# On Continuity and Boundedness of Fuzzy Syntopogenous Spaces

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## Abstract

In this paper , we combine fuzzy topological structures with algebraic structures on X , and investigate their corresponding structures and properties. In the section 3, the relationship between fuzzy topological structure and increasing(decreasing) fuzzy syntopogenous structure is studied. In the section 4 the definition of continuity of a preordered L-fuzzy syntopogenous space $(X, S, \leq)$ and an example is given. In the section 5 the equivalent depictions of continuity are researched. In the section 6 the boundedness and its properties are discussed.

**Keywords**: Fuzzy topology , Algebra , Order , Continuity, Boundedness

## 1. Introduction

In [39] A·Csaszar introduced the concept of a syntopogenous structure to develop a unified approach to the three main structures of set-theoretic-topology: topologies, uniformities and proximities. This enable him to evolve a theory including the foundations of the three classical theories of topological spaces,uniform spaces and proximity spaces.In the case of the fuzzy structures there are at least three notions of fuzzy syntopogenous structures. The first notion worked out in [4,5,6,12] presents aunified approach to the theories of Chang fuzzy topological spaces[1], Hutton fuzzy uniform spaces[2] and Liu fuzzy proximity spaces[7]. The second notion worked out in [36,37] agree very well with Lowen fuzzy topological spaces[20], Lowen-Hohle fuzzy uniform spaces [17,21] and Artico-Moresco fuzzy proximity spaces[14]. The third notion worked out in [23] agree with the framework of a fuzzifying topology[25-28]. In [29], sostak introduced a new approach for a fuzzy topology as a fuzzy subset of the fuzzy powerset $I^X$(i.e. a mapping $\tau : I^X \to I$) satisfying certain axiom, the corresponding theory of fuzzy topological spaces containing Chang's approach as special, in a certain sense crisp case, was developed in a series of subsequent papers[29-35]. Based on the first notion of fuzzy syntopogenous structure the authors [9]established the general theory of syntopogenous structures on a completely distributive lattice and researched the unified question of cotopology, quasi-uniformity and T-structure. In [10], we combined fuzzy topological structure on X with algebraic structure on X, and investigated their corresponding structures and their properties. Concretely , we studied the preorder relation generated by an L-fuzzy syntopogenous structure. Conversely we researched the L-fuzzy syntopogenous structure defined by a preorder relation. And the increasing (decreasing) L-fuzzy syntopogenous space with preorder were defined , their properties were studied and an important example of an increasing L-fuzzy syntopogenous space $(R_\phi, S_R, \leq)$ was given. In this paper, we continue the research of [10]. In the section 3, the relationship between fuzzy topological structures and increasing(decreasing)fuzzy syntopogenous structures is studied. In the section 4 the definition of continuity of a preordered L-fu$t$ syntopogenous space $(X, S, \leq)$ and two examples are given. In the section 5 the equivalent depictions of continuity are researched. In the section 6 the boundedness and its properties are discussed. In this paper we use notation,which is standard for the "fuzzy mexthematics", usually without explanation.

## 2. Preliminaries

In this paper, $L = < L, \leq, \wedge, \vee, ' >$ always denotes a completely distributive lattice with order-reversing involution "′" . Let O be the least element and 1 be the greatest one in L . Suppose X is a nonempty(usual)set, an L-fuzzy set in X is a mapping $A: X \to L$, and $L^X$ will denote the family of all L-fuzzy sets in X . It is clear that $L^X = < L^X, \leq, \wedge, \vee, ' >$ is a fuzzy lattice, which has the least element $\underline{0}$ and the greatest one $\underline{1}$, where $\underline{0}(x) = 0, \underline{1}(x) = \underline{1}$, for any $x \in X$.

**Definition 2.1.[4])** A binary relation << on $L^X$ is called an L-fuzzy semi-topogenous order if it satisfies the following axioms : (1)$\underline{0}<<\underline{0}$ and $\underline{1}<<\underline{1}$ ; (2) $A<<B$ implies $A\le B$ ; (3) $A_1\le A<<B\le B_1$ implies $A_1<<B_1$ . The complement of an L-fuzzy semi-topogenous order $<<$ is the L-fuzzy semi-topogenous order $<<^c$ defined by $A<<^c B$ iff $B'<<A'$ . An L-fuzzy semi-topogenous order $<<$ is called : (i)symmetrical if $<<=<<^c$ ; (ii)topogenous if $A_1<<B_1$ and $A_2<<B_2$ imply $A_1\wedge A_2<<B_1\wedge B_2$ and $A_1\vee A_2<<B_1\vee B_2$ ; (iii)perfect if $A_j<<B_j, j\in J$ implies $\vee A_j<<\vee B_j$ ; (iv)co-perfect if $A_j<<B_j, j\in J$ implies $\wedge A_j<<\wedge B_j$ ; (v)biperfect if it is perfect and co-perfect.

