The Shapley value for fuzzy bi-cooperative games

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Abstract

In this article, we investigated a Shapley function on the class of crisp bi-cooperative games Firstly, we redefine bi-cooperative games axiom which was introduced by Grabisch and proposed the Shapley function that satisfies the four axioms. Then, the concepts related to a Shapley function have been extended to the case of fuzzy bi-cooperative games by choquet integral. Similarly, we define the four fuzzy Shapley axioms which correspond to four axioms crisp bi-cooperative Shapley, respectively. Finally, for the purpose of bridging the results to a real world problem, we gave a concrete example.

Keywords: Cooperative game, Fuzzy cooperative game, Bi-capacity, Shapley value, Choquet integral

1. Introduction

The Shapley value [1]-[2] is one of the most appealing solution concepts in cooperative game theory. Butnariu[14] defined a Shapley value and showed the explicit form of the Shapley function on a limited class of fuzzy games. Tsurmmi [5] defined new Shapley axioms and a new class of fuzzy games with Choquet integral form.But this two kinds of Shapley value can only be applied to cooperative games without competition. In fact, when player prepare to invest, there are always more than one economic item for him to choose. In this situation, Butnariu and Tsurmmi's games are not applicable any more. Therefore, Bibao [6]has proposed bi-cooperative games, which can compute relative value for two games at the same time. Ternary voting games of Felsenthal and Machover [7] are a particular case of bi-cooperative games. Also, independently, Greco et al [8],have proposed bipolar capacities, where they consider that the characteristic function is a pair of real numbers. Grabisch[9]has also give the Shapley value for crisp bi-cooperative games, but he did not consider the situation in which some players take part in a coalition to a certain extent.

In this paper, we follow Grabisch's axioms of Shapley function and introduce a new class of fuzzy games whose coalition is fuzzy. Moreover, we propose a new explicit Shapley function which is applicable to this new kind of fuzzy bi-cooperative games.

2. The Shapley value of crisp bi-cooperative games

We consider cooperative games with the set of players N={1,…, n}. We denote the set of all crisp subsets of a crisp set $W \subseteq N$ by $P(W)$, and $\wp(W) := \{(A, B) \in P(W) \times P(W) \mid A \cap B = \Phi\}$. A (cooperative)game $v : 2^N \to R$ is a set function such that $v(\Phi) = 0$. And a capacity v is a game such $A \subseteq B \subseteq N$ implies $v(A) \le v(B)$. The capacity is normalized if in addition $v(N) = 1$. A capacity v is additive if $v(A) = \sum_{i \in A} v(\{i\})$ for every $A \subseteq N$.

Definition 1 A function $v : \wp(N) \to R$ is a bi-capacity if it satisfies:

- $v(\Phi, \Phi) = 0$
- $A \subseteq B$ implies $v(A,\cdot) \le v(B,\cdot)$, $v(\cdot,A) \ge v(B,\cdot)$.

And v is normalized if $v(N, \Phi) = 1 = -v(\Phi, N)$.

In the sequel, we will consider that bi-capacities are normalized. Note that the definition implies that $v(\cdot, \Phi) \ge 0$ and $v(\Phi, \cdot) \le 0$. We say that a bi-capacity is of CPT type [12] if there exist two (normalized)capacities v_1, v_2 such that $v(A, B) = v_1(A) - v_2(B)$ for any $(A, B) \in \wp(N)$. By analogy with the classical case, a bi-capacity is said to be additive if it is of the CPT type with v_1, v_2 being additive, i.e., it satisfies for $(A, B) \in \wp(N)$, $v(A, B) = \sum_{i \in A} v_1(\{i\}) - \sum_{i \in B} v_2(\{i\})$.

We consider now bi-capacity as games, i.e., the mototonicity assumption (2) of Definition 1 is no more required. As Bilbao at al [6] , we could call such games bi-cooperative games. Let us denote by G(N) the set of all bi-cooperative games on N. For a bi-cooperative game, it can be interpreted like this : $v(S, T)$ is the worth of coalition S when T is the opposite coalition, and $N / S \cup T$ is the set of indifferent (indecisive) players. We call S the defender coalition, and T the defeater coalition. Hence, a bi-cooperative game reduces to an ordinary cooperative game v if it is equivalent to know either the defender coalition S or defeater coalition T, i.e. $v(S,T) = v(S,T^{/}) =: v(S)$ for all $T, T^{/} \subset N/S$ or $v(S,T) = v(S^{/},T) =: v(N/T)$ for all $S, S^{/} \subset: N/T$.

In bi-cooperative games, we denote by $\phi_{i,\Phi}^{v}$ and $\phi_{\Phi,i}^{v}$ the coordinates of the Shapley value for player i for the defender and the defeater parts, respectively. Hence, we consider the Shapley value as an operator on the set of bi-cooperative game $\phi : G(N) \to R^{2^n}$; $v \mapsto \phi^{v}$, for any finite support N, and coordinates of ϕ^{v} are either of $\phi_{i,\Phi}^{v}$ or $\phi_{\Phi,i}^{v}$ type. Grabisch also gave the axioms the Shapley value for bi-cooperative games should satisfy, but his axioms was not fully consistent with the Shapley axioms which was proposed by L. S. Shapley in 1953. Hence, we need to readjust the Shapley axioms.

