. We will find the scale of its self-similar and its convergence and divergence in detail.

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