Suppose that $<<_1$ , $<<_2$ are L-fuzzy semi-topogenous orders on $X$ , we call $<<_1$ finer than $<<_2$ ( *i.e.* $<<_2$ is coarser than $<<_1$ ) if for any $A,B\in L^X$, $A<<_2 B$ implies $A<<_1 B$ , it is denoted by $<<_2\le<<_1$ . If $<<_1\le<<_2$ and $<<_2\le<<_1$ , then $<<_1=<<_2$. For a given L-fuzzy semi-topogenous order $<<$ , we define $<<^p,<<^i,<<^b$ as follows : $A<<^p B$ iff there exist $A_i, i\in I$ , such that $A=\vee A^i$ , and for any $i, A_i<<B; A<<^i B$ iff there exist $B_j, j\in J$ such that $B=\wedge B_j$ and $B<<B_j$ for any $j\in J; A<<^b B$ iff there exist $A_i$ , $i\in I, B_j, j\in J$ such that $A=\vee A_i, B=\wedge B_j$ and $A_i<<B_j$ for any $i\in I, j\in J$ .

**Definition 2.2.([6])** An L-fuzzy syntopogenous structure on $X$ is a nonempty family S of L-fuzzy topogenous orders on X having the following two properties:(LFS1)S is directed in the sense that given any two members of S there exists a member of S finer than both ; (LFS2)for any $<<$ in S there exists $<<_1$ in S such that $A<<B$ implies the existence of an L-fuzzy set $C$ with $A<<_1 C<<_1 B$ .

Let $S(X)$ be the set of all L-fuzzy syntopogenous structures on $X$ . If $S$ is an L-fuzzy syntopogenous structure on $X$ , then the pair $(X,S)$ is called an L-fuzzy syntopogenous space . An L-fuzzy syntopogenous structure S consisting of a single topogenous order is called a topogenous structure and the pair $(X,S)$ is called an L-fuzzy topogenous space . $S$ is called perfect (resp. co-perfect, biperfect) if each member of S is perfect(resp . co-perfect , biperfect) . An L-fuzzy syntopogenous structure $S_1$ is called finer than another one $S_2$ , if for each $<<$ in $S_2$ there exists a member of $S_1$ finer than $<<$. In this case we also say that $S_2$ is coarser than $S_1$ , denoted by $S_2\le S_1$ . If $S_1$ is finer than $S_2$ and $S_2$ finer than $S_1$ , then $S_1$ , $S_2$ are called equivalent, denoted by $S_1\sim S_2$ . To every L-fuzzy syntopogenous structure correspond two L-fuzzy topologies $\tau_{s_0}$ 、 $\tau_s^*$ , $\tau_s$ given by the interior operator $\mu^0=\sup\{\rho: \rho<<\mu$ for some $<<\in S\}$ , $\tau_s^*$ given by the closure operator $\overline{\mu}=\wedge\{\rho:\mu<<\rho$ for some $<<\in S\}$ . If $\{<<_a: a\in I\}$ is a family of L-fuzzy semi-topogenous orders on X then $<<=\vee_{a\in I}<<_a$ is the fuzzy semi-topogenous order defined by $\mu<<\rho$ iff $\mu<<_a\rho$ for some $a\in I$ . If S is a fuzzy syntopogenous structure, then it is easy to see that $<<_s=\vee\{<<:<<\in S\}$ is an L-fuzzy topogenous order and than $\{<<_s\}$ is an L-fuzzy topogenous structure . Moreover , $\mu\in\tau_s$ iff $\mu<<_s^p\mu$ and $\mu\in\tau_s^*$ iff $\mu'<<_s^i\mu'$ . To every L-fuzzy topology $\tau$ on $X$ corresponds a perfect L-fuzzy topogenous structure , $S_\tau=\{<<\}$ where $\mu<<\rho$ iff there exists $\sigma\in\tau$ with $\mu\le\sigma\le\rho$ , and a co-perfect L-fuzzy topogenous structure $S_\tau^*=\{<<\}$ , where $\mu<<\rho$ iff there exists $E'\in\tau$ with $\mu\le E\le\rho$ . Moreover , $\tau=\tau_{S_c}$ and $\tau=\tau^*_{S_\tau^*}$ , conversely , to every perfect(or co-perfect)L-fuzzy topogenous structure $S=\{<<\}$ corresponds the L-fuzzy topology $\tau=\tau_s$ (or $\tau=\tau_s^*$ ) where $\mu\in\tau_s$ iff $\mu<<\mu$ (or $\mu'<<\mu'$ ). To two different L-fuzzy topologies correspond different perfect(or co-perfect)L-fuzzy topogenous structures.

# 3. Topology and preorder

A preorder on $X$ is a binary relation "≤" on X which is reflexive and transitive. A preorder on X which is also anti-symmetric is called a partial order or simply an order. By a preordered (resp. an ordered) set we mean a set with a preorder (resp. a partial order) on it.