Definition 2 Let $v \in G(N)$, $W \in P(N)$. Player $i \in W$ is called a left-null (resp. right-null) player in $(S,T) \in \wp(W \setminus \{i\})$ for v if the following holds, ie. $v(S \cup \{i\}, T) = v(S,T)$ (resp. $v(S, T \cup \{i\}) = v(S,T)$).

Definition 3 Let $v \in G(N)$, $W \in P(N)$. If $S \in P(W)$ (resp. $T \in P(W)$)is called a left(resp.right) -carrier in W for a game v if $v(S \cap S^{/}, T) = v(S^{/}, T), \forall T \in P(W \setminus S)$, $\forall S^{/} \in P(W \setminus T)$.(resp. $v(S, T \cap T^{/}) = v(S, T^{/})$, $\forall S \in P(W \setminus T), \forall T^{/} \in P(W \setminus S)$).
We denote the set of all left-carriers (resp.right -carriers)in W for v by $CL(W|v)$ (resp. $CR(W|v)$) .

Definition 4 Function $\phi : G(N) \to R^{2^n}$ is Shapley function on $G(N)$ if it satisfies the four axioms:

***Axiom b*1** (Efficiency): If $W \in P(N)$, $S, T \in P(W)$ is a left-carrier and right-carrier for game $v \in G(N)$ respectively, then $\sum_{i \in W} (\phi_{i,\Phi}^{v} + \phi_{\Phi,i}^{v}) = \sum_{i \in S} \phi_{i,\Phi}^{v} + \sum_{i \in T} \phi_{\Phi,i}^{v} = v(S,\Phi) - v(\Phi,T)$; if $i \notin S$, then $\phi_{i,\Phi}^{v}(W) = 0$, and if $i \notin T$, $\phi_{\Phi,i}^{v}(W) = 0$.

***Axiom b*2** (Fairness): If $W \in P(N),\ i, j \in P(W)$ and $v(S \cup \{i\}, W \setminus S \cup \{i\}) = v(S \cup \{j\}, W \setminus S \cup \{j\})$ holds for any $S \in P(W \setminus \{i, j\})$, then $\phi_{i,\Phi}^{v}(W) = \phi_{j,\Phi}^{v}(W)$; If $W \in P(N),\ i, j \in P(W)$ and $v(W \setminus T \cup \{i\}, T \cup \{i\}) = v(W \setminus T \cup \{j\}, T \cup \{j\})$ holds for any $T \in P(W \setminus \{i, j\})$, then $\phi_{\Phi,i}^{v}(W) = \phi_{\Phi,j}^{v}(W)$;

***Axiom b*3** (Symmetry): For $v_1, v_2 \in G(N)$, $W \in P(N)$ .If for some $i \in W$, and $\forall (S,T) \in \wp(W \setminus i)$, $v_2(S \cup \{i\}, T) - v_2(S,T) = v_1(S,T) - v_1(S, T \cup \{i\})$, then $\phi_{i,\Phi}^{v_2}(W) = \phi_{\Phi,i}^{v_1}(W)$.

***Axiom b*4** (Additivity): For any two games $v_1, v_2 \in G(N)$, $W \in P(N)$, define $v_1 + v_2$ by $(v_1 + v_2)(S,T) = v_1(S,T) + v_2(S,T)$ for any $(S,T) \in \wp(W)$. If $v_1 + v_2 \in G(W)$ then $\phi_{i,\Phi}^{v_1+v_2}(W) = \phi_{i,\Phi}^{v_1}(W) + \phi_{i,\Phi}^{v_2}(W)$, $\phi_{\Phi,i}^{v_1+v_2}(W) = \phi_{\Phi,i}^{v_1}(W) + \phi_{\Phi,i}^{v_2}(W)$, for all $i \in W$.

Theorem 1 Under the axioms in Definition 4, the Shapley value for bi-cooperative games is :

$$\phi_{i,\Phi}^{v}(W) = \sum_{S \in P_i(W) \setminus i} \frac{(|W| - s - 1)!\,s!}{|W|!} \times [v(S \cup \{i\}, W \setminus (S \cup \{i\})) - v(S, W \setminus (S \cup \{i\}))] \qquad (1)$$

$$\phi_{\Phi,i}^{v}(W) = \sum_{S \in P_i(W) \setminus i} \frac{(|W| - s - 1)!\,s!}{|W|!} [v(S, W \setminus (S \cup \{i\})) - v(S, W \setminus S)] \qquad (2)$$

where $W \in P(N), P_i(W) = \{S \in P(W) \mid i \in S\}$.

***Proof*:** We shall prove that the function $\phi_{i,\Phi}, \phi_{\Phi,i}$ defined by (1),(2) satisfy *Axiom b*1-4 .
*Axiom b*1: Let $S \in CL(W \mid v)$. If $i \notin S$, then we can get that
$v(S \cup \{i\}, W \setminus S \cup \{i\}) = v(S \cap (S \cup \{i\}), W \setminus S \cup \{i\})$
$= v(S, W \setminus S \cup \{i\})$
Therefore, player i is a left-null in $(S, W \setminus S \cup \{i\})$ for game v. According to formula (1), we get that $\phi_{i,\Phi}^{v}(W) = 0$.