**Definition 3.1.([3])** Let $(X,\le)$ be a preordered set. $A\in L^X$ is called:

(i)increasing, if $x\le y$ implies $A(x)\le A(y)$ ; (ii)decreasing , if $x\le y$ implies $A(y)\le A(x)$ ; (iii)order convex , if $y\le x\le z$ implies $A(y)\wedge A(z)\le A(x)$ .

**Definition 3.2.** Let $(X,\le)$ be a preordered set, define mappings p, a, c: $L^X\to L^X$ . as follows : for any $A\in L^X, x\in X, p(A)(x)=\vee\{A(y): y\le x\}$; $a(A)(x)=\wedge\{A(y): x\le y\}; c(A)=p(A)\wedge a(A)$.

**Lemma 3.1.**([10] theorem 3.1) Let $(X,S)$ be an L-fuzzy syntopogenous space, define a binary relation $\le_s$ on $X$ as follows : for any $x,y\in X, x\le_s y$ iff for $A\in L^X, <<\in S, \lambda\in L$, $\lambda\ne 0$ and $y_\lambda<<A$ implies $x_\lambda\le A$ . Then "≤"is a preorder on X, it is called the preorder generated by S on X.

**Lemma 3.2.**([10] theorem 3.5) Let $(X,\le)$ be a preordered set, define $<<$ on $L^X$ as follows : for any $A,B\in L^X$ . $A<<B$ iff $x\le y$ implies $A(y)\le B(x)$ , then $S_\le=\{<<\}$ is an L-fuzzy biperfect topogenous structure , and for any $x,y\in X, x\le y$ implies $x\le_{s\le} y$ .

**Definition 3.3.**([10] Def 4.1) Let $(X,\leq)$ be a preordered set, $S$ be an L-fuzzy syntopogenous structure on $X$, then $(X,S,\leq)$ is called increasing (decreasing)if for $x,y\in X, x\leq y$ implies $x\leq_s y(y\leq_s x)$.

**Lemma 3.3.**([10] Theorem 4.3) $S^{\mu}(S^{1})$ is the finest one of all increasing (decreasing)L-fuzzy syntopogenous structures which are coarser than $S$ on $S(X)$, where $S^{\mu}=\vee\{S'\in S(X):(X,S',\leq)$ increasing, $S'\leq S\}$;

$S^{1}=\vee\{S'\in S(X):(X,S',\leq)$ decreasing, $S'\leq S\}$.

**Proposition 3.4.** Let $(X,\tau)$ be an L-fuzzy topological space, then the set of increasing(decreasing) τ-open set is an L-fuzzy topology on $X$, denoted by $\tau^{\mu}(\tau^{1})$.

**Proof .** We prove Proposition 3.4 by [3] Proposition 3.2.

**Theorem 3.5.** Let $(X,\tau)$ be an L-fuzzy topological space, and $S_{\tau}^{*}=\{<<\}$, then $\tau_{S_{\tau}^{*\mu}}^{*}=\tau^{\mu}, \tau_{S_{\tau}^{*l}}^{*}=\tau^{l}$.

**Proof .** If $A\in\tau_{S_{\tau}^{*\mu}}^{*}$, as $S_{\tau}^{*\mu}\leq S_{\tau}^{*}$ by [10] Proposition 2.1 $\tau_{S_{\tau}^{*\mu}}^{*}\leq\tau_{S_{\tau}^{*}}^{*}=\tau$, also by [10] Proposition 4.1, $S_{\tau}^{*}\leq S_{\leq}$, hence $A\in\tau, A'<<_{\leq}A'$, $i.e. A\in\tau$ and $x\leq y$ implies $A'(y)\leq A'(x)$ (i. e. A increasing). So we have completed the proof of $\tau_{S^{*\mu}}^{*}\leq\tau^{\mu}$. Conversely, if $B\in\tau^{\mu}$ (i.e. $B\in\tau$ and B increasing), then $B'<<_{\tau}^{*}B'$ and by [10] Corollary 3.6 $B'<<_{\leq}B'$, let $<<_{0}=<<_{\tau}^{*}\cap<<_{\leq}$, as $<<_{0}\leq<<_{\leq}, <<_{0}\leq<<_{\tau}^{*}$ by Proposition 3.5 [5], we know that (X, $S_0$= $\{<<_{0}\}$, ≤) is increasing, and $S_{0}\leq S_{\tau}^{*}$, thus $S_{0}\leq S_{\tau}^{*\mu}$. But $B'<<_{0}B'$ implies $B'(\bigcup_{<<\in S_{\tau}^{*\mu}}<<)^{i}B'$, and by Proposition 2.1 [10] $B'\in\tau_{S_{\tau}^{*\mu}}^{*}$ so $\tau^{\mu}\leq\tau_{S_{\tau}^{*\mu}}^{*}$ thus $\tau_{S_{\tau}^{*\mu}}^{*}=\tau^{\mu}$. We can similarly prove $\tau^{1}=\tau_{S_{\tau}^{*l}}^{*}$.

Let $(X,\delta)$ be an L-fuzzy proximity space (see [7]). by Proposition 2.2[10]. $S=\{<<_{\delta}\}$ is an L-fuzzy symmetrical_topogenous structure, where $A<<_{\delta}B$ iff $A\delta B'$. Let's define $\delta_{s^{\mu}}$ as follows: $A\delta_{s^{\mu}}B$ iff $A<<B'$, for some $<<\in S^{\mu}$. It is easy to prove that $\delta_{s^{\mu}}$ is an L-fuzzy proximity, denoted by $\delta^{\mu}$. We have the following $\delta^{\mu}$ depiction theorem.

**Theorem 3.6.** $A\delta^{\mu}B$ iff there exist an increasing L-fuzzy proximity $\delta_{\alpha}$ (i.e. $(X,\{<<_{\delta\alpha}\},\leq)$ increasing) which is coarser than δ and families of L-fuzzy sets $\{A_{i}:i=1,\ldots,m\}$ and $\{B_{j}:j=1,\ldots,n\}$ such that $A=\vee_{i=1}^{m}A_{i}$, $B=\vee_{j=1}^{n}B_{j}'$ and $A_{i}\delta_{a}B_{j}'$ for $i\in\{1,2,\ldots,m\}, j\in\{1,2,\ldots,n\}$.

**Proof.** As $A\delta^{\mu}B$ iff $A<<B'$, $\{<<\}=(S^{\mu})^{t}$, $\left\{(\bigcup_{i=1}^{n}<<_{\alpha_{i}})^{q}:a_{i}\in I\right\}=S^{\mu}$, so $A<<B'$ iff $A(\bigcup_{i=1}^{n}<<_{a_{i}})^{q}B'$, by Proposition 3.1[6] there exist $\{A_1,\ldots,A_m\}$ $\{B'_1,\ldots,B'_n\}$ and $A=\vee_{i=1}^{m}A_{i}, B'=\wedge_{j=1}^{n}B_{j}$ for any i, j, $A_{i}(\bigcup_{i=1}^{n}<<_{\alpha_{i}})B_{j}$, thus there is $a_{i_0}\in I$ such that $A_{i}<<_{\alpha_{i_0}}B_{j}$ i.e. $A_{i}\delta_{\alpha_{i_0}}B_{j}'$.

**Theorem 3.7.** Let $(X,\leq)$ be a preorder set, define a binary relation $<<_{0}$ on $L^{X}$ as follows: for any $A,B\in L^{X}, A<<_{0}B$ iff $x\leq y$ implies $A(x)\leq A(y)$. Then $S_{\leq}^{*}=\{<<_{0}\}$ is an L-fuzzy biperfect topogenous structure, and for any $x,y\in X$, $x\leq y$ implies $x\leq_{S_{\leq}^{*}}y$.

**Proof .** Please see the proof of theorem 3.5 [10].

**Theorem 3.8.** Let $(X,\leq)$ be a preordered set, $H_{i}=\{E\in L^{X}:E$ is increasing on $(X,\leq)\}$, $H_{d}=\{E\in L^{X}:E$ is decreasing on $(X,\leq)\}$. We define binary relations $<<_{H_i}, <<_{H_d}$ as follows : $A<<_{H_i}B$ iff there exists $E\in H_{i}$ such that $A\leq E\leq B$; $A<<H_{d}B$ iff there exists $E\in H_{d}$ such that $A\leq E\leq B$. Then(1) $<<_{H_i}, <<_{H_d}$ are L-fuzzy biperfect topogenous orders; (2) $\{<<_{H_i}\}=S_{\leq}^{*}, \{<<_{H_d}\}=S_{\leq}$.

**Proof.** (1)We can prove the results immediately by Proposition 3.2[3]. (2)As $A<<_{H_i}B$ iff there exists $E\in H_{i}$ such that $A\leq E\leq B$, , also $x\leq y$ implies $A(x)\leq E(x)\leq E(y)\leq B(y)$, so $A(x)\leq B(y)$, thus $A<<_{0}B$, i.e. $\{<<_{H_i}\}\leq\{<<_{0}\}=S_{\leq}^{*}$.

Conversely, if $A<<_{0}B$, choose $E=p(A)\in H_{i}$, as $A<<_{0}B$, so $x\leq y$ implies $A(x)\leq B(y)$, hence $A\leq p(A)\leq B$, thus $A<<_{H_i}B$, i.e. $\{<<_{0}\}=S_{\leq}^{*}\leq\{<<_{H_i}\}$. Therefore $\{<<_{H_i}\}=S_{\leq}^{*}$.