Similarly, if $T \in CR(W \mid v)$, then we get that $\phi_{\Phi,i}^{v}(W) = 0$. Hence, we have

$$\sum_{i \in W} (\phi_{i,\Phi}^{v} + \phi_{\Phi,i}^{v}) = \sum_{i \in S} \phi_{i,\Phi}^{v} + \sum_{i \in T} \phi_{\Phi,i}^{v} .$$

And by [9], we know that $\sum_{i \in W} (\phi_{i,\Phi}^{v} + \phi_{\Phi,i}^{v}) = v(W,\Phi) - v(\Phi,W)$. Because $S \in CL(W \mid v)$ and $T \in CR(W \mid v)$, we obtain $v(W,\Phi) - v(\Phi,W) = v(S,\Phi) - v(\Phi,T)$. Thus,

$$\sum_{i \in W} (\phi_{i,\Phi}^{v} + \phi_{\Phi,i}^{v}) = \sum_{i \in S} \phi_{i,\Phi}^{v} + \sum_{i \in T} \phi_{\Phi,i}^{v} = v(S,\Phi) - v(\Phi,T)$$

*Axiom b*2:

If for any $S \in P(W \setminus \{i, j\})$,
$v(S \cup \{i\}, W \setminus S \cup \{i\}) = v(S \cup \{j\}, W \setminus S \cup \{j\})$,
then

$$\phi_{i,\Phi}^{v}(W) = \sum_{S \in P_{i}(W)\setminus i} \begin{array}{l}(|W|-s-1)!s!/|W|!\times \\ [v(S \cup \{i\}, W \setminus (S \cup \{i\})) - v(S, W \setminus (S \cup \{i\}))]\end{array}$$

$$= \sum_{S \in P_{j}(W)\setminus i} \begin{array}{l}(|W|-s-1)!s!/|W|!\times \\ [v(S \cup \{j\}, W \setminus (S \cup \{j\})) - v(S, W \setminus (S \cup \{j\}))]\end{array}$$

$$= \phi_{j,\Phi}^{v}(W)$$

Similarly, $\phi_{\Phi,i}^{v}(W) = \phi_{\Phi,j}^{v}(W)$ if
$v(W \setminus T \cup \{i\}, T \cup \{i\}) = v(W \setminus T \cup \{j\}, T \cup \{j\})$

*Axiom b*3: If for some $i \in W$, $\forall (S,T) \in \wp(W \setminus i)$,

$$v_{2}(S \cup \{i\}, T) - v_{2}(S,T) = v_{1}(S,T) - v_{1}(S, T \cup \{i\}),$$

$$\begin{array}{l} v(S \cup \{i\}, W \setminus (S \cup \{i\})) - v(S, W \setminus (S \cup \{i\}) \\ = v(S, W \setminus (S \cup \{i\})) - v(S, W \setminus S) \end{array}.$$

$$\phi_{i,\Phi}^{v_1}(W)$$
$$= \sum_{S \in P_{i}(W)\setminus i} \begin{array}{l}(|W|-s-1)!s!/|W|!\times \\ [v(S \cup \{i\}, W \setminus (S \cup \{i\})) - v(S, W \setminus (S \cup \{i\}))]\end{array}$$
$$= \sum_{S \in P_{i}(W)\setminus i} \begin{array}{l}(|W|-s-1)!s!/|W|!\times \\ [v(S, W \setminus (S \cup \{i\})) - v(S, W \setminus S)]\end{array} = \phi_{\Phi,i}^{v_2}(W)$$

*Axiom b*4: If $(v_{1}+v_{2})(S,T) = v_{1}(S,T) + v_{2}(S,T)$ for

any $(S,T) \in \wp(W)$, then

$$\phi_{i,\Phi}^{v_1+v_2}(W)$$
$$= \sum_{S \in P_{i}(W)\setminus i} \begin{array}{l}(|W|-s-1)!s!/|W|!\times[(v_1+v_2)(S \cup i, W \setminus (S \cup \{i\})) \\ -(v_1+v_2)(S, W \setminus (S \cup \{i\}))]\end{array}$$
$$= \sum_{S \in P_{i}(W)\setminus i} (|W|-s-1)!s!/|W|!\times \left\{\begin{array}{l}[v_1(S \cup i, W \setminus (S \cup \{i\})) - v_1(S, W \setminus (S \cup \{i\}))] \\ +[v_2(S \cup i, W \setminus (S \cup \{i\})) - v_2(S, W \setminus (S \cup \{i\}))]\end{array}\right\}$$
$$= \phi_{i,\Phi}^{v_1}(W) + \phi_{i,\Phi}^{v_1}(W)$$

Similarly, $\phi_{\Phi,i}^{v_1+v_2} = \phi_{\Phi,i}^{v_1} + \phi_{\Phi,i}^{v_2}$.

Definition 5 Game $v \in G(N)$ is superadditive, if for $\forall (S,T) \in \wp(N)$, $S' \in N \setminus S \cup T$,
$v(S \bigcup S', T) \geq v(S,T) + v(S', \Phi)$
and $v(S, T \cup S') \leq v(S,T) + v(\Phi, S')$.

Definition 6 A function $\boldsymbol{x} : G(N) \rightarrow R^{2^n}$ is a payoff function of game $v \in G(N)$ if it satisfies

(1) $\sum_{i \in N} x_{i,\Phi} + x_{\Phi,i} = v(N,\Phi) - v(\Phi,N)$;

(2) $x_{i,\Phi} \geq v(\{i\},\Phi), x_{\Phi,i} \leq v(\Phi,\{i\})$

where $W \in P(N)$, $\boldsymbol{x} = (\boldsymbol{x}_{1,\Phi}, \boldsymbol{x}_{\Phi,1}, \boldsymbol{x}_{2,\Phi}, \boldsymbol{x}_{\Phi,2}, \ldots$
$\ldots, x_{i,\Phi}, x_{\Phi,i}, \ldots, x_{n,\Phi}, x_{\Phi,n})$.

Theorem 2 If a game $v \in G(N)$ is superadditive, the Shalpley value defined by the Formula (1) and (2) is a payoff function on $G(N)$.

Proof: By the proof of *Axiom b*1, we get that $\sum_{i \in N} \phi_{i,\Phi}^{v} + \phi_{\Phi,i}^{v} = v(N,\Phi) - v(\Phi,N)$ since N is both the left-carrier and right-carrier in N for v. Hence, if we can show that $x_{i,\Phi} \geq v(\{i\},\Phi)$ and $x_{\Phi,i} \leq v(\Phi,\{i\})$, then the Shalpley value defined by (1) and (2) is a payoff function on $G(N)$. If v is superadditive, then for $\forall S \in N \setminus \{i\}$,
$v(S \cup \{i\}, N \setminus (S \cup \{i\})) - v(S, N \setminus (S \cup \{i\})) \geq v(\{i\},\Phi)$
By (1), we get that

$$\phi_{i,\Phi}^{v} \geq \left\{\sum_{S \subseteq N \setminus i} (n-s-1)!s!/n!\right\} \cdot [v(\{i\},\Phi)] = v(\{i\},\Phi).$$

Similarly, we prove $\phi_{\Phi,i} \leq v(\Phi,\{i\})$.

Lemma 1 Let $v \in G(N)$. If v is superadditive, then for $\forall (S,T) \in \wp(N), S' \in N \setminus S \cup T$, $v(S \bigcup S', T) \geq v(S,T)$, $v(S, T \cup S') \leq v(S,T)$.

Proof:
$v(S \bigcup S', T) \geq v(S,T) + v(S', \Phi)$, where
$S' \in N \setminus S \cup T$. Thus, $v(S \bigcup S', T) \geq v(S,T)$, since $v(S',\Phi) \geq 0$. Similarly, $S' \in N \setminus S \cup T$ for $v(S, T \cup S') \leq v(S,T)$ since $v(\Phi, S') \leq 0$.

3. The Shapley value of fuzzy bi-cooperative games

A fuzzy coalition of bi-cooperative games is a fuzzy subset of *N*, which is a vector ***S***={*S*(1),…,*S*(*n*)}with coordinates *S*(*i*) contained in the interval [0,1]. The number *S*(*i*) indicates the membership grade of in *S*. If the player participation number *S*(*i*)>0, then *S*(*i*) are consider as player *i* being in defender coalition *S* and the value -*S*(*i*) are considered to be in defeater coalition. And *S*(*i*)=0 means that player *i* does not take part in neither coalition. We denote the class of all fuzzy subsets of a fuzzy set $S \subseteq N$ by *F*(*S*). For $\forall S,T \in F(N)$, union, intersection and inclusion of two fuzzy sets are defined as usual, $(S \bigcup T)(i) = \max\{S(i), T(i)\}$, $(S \bigcap T)(i) = \min\{S(i), T(i)\}$, where $i \in N$. We denote
$\aleph(N) := \{(A,B) \in F(N) \times F(N) \mid A \cap B = \Phi\}$.
Given $U \in F(N)$, let
$Q(U) = \{U(i) \mid U(i) > 0, i \in N\}$ and $q(U)$ be the cardinality of *Q*(*U*). The elements of in *Q*(*U*) is in the increasing order as $h_1 \leq \ldots, \leq h_{q(S)}$.

Definition 7 Let $U \in F(N)$ .Then a game C_v is said to be a fuzzy bi-cooperative game if and only if for any $U \in F(N)$, $(S,T) \in \aleph(N)$,

$$C_{v(U)}(S,T) = \sum_{l=1}^{q(U)} v(S \cap U_{h_l}, T \cap U_{h_l}) \times (h_l - h_{l-1}) \quad (3)$$

The set of all fuzzy bi-cooperative games is denoted by $G_F(N)$ .There is one-to-one correspondence between a crisp bi-cooperative game and a fuzzy bi-cooperative game. It is apparent that (3) is Choquet integral [13]of the function U with regard to bi-capacity v . By Grabisch[13], we know that if v is special bi-capacity , i.e.,CPT type, then the Choquet integral $C_{v(U)}(S,T)$ has the following theorem.

Theorem 3 Give $S \in F(N)$, $v \in G(N)$. If v is of the CPT type, with $v(A,B) = v_1(A) - v_2(B)$, for $\forall (A,B) \in \wp(N)$, then for any $(S,T) \in \aleph(N)$, correlative fuzzy bi-cooperative game $C_v(S,T) = C_{v_+}(S) - C_{v_-}(T)$.