Similarly for $<<_{H_d}=S_{\leq}$.

**Corollary 3.9.** Let $(X,\leq)$ be a preordered set, $S$ be an L-fuzzy syntopogenous structure on $X$, then $(X,S,\leq)$ is increasing iff $S\leq S_{H_d}=\{<<_{H_d}\}$.

**Proof.** We can prove the result immediately by Proposition 4. 2(3)[10].

# 4. Definition of continuity

A preordered (resp, an ordered) set $(X,\leq)$ on which there is a given L-fuzzy syntopogenous structure S is called a preordered (resp , an ordered) L-fuzzy syntopogenous space, denoted by $(X,S,\leq)$.

**Definition 4.1.** A preordered L-fuzzy syntopogenous space $(X,S,\leq)$ is called continuous iff for every $<<\in S$, there exists $<<_{1}\in S$ such that $A<<B$ implies $p(A)<<_{1}p(B)$ and $a(A)<<_{1}a(B)$.

**Example 4.1.** Any L-fuzzy syntopogenous space $(X,S)$ can be regarded as a preordered L-fuzzy syntopogenous space $(X,S,=)$ where "=" is the relation of the equality on X. Such space is always continuous.

**Example 4.2.** Let $(X,\leq)$ be a preordered set, define binary relation $<<_{0}$ on $L^{X}$ as follows: for any

$A, B \in L^X, A <<_0 B$ iff $x \le y$ implies $A(x) \le A(y)$, by Theorem 3. 7, $S_{\le}^* = \{<<_0\}$ is an L-fuzzy biperfect topogenous structure. Then $(X, S_{\le}^*, \le)$ is continuous.

In fact, if $A <<_0 B$ and $x \le y$, then $A \le B, z \le x$ implies $z \le y$, $A(z) \le B(z)$, therefore $p(A)(x) = \vee\{A(z) : z \le x\} \le p(B)(y) = \vee\{B(z) : z \le y\}$, $p(A) <<_0 p(B)$. Similarly $a(A) <<_0 a(B)$.

**Theorem 4.1.** Let $(X, \tau)$ be an L-fuzzy topological space, $S = \{<<_\tau\}$, then preordered L-fuzzy syntopogenous space $(X, S, \le)$ is continuous iff for any $\mu \in \tau$ implies $p(\mu) \in \tau$ and $a(\mu) \in \tau$.

**Proof.** Sufficiency: for any $\mu \in \tau, \mu <<_\tau \mu$, implies $p(\mu) <<_\tau p(\mu)$, $a(\mu) <<_\tau a(\mu)$, so $p(\mu) \in \tau$ and $a(\mu) \in \tau$. Necessity: if $\mu \in_\tau \rho$, then there exists $\mu_0 \in \tau$ such that $\mu \le \mu_0 \le \rho$, so $p(\mu) \le p(\mu_0) \le p(\rho)$ and $a(\mu) \le a(\mu_0) \le a(\rho), p(\mu_0) \in \tau, a(\mu_0) \in \tau$, hence $p(\mu) <<_\tau p(\rho)$ and $a(\mu) <<_\tau a(\rho)$.

**Proposition 4.2.** Let $(X, \delta)$ be an L-fuzzy proximity space([7]), $S = \{<<_\delta\}$, then $(X, S, \le)$ is continuous iff for every increasing (or decreasing) set $\rho \in L^X$, we have that $\rho(\mu)\delta\rho$ implies $\mu\delta\rho$ (or $a(\mu)\delta\rho$ implies $\mu\delta\rho$).

The proof is omitted.

**Proposition 4.3.** Let $U$ be an L-fuzzy uniformity structure ([2])on $X$, $S_u$ (see[5]) is induced by $U$, then $(X, S_u, \le)$ is continuous iff for every $u \in U$, there exists $u_1 \in U$, such that $u(A) \le B$ implies $u_1(p(A)) \le p(B)$ and $u_1(a(A)) \le a(B)$.

The proof is omitted.

# 5. Definition of continuity

**Theorem 5.1.** Let $(X, \le)$ be a preordered set, $<<$ be an L-fuzzy semi-topogenous order on X, we define binary relations $<<'$ and $<<''$ on $X$ as follows: $\mu <<' \rho$ iff there exist $\mu_1 \in L^X, \rho_1 \in L^X$ such that $\mu_1 << \rho_1$ and $\mu \le a(\mu_1), a(\rho_1) \le \rho, \mu <<'' \rho$ iff there exist $\mu_1, \rho_1 \in L^X$ such that $\mu_1 << \rho_1$ and $\mu << p(\mu_1), p(\rho_1) \le \rho$.

Then $<<'$ and $<<''$ are L-fuzzy semi-topogenous orders on $X$. Let $p(<<) = <<'^q, a(<<) = <<''^q$. If $S$ is an L-fuzzy syntopogenous structure on $X$, the $p(S) = \{p(<<) : << \in S\}$ and $a(S) = \{a(<<) : << \in S\}$ are L-fuzzy syntopogenous structures on $X$.