Remark 1 Let $C_v \in G_F(N)$. Given $U \in F(N)$, consider a set $Q(U) \subseteq \{k_1,\ldots,k_m\}$ such that $0 \le k_1 < \ldots < k_m \le 1$. Let $k_0 = 0$, $(S,T) \in \aleph(N)$, then the following holds:

$$C_{v(U)}(S,T) = \sum_{i=1}^{m} v([S]_{k_l},[T]_{k_l}) \cdot (k_l - k_{l-1})$$

Preparatory to the definitions of the fuzzy left-carrier and right-carrier, we define $S_i^U, U \setminus S_i^U \in F(U)$ for any $U \in F(N)$, $S \in F(U)$ and $i \in N$ as follows:

$$S_i^U(j) = \begin{cases} U(i), & if \ j = i, \\ S(j), & or, \end{cases}$$

$$(U \setminus S)(k) = \begin{cases} 0, & if \ S(k) > 0 , \\ U(k) , & otherwise. \end{cases}$$

Definition 8

Let $U \in F(N)$, $0 \le \gamma < U(i)$. Player i is called a γ -left(resp.right)-null in $(S,T) \in \aleph(U)$ for $C_v \in G_F(N)$ if $C_{v(U)}(S_i^U,T) = C_{v(U)}(S,T)$, $\forall S \in F(U)$, $(S,T) \in \aleph(N)$.

(resp. $C_{v(U)}(S,T_i^U) = C_{v(U)}(S,T)$, $\forall T \in F(U)$, $(S,T) \in \aleph(N)$).)

Definition 9 Let $C_v \in G_F(N)$, $U \in F(N)$. S(resp.T)is called a fuzzy left(resp.)-carrier for C_v if for $\forall T \in F(U \setminus S), \forall S' \in F(U \setminus T)$

$C_{v(U)}(S \cap S',T) = C_{v(U)}(S',T)$.

(resp. $\forall S \in F(U \setminus S)$, $\forall T' \in F(U \setminus S)$

$C_{v(U)}(S,T \cap T') = C_{v(U)}(S,T')$.)

We denote the set of all fuzzy left-carriers (resp.right-carriers) in U for C_v by $CL(U|C_v)$(resp. $CR(U|C_v)$) . Note that the definitions above can be applicable to crisp bi-cooperative games by restricting the domain. Preparatory to the definition of a fuzzy Shapley function, we define the following denotations. Let $U \in F(N)$. For any $S,T \in F(U)$, define S_i^U, $p(S_i^U) \in F(U)$ by

$$S_i^U(k) = \begin{cases} \min\{S(i),U(j)\}, & if \ k = i , \\ \min\{S(j),U(i)\}, & if \ k = j, \\ S(k) , & otherwise. \end{cases}$$

$$p_{ij}[U](k) = \begin{cases} U(j), & if \ k = i , \\ U(i), & if \ k = j, \\ U(k) , & otherwise. \end{cases}$$

Lemma 2 Let $C_v \in G_F(N), U \in F(N)$, $(S_1,T_1),(S_2,T_2) \in \aleph(U)$ such that $S_1 \subseteq S_2, T_1 \supseteq T_2$. Then $C_{v(U)}(S_1,T_1) = C_{v(U)}(S_2,T_2)$ if and only if $v([S_1]_h,[T_1]_h) = v([S_2]_h,[T_2]_h)$ for any $h \in (0,1]$.

Proof: $C_{v(U)}(S_1,T_1) = C_{v(U)}(S_2,T_2)$ if $v([S_1]_h,[T_1]_h) = v([S_2]_h,[T_2]_h)$ for any $h \in (0,1]$. We shall the reverse relationship. From Remark 1, we have

$$C_{v(U)}(S_1,T_1) - C_{v(U)}(S_2,T_2) = \sum_{l=1}^{q(U)} \{v([S_1]_h,[T_1]_h) - v([S_2]_h,[T_2]_h)\} \cdot (h_l - h_{l-1})$$

We have known that $S_1 \subseteq S_2, T_1 \subseteq T_2$ if and only if $[S_1]_h \subseteq [S_2]_h$, $[T]_{h1} \subseteq [T_2]_h$ for any $h \in [0,1]$. By Lemma 1, we get that $v([S_1]_h,[T_1]_h) \le v([S_2]_h,[T_1]_h) \le v([S_2]_h,[T_2]_h)$ holds for any $h \in [0,1]$. Thus, if $C_{v(U)}(S_1,T_1) = C_{v(U)}(S_2,T_2)$,

$v([S_1]_h,[T_1]_h)=v([S_2]_h,[T_2]_h)$ for any $h_l \in Q(U)$.

Note that $[S_1]_h=[S_1]_{h_l}$ $[S_2]_h=[S_2]_{h_l}$, $[T_1]_h=[T_1]_{h_l}$ and $[T_2]_h=[T_2]_{h_l}$ holds for any h satisfying $h_{l-1}<h\le h_l$. Hence, $v([S_1]_h,[T_1]_h)=v([S_2]_h,[T_2]_h)$ holds for any $h\in(0,h_{q(U)}]$. For any h satisfying $h>h_{q(U)}$, $[S_1]_h=[S_2]_h=[T_1]_h=[T_2]_h=\Phi$. Thus, $v([S_1]_h,[T_1]_h)=v([S_2]_h,[T_2]_h)=v(\Phi,\Phi)$.

Consequently, if $C_{v(U)}(S_1,T_1)=C_{v(U)}(S_2,T_2)$, then for any $h\in(0,1]$,

$v([S_1]_h,[T_1]_h)=v([S_2]_h,[T_2]_h)$.

Theorem 4 Let $C_v\in G_F(N)$ and $U\in F(N)$. If S is an fuzzy left-carrier(resp. right-carrier) in U for C_v then $[S]_h$ is a left-carrier(resp. right-carrier) in $[U]_h$ for the relative crisp bi-cooperative game $v\in G(N)$ for any $h\in(0,1]$.

Proof: The following always holds: $\{[T]_h \mid T\in L(U)\}=P([U]_h),\forall h\in(0,1]$.
By using Lemma 2 and the above relationship,

$$
\begin{aligned}
& S\in CL(U\mid C_v)\\
&\Leftrightarrow C_{v(U)}(S\cap S',T)=C_{v(U)}(S',T),\ \forall T\in F(U\setminus S),\forall S'\in F(U\setminus T)\\
&\Leftrightarrow v([S\cap S']_h,[T]_h)=v([S]_h{}',[T]_h),\forall h\in(0,1],\forall S'\in F(U)\\
&\Leftrightarrow v([S]_h\cap[S']_h,[T]_h)=v([S]_h{}',[T]_h),\ \forall h\in(0,1],\forall S'\in F(U)\\
&\Leftrightarrow [S]_h\in CL(U\mid v),\qquad \forall h\in(0,1]
\end{aligned}
$$

Hence, if $S\in CL(U\mid C_v)$, then

$[S]_h\in CL(U\mid v)$ for any $h\in(0,1]$. Similarly,

if $S\in CR(U\mid C_v)$, then $[S]_h\in CR(U\mid v)$.

Definition 10 Function $\varphi: G_F(N)\to R^{2n}$ is Shapley value on $C_v\in G_F(N)$ if it satisfies four axioms:

Axiom c1 (Efficiency): If $U\in F(N)$, $S,T\in F(U)$ is fuzzy left-carrier and right-carrier respectively, then

$$
\begin{aligned}
&\sum_{i\in N}(\varphi^v_{i,\Phi}(U)+\varphi^v_{\Phi,i}(U))\\
&=\sum_{i\in N}\varphi^v_{i,\Phi}(S)+\sum_{i\in N}\varphi^v_{\Phi,i}(T)\\
&=C_{v(U)}(S,\Phi)-C_{v(U)}(\Phi,T);
\end{aligned}
$$

If $i\notin Supp(S)$, then $\phi^v_{i,\Phi}(U)=0$, and if $i\notin Supp(T)$, then $\phi^v_{\Phi,i}(U)=0$.

Axiom c2 (Fairness): If $U\in F(N)$, and $C_v(S,U^U_{ij}\setminus S)=C_v(p_{ij}[S],U^U_{ij}\setminus p_{ij}[S])$ for any $S\in F(U^U_{ij})$, then $\phi^v_{i,\Phi}(U^U_{ij})=\phi^v_{j,\Phi}(U^U_{ij})$; and if $C_v(U^U_{ij}\setminus T,T)=C_v(U^U_{ij}\setminus p_{ij}[T],p^U_{ij}[T])$ holds for any $T\in F(U^U_{ij})$, then $\phi^v_{\Phi,i}(U^U_{ij})=\phi^v_{\Phi,j}(U^U_{ij})$.

Axiom c3 (Symmetry): Let two games $C_{v_1},C_{v_2}\in G_F(N)$, $\forall(S,T)\in\aleph(U\setminus i)$, $\forall h\in[0,1]$. If $i\in N$, satisfies

$C_{v_2}(S^U_i,T)-C_{v_2}(S,T)=C_{v_1}(S,T)-C_{v_1}(S,T^U_i)$ and

$C_{v_2}(\{i\},\Phi)+C_{v_1}([S]_h,[T]_h\cup\{i\})-C_{v_2}([S]_h,[T]_h)\ge 0$

or

$C_{v1}(\Phi,\{i\})+C_{v_2}([S]_h,[T]_h)-C_{v_2}([S]_h\cup\{i\},[T]_h)\ge 0$,

then $\varphi^{C_{v_2}}_{i,\Phi}=\varphi^{C_{v1}}_{\Phi,i}$.

Axiom c4 (Additivity): For any two games $C_{v1},C_{v_2}\in G_F(N)$, define a game $C_{v_1}+C_{v_2}$ by $(C_{v_1}+C_{v_2})(S,T)=C_{v_1}(S,T)+C_{v_2}(S,T)$ for any $(S,T)\in\aleph(N)$. If $C_{v_1}+C_{v_2}\in G_F(N)$ then $\varphi^{C_{v_1}+C_{v_2}}_{i,\Phi}=\varphi^{C_{v_1}}_{i,\Phi}+\varphi^{C_{v_2}}_{i,\Phi}$, $\varphi^{C_{v_1}+C_{v_2}}_{\Phi,i}=\varphi^{C_{v_1}}_{\Phi,i}+\varphi^{C_{v_2}}_{\Phi,i}$, where $i\in N$.

Note that the definition above is equivalent to Shapley axioms of crisp bi-cooperative games by restricting the fuzzy coalition to crisp coalition.

Definition 11 A function

$\boldsymbol{x}: G_F(N)\to R^{2n}$ is said to be a payoff function

of a game $C_v\in G_F(N)$ if it satisfies

(1) $\sum_{i\in Supp(U)} x_{i,\Phi}+x_{\Phi,i}=C_{v(U)}(N,\Phi)-C_{v(U)}(\Phi,N)$;

(2) $x_{i,\Phi}\ge C_v(\{i\},\Phi), x_{\Phi,i}\le C_v(\Phi,\{i\})$

where $U\in F(N)$, $\boldsymbol{x}=(\boldsymbol{x}_{1,\Phi},\boldsymbol{x}_{\Phi,1}\boldsymbol{x}_{2,\Phi},\boldsymbol{x}_{\Phi,2},\ldots$

$\ldots,x_{i,\Phi},x_{\Phi,i},\ldots,x_{n,\Phi},x_{\Phi,n})$.