**Proof.** We can verify straightly from definition 2.1, 2.2.

In the same way we can define $c(<<)$ and $c(S)$.

**Proposition 5.2.** Let $(X, \le)$ be a preordered set, $<<$ be an L-fuzzy semi-topogenous order, then $p(<<^p) \le p(<<)^p$, and $a(<<^p) \le a(<<)^p$, $c(<<^p) \le c(<<)^p$. And if $S$ is an L-fuzzy syntopogenous structure on $X$, then $p(S^p) \le p(S)^p, a(S^p) \le a(S)^p, c(S^p) \le c(S)^p$ and $c(S^t) \sim c(S)^t$.

Then proof is omitted.

**Proposition 5.3.** If a preordered L-fuzzy syntopogenous space $(X, S, \le)$ is continuous, then $(X, S^p, \le), (X, S^t, \le)$ and $(X, S^{tp}, \le)$ are continuous.

The proof is omitted.

**Proposition 5.4.** For every $j \in J, (X, S_j, \le)$ is continuous, then $(X, \vee_{j \in J} S_j, \le)$ is continuous.

The proof is omitted.

**Theorem 5.5.** Let $(X, S, \le)$ be a preordered L-fuzzy syntopogenous space, then the following conditions are equivalent: (1) $(X, S, \le)$ is continuous; (2) $p(S) \le S$ and $a(S) \le S$; (3) $c(S) \le S$; (4) $c(S)^a \le S$, where $a \in \{t, p, b, tp\}$.

**Proof**. (1)iff(2). If $(X, S, \le)$ is continuous, then for $<< \in S$, there exists $<<_1 \in S$ such that $A << B$ implies $p(A) <<_1 p(B), a(A) <<_1 a(B)$, hence $p(<<) \le <<_1^q = <<_1, a(<<) \le <<_1^q = <<_1$. Conversely, if $p(S) \le S$ and $a(S) \le S$, then $<< \in S$, there exist $<<_1 \in S, <<_2 \in S$ such that $p(<<) \le <<_1$ and $a(<<) \le <<_2$, by Definition 2.2 there exists $<<' \in S$ Such that $<<_1 u <<_2 \le <<'$. Thus if $A << B$, then $p(A) p(<<) p(B)$ and $a(A) a(<<) a(B)$, so $p(A) <<' p(B)$ and $a(A) <<' a(B)$.

(2)iff (3) is obvious.

(3)iff (4). Because of $c(S) \le c(S)^a$, but $c(S)^a \le S$, so $c(S) \le S$, conversely, if $c(S) \le S$, then $c(S)^a \le S^a \sim$S.

**Corollary 5.6.** Let $(X, \le)$ be a preordered set, τ be a topology on $X$, $S = \{<<_\tau\}$. Let $G_p = \{p(\mu) : \mu \in \tau\}, G_a = \{a(\mu) : \mu \in \tau\}$. $G_c = G_a \vee G_p$, the topology generated by the subbase $G_p$ (resp. $G_a$, $G_c$) is denoted by $\tau_p$ (resp. $\tau_a, \tau_c$), then $\tau_c = \tau_p \vee \tau_a$, $\{<<_{\tau_p}\} \sim \{p(<<_\tau)^p\}, \{<<_{\tau_a}\} \sim \{a(<<_\tau)^p\}, \{<<_{\tau_c}\} \sim \{c(<<_\tau)^p\}$.

The proof is omitted.

**Corollary 5.7.** Let $(X, \delta)$ be an L-fuzzy proximity space, "$\le$"be a preorder on $X, S = \{<<_\delta\}$, Let $\delta_p = \delta_{p(<<_\delta)}, \delta_a = \delta_{a(<<_\delta)}$, then (1) $A\delta B$ iff $A \le \wedge_{j=1}^{n} p(x_j), B \le \vee_{j=1}^{n} y_j$, where $x_j \delta y_j$, and $y_j$ is decreasing, $A, B, x_j, y_j \in L^x, 1 \le j \le n$.

(2) $A\delta_a B$ iff $A \le \bigwedge_{j=1}^{n} a(x_j)$, $B \le \bigvee_{j=1}^{n} y_j$, where $x_j \delta y_j$ and $y_j$ is increasing, $A, B, x_j, y_j \in L^x, 1 \le j \le n.$

The proof is omitted.

**Corollary 5.8.** Let $(X, \le)$ be a preordered set, U be an L-fuzzy quasi-uniformity structure, let $p(S_\mu)^b = \{p(<<)^b \ : \ <<\in S_\mu\}$, the L-fuzzy quasi-uniformity structure induced by $p(S_\mu)^b$ denoted by $\mu_p$, similarly $\mu_a, \mu_c = \mu_p \vee \mu_a, S_{\mu_p}, S_{\mu_c}, S_{\mu_a}$ then $S_{\mu_p} = p(S_\mu)^b$, $S_{\mu_a} = a(S_\mu)^b$ and $S_{\mu_c} = c(S_\mu)^b$.