Definition 12 A game $C_v\in G_F(N)$ is s superadditive, if for $\forall(S,T)\in\aleph(N)$ and $S'\in N\setminus S\cup T$, $C_v(S\cup S',T)\ge C_v(S,T)+C_v(S',\Phi)$ and $C_v(S,T\cup S')\le C_v(S,T)+C_v(\Phi,S')$.

Proposition 1 Under the four axioms in Definition 10, the Shapley value for superadditive fuzzy bi-cooperative games is as follows:

$$\varphi_{i,\Phi}^{C_v}(U) = \sum_{l=1}^{q(U)} \phi_{i,\Phi}^{v}([U]_{h_l}) \cdot (h_l - h_{l-1}) \quad (4)$$

$$\varphi_{\Phi,i}^{C_v}(U) = \sum_{l=1}^{q(U)} \phi_{\Phi,i}^{v}([U]_{h_l}) \cdot (h_l - h_{l-1}) \quad (5)$$

where $U \in F(N)$, $\phi_{i,\Phi}, \phi_{\Phi,i}$ is given in Theorem 1.

***Proof*:** We shall prove that function $\varphi_{i,\Phi}, \varphi_{\Phi,i}$ defined by (4), (5) satisfy *Axiom c*1-4 in Definition 10.

*Axiom c*1: Let $C_v \in G_F(N), U \in F(N)$. Using Theorem 4, if $S \in CL(U \mid C_v)$ and $T \in CR(U \mid C_v)$, then $[S]_h \in CL(U \mid v)$, $[T]_h \in CR(U \mid v)$.

If $i \notin Supp(S)$, then $i \notin [S]_h \in CL(U \mid v)$. Thus, $\phi_{i,\Phi}^{v}(U) = 0$. And if $i \notin Supp(T)$, then $\phi_{\Phi,i}^{v}(U) = 0$. Since

$$\sum_{i \in N} \left\{ \phi_{i,\Phi}^{v}([U]_{h_l}) + \phi_{\Phi,i}^{v}([U]_{h_l}) \right\} = v([U]_{h_l}, \Phi) - v(\Phi, [U]_{h_l})$$

holds for any $l \in \{1, \dots, q(U)\}$ from *Axiom b*1, we obtain

$$\begin{aligned}&\sum_{i \in N} [\varphi_{i,\Phi}^{C_v}(U) + \varphi_{\Phi,i}^{C_v}(U)] = \\ &\sum_{l=1}^{q(U)} \left\{ v([S]_{h_l}, \Phi) - v(\Phi, [T]_{h_l}) \right\} \cdot (h_l - h_{l-1}) \\ &= C_{v(U)}(S, \Phi) - C_{v(U)}(\Phi, T)\end{aligned}$$

*Axiom c*2: Let $U \in F(N)$. Note that $U_{ij}^{U}(i) = U_{ij}^{U}(j)$. If $U_{ij}^{U}(i) = U_{ij}^{U}(j) = 0$, then $\varphi_{i,\Phi}^{C_v}(U_{ij}^{U}) = \varphi_{j,\Phi}^{C_v}(U_{ij}^{U}) = 0$ from Axiom *c*1 proved above. If $U_{ij}^{U}(i) = U_{ij}^{U}(j) > 0$, i.e., $U(i), U(j) > 0$, then the following is valid.

$$C_v(S, U_{ij}^{U} \setminus S) - C_v(p_{ij}[S], U_{ij}^{U} \setminus p_{ij}[S]) = 0, \forall S \in F(U_{ij}^{U}),$$

$$\Rightarrow \begin{bmatrix} C_v(S, U_{ij}^{U} \setminus S) - C_v(p_{ij}[S], U_{ij}^{U} \setminus S) = 0, \\ \forall S \in F(U_{ij}^{U}), \quad s.t.\ S(j) = 0, S(k) \in \{S(i), 0\}, \\ \forall k \in Supp(U) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} C_v(S, U_{ij}^{U} \setminus S) - C_v(p_{ij}[S], U_{ij}^{U} \setminus S) = 0, \\ \forall S \in F(U_{ij}^{U}), \text{s.t. } S(i) = h,\ S(j) = 0 \\ and\ S(k) \in \{h, 0\}, \forall k \in Supp(U) \end{bmatrix}, \ \forall h \in (0, U_{ij}^{U}(i)],$$

$$\Rightarrow \begin{bmatrix} v([S']_h \cup \{i\}, [U_{ij}^{U}]_h \setminus [S']_h \cup \{i\}) - v([S']_h \cup \{j\}, \\ [U_{ij}^{U}]_h \setminus [S']_h \cup \{j\}) = 0, \forall S' \in F(U_{ij}^{U}), \\ \text{s.t. } S'(i) = S'(j) = 0\ \ and\ \ S'(k) \in \{h, 0\}, \\ \forall k \in Supp(U) \end{bmatrix}, \quad \forall h \in (0, U_{ij}^{U}(i)],$$

$$\Rightarrow \begin{bmatrix} v(R \cup \{i\}, [U_{ij}^{U}]_h \setminus R \cup \{i\}) - \\ v(R \cup \{j\}, [U_{ij}^{U}]_h \setminus R \cup \{j\}) = 0, \\ \forall T \in P([U_{ij}^{U}]_h \setminus \{i, j\}) \end{bmatrix}, \quad \forall h \in (0, U_{ij}^{U}(i)],$$

Hence, we have $\phi_{i,\Phi}^{v}([U_{ij}^{U}]_h) = \phi_{j,\Phi}^{v}([U_{ij}^{U}]_h)$ for any $h \in (0, U_{ij}^{U}(i)]$ from *Axiom b*2.