The proof is omitted.

# 6. Boundedness and its depictions

**Definition 6.1.** A preordered L-fuzzy syntopogenous space $(X, S, \le)$ is called bounded iff for every $<<\in S$, there exists $\varepsilon \subseteq L^x$ such that $A << B$ implies there exists $c \in \varepsilon$ with $A \le C \le B$.

**Theorem 6.1**. If $(X, S, \le)$ is totally bounded, then $(X, S, \le)$ is bounded.

**Proof**. We can prove the result immediately by theorem 3.4[12].

**Theorem 6.2**. A preordered L-fuzzy syntopogenous space $(X, S, \le)$ is bounded iff for every $<<\in S$, there exists $<<_1\in S$ and $\sigma \subseteq L^x$ such that $A << B$ implies there exist $C, D \in \sigma$ with $A \le C \le_1 D \le B$.

**Proof**. Necessity : We can prove the result by choose $\varepsilon = \sigma$.

Sufficiency : For $<<\in S$, by (LFS$_2$) there exists $<<_1$ with $<<\le<<_1 0 <<_1 0 <<_1$. As $<<_1\in S$ and $(X, S, \le)$ is bounded, So there is $\varepsilon \subseteq L^X$ with $A_0 <<_1 B_0$, therefore exists $C_0 \in \varepsilon$ with $A_0 \le C_0 \le B_0$. Now for $A << B$, $<<\le<<_1 0 <<_1 0 <<_1$, so there exist $C, D \in L^X$ with $A <<_1 C <<_1 D <<_1 B$. Thus there are $C_1, D_1 \in \varepsilon$ such that $A \le C_1 \le C$, $D \le D_1 \le B$, also $C_1 \le C <<_1 D \le D_1$ So $C_1 <<_1 D_1$. Above all, give $<<\in S$, there exist $<<_1\in S$ and $\varepsilon \subseteq L^X$ such that $A << B$ implies there is $C_1, D_1 \in \varepsilon$ satisfying $A \le C_1 <<_1 D \le B$, therefore the theorem is obtained.

**Proposition 6.3.** (1) If $S_0, S \in S(X), S \le S_0$ and $S_0$ is bounded, then $S$ is bounded. (2)If $S \in S(X)$, then $S$ is bounded iff $S^c$ is bounded.

The proof is omitted.

**Theorem 6.4.** Suppose $f: x \to y$ is a mapping, $S' \in S(Y)$ and $S = f^{-1}(S')$. (1)If $S'$ is bounded, then S is bounded. (2)If f is onto and $S'$ is bounded, then S′ is bounded.

**Proof**. (1)Suppose $<<' \in S', << = f^{-1}(<<')$, by def. 6. 1 there is $\varepsilon \subseteq L^Y$ such that $A_0 <<' B_0$ implies that there exists $c_0 \in \varepsilon$ with $\mathrm{A}_0 \le \mathrm{C}_0 \le \mathrm{B}_0$.

Now suppose $A << B$, so $f(A) << [f(B')]'$, therefore for some $c \in \varepsilon$ such that $f(A) \le c \le [f(B')]'$, and $A \le f^{-1}(C) \le B$.

(2)Suppose $<<' \in S'$ and $<< = f^{-1}(<<'), \varepsilon \subseteq L^X$ such that $A_0 << B_0$ implies there exists $C_0 \in \varepsilon$ with $\mathrm{A}_0 \le \mathrm{C}_0 \le \mathrm{B}_0$. Now suppose $A << B'$, so $f^{-1}(A) << f^{-1}(B)$, thus for some $c \in \varepsilon, f^{-1}(A) \le \mathrm{C} \le \mathrm{f}^{-1}(B).$ therefore $A \le f(C) \le B$.

**Theorem 6.5.** Suppose $S \in S(X), S' \in S(Y)$ and $f: (X, S) \to (Y, S')$ is $(S, S')$ -continuous onto mapping, If $S$ is bounded, then $S'$ is bounded.

**Proof** . As $f$ is $(S, S')$ -continuous , so $f^{-1}(S') \le S$, therefore $f^{-1}(S')$ is bounded. By theorem 6.4(2) $S'$ is bounded.

**Theorem 6.6.** $S \in S(X), S$ is bounded iff for every $<<\in S$, there exist $<_{\mu_i, \rho_i}, i \in I$, and $<<_1\in S$ such that $<<\le<<_2 = (\bigcup <_{\mu_i, \rho_i})^q \le<<_1$, where $<_{\mu_i, \rho_i}$ defined as follows: $A <_{\mu_i, \rho_i} B$ iff $A \le \mu_i \le \rho_i \le B, \mu_i, \rho_i \in L^X$.

**Proof**. Sufficiency : For $<<\in S$, by theorem 6.2 , there exist $<<_1\in S$ and $\sigma \subseteq L^x$, set $<<_2 = (\bigcup_{(\sigma_1, \sigma_2) \in \sigma \times \sigma} <_{\sigma_1, \sigma_2})^q$ where $(\sigma_1, \sigma_2) \in \sigma \times \sigma$ and $\sigma_1 <<_1 \sigma_2$. We easily prove $<<\le<<_2\le<<_1$.

**Necessity:** For $<<\in S$, there is $<<_1\in S$ and $<<_2 = (\bigcup_{i \in I} <_{\mu_i, \rho_i})^q$ such that $\le<<_2\le<<_1$. Set $\sigma = \{\mu \in L^X, \mu = \underline{0}$, or $\mu = \underline{1}$, or $\mu = \bigvee_k \bigwedge_j \mu_{m_{kj}}$ or $\mu = \bigvee_k \bigwedge_j \rho_{m_{kj}}$, where $m_{kj} \in I\}$, We can prove $\sigma$ satisfying the condition. In fact, if $\mu << \rho$, as $<<\le<<_2$, so $\mu <<_2 \rho,$ therefore exist $A_k, B_j, k = 1, 2, \ldots, K, j = 1, 2, \ldots N$ such that $\mu = \bigvee_{k=1}^{K} A_k, \rho = \bigwedge_{j=1}^{N} B_j$, $A_k <<' B_j$, where $<<' = \bigcup_{i \in I} <_{\mu_i, \rho_i}$; for $A_k <<' B_j$, there exists $m_{kj} \in I, A_k <_{\mu_{m_{kj}}, \rho_{m_{kj}}} B_j, A_k \le \mu_{m_{kj}} \le \rho_{m_{kj}} \le B_j \Rightarrow \mu = \bigvee_{k=1}^{K} A_k \le \bigvee_{k=1}^{K} \mu_{m_{kj}} \le \bigvee_{k=1}^{K} \rho_{m_{kj}} \le B_j$ $\Rightarrow \mu = \bigvee_{k=1}^{K} A_k \le \bigwedge_{j=1}^{N} \bigvee_{k=1}^{K} \mu_{m_{kj}} \le \bigwedge_{j=1}^{N} \bigvee_{k=1}^{K} \rho_{m_{kj}} \le \bigwedge_{j=1}^{N} B_j = \rho,$ choose $C = \bigwedge_{j=1}^{N} \bigvee_{k=1}^{K} \mu_{m_{kj}} \in \sigma, D = \bigwedge_{j=1}^{N} \bigvee_{k=1}^{K} \rho_{m_{kj}} \in \sigma$, So $\mu \le C \le D \le \rho$, also as $<<_2\le<<_1$, hence $\mu_i <<_1 \rho_i, i \in I$, and $C <<_1 D$, so the theorem is obtain.

**Theorem 6.7.** Suppose $(X, S_\lambda, \le)_{\lambda \in \wedge}$ is a family of preordered L-fuzzy

Syntopogenous spaces . so $S = \vee_{\lambda \in \vee} S_\lambda$ is bounded $\Leftrightarrow$ for every $\lambda \in \wedge, S_\lambda$ is bounded.

**Proof** . If S is bounded , as $S_\lambda \le S$, by proposition 6 . 3(1) $S_\lambda$ is bounded. Conversely, for $<<\in S$, suppose $<< = (\bigcup_{i=1}^{n} <<_{\lambda_i})^q$, where

$<<_{\lambda_i} \in S_{\lambda_i}$, $\lambda_i \in \wedge$, as $S_{\lambda_i}$ is bounded, order $\varepsilon_{\lambda_i} \subseteq L^{X^{\lambda_i}}$ satisfying $A_i <<_{\lambda_i} B_i \Rightarrow$ for some $C_{\lambda_i} \in \varepsilon_{\lambda_i}$ with $A_i \le C_{\lambda_i} \le B_i$. Now suppose A<<B, so $A = \vee_{j=1}^{m} A_j, B = \wedge_{k=1}^{n} B_k$ and $A_j <<_0 B_k$, where $<<_0 = \bigcup_{i=1}^{n} <<_i$, for (j, k) exists $C_{jk} \in \varepsilon_m = \bigcup_{i=1}^{n} \varepsilon_{\lambda_i}$ with $A_j \le C_{jk} \le B_k$, therefore $A \le \vee_{j=1}^{m} \wedge_{k=1}^{n} C_{jk} \le B$. By the definition 6.1, we obtain the theorem.

## Acknowledgement

This work is supported by National Nature Science Foundation of China (Grant No. 10671031), and partially Youth Science & Technology Foundation of Sichuan Province (Grant NO. 07ZQ026-114).

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