$\phi_{i,\Phi}^{v}([U_{ij}^{U}]_h) = \phi_{j,\Phi}^{v}([U_{ij}^{U}]_h) = 0$ holds for any $h \in (U_{ij}^{U}(i), 1]$ from *Axiom b*1. Hence,

$\phi_{i,\Phi}^{v}([U_{ij}^{U}]_h) = \phi_{j,\Phi}^{v}([U_{ij}^{U}]_h)$ for any $h \in (0, 1]$.

It follows that

$$\begin{aligned}\varphi_{i,\Phi}^{C_v}(U_{ij}^{U}) &= \sum_{l=1}^{q(U_{ij}^{U})} \phi_{j,\Phi}^{v}([U_{ij}^{U}]_{h_l}) \cdot (h_l - h_{l-1}) \\ &= \sum_{l=1}^{q(U_{ij}^{U})} \phi_{j,\Phi}^{v}([U_{ij}^{U}]_{h_l}) \cdot (h_l - h_{l-1}) = \varphi_{j,\Phi}^{C_v}(U_{ij}^{U})\end{aligned}$$

Similarly, $\phi_{\Phi,i}^{v}(U_{ij}^{U}) = \phi_{\Phi,j}^{v}(U_{ij}^{U})$ holds if $C_v(U_{ij}^{U} \setminus T, T) = C_v(U_{ij}^{U} \setminus p_{ij}[T], p_{ij}^{U}[T])$.

*Axiom c*3: For $C_{v_1}, C_{v_2} \in G_F(N)$, if $C_{v_2}(S_i^{U}, T) - C_{v_2}(S, T) = C_{v_1}(S, T) - C_{v_1}(S, T_i^{U})$, then

$$\begin{aligned}&\sum_{l=1}^{q(U)} \left\{ v_2([S_i^{U}]_{h_l}, [T]_{h_l}) - v_2([S]_{h_l}, [T]_{h_l}) \right\} \cdot (h_l - h_{l-1}) \\ &= \sum_{l=1}^{q(U)} \left\{ v_1([S]_{h_l}, [T]_{h_l}) - v_1([S]_{h_l}, [T_i^{U}]_{h_l}) \right\} \cdot (h_l - h_{l-1})\end{aligned}.$$

If $U(i) \geq h_l$, then $[S_i^{U}]_{h_l} = [S]_{h_l} \cup \{i\}$, $[T_i^{U}]_{h_l} = [T]_{h_l} \cup \{i\}$. Hence,

$$\begin{aligned}&v_2([S_i^{U}]_{h_l}, [T]_{h_l}) - v_2([S]_{h_l}, [T]_{h_l}) \\ &= v_2([S]_{h_l} \cup \{i\}, [T]_{h_l}) - v_2([S]_{h_l}, [T]_{h_l})\end{aligned}.$$

If $0 \le U(i) \le h_l$, then $[S_i^U]_{h_l} = [S]_{h_l}$, $[T_i^U]_{h_l} = [T]_{h_l}$,

and then

$$v_2([S_i^U]_{h_l},[T]_{h_l}) - v_2([S]_{h_l},[T]_{h_l}) \\ = v_2([S]_{h_l} \cup \{i\},[T]_{h_l}) - v_2([S]_{h_l},[T]_{h_l}) = 0 .$$

Therefore, for $0 \le h_l \le 1$,

$$v_2([S_i^U]_{h_l},[T]_{h_l}) - v_2([S]_{h_l},[T]_{h_l}) \\ = v_2([S]_{h_l} \cup \{i\},[T]_{h_l}) - v_2([S]_{h_l},[T]_{h_l}) .$$

Similarly ,

$$v_1([S]_{h_l},[T]_{h_l}) - v_1([S]_{h_l},[T_i^U]_{h_l}) \\ = v_1([S]_{h_l},[T]_{h_l}) - v_1([S]_{h_l},[T]_{h_l} \cup \{i\}) .$$

Thus, we get that

$$\sum_{l=1}^{q(U)} \left\{ \begin{array}{l} v_2([S]_{h_l} \cup \{i\},[T]_{h_l}) - v_2([S]_{h_l},[T]_{h_l}) - \\ v_1([S]_{h_l},[T]_{h_l}) + v_1([S]_{h_l},[T]_{h_l} \cup \{i\}) \end{array} \right\} \cdot (h_l - h_{l-1}) = 0 \quad (6)$$

Consequently, if

$v_2(\{i\},\Phi) + v_1([S]_h,[T]_h \cup \{i\}) - v_2([S]_h,[T]_h) \ge 0$

and $0 \le h \le 1$, then

$$\left\{ \begin{array}{l} v_2([S]_{h_l} \cup \{i\},[T]_{h_l}) - v_2([S]_{h_l},[T]_{h_l}) \\ -v_1([S]_{h_l},[T]_{h_l}) + v_1([S]_{h_l},[T]_{h_l} \cup \{i\}) \end{array} \right\} \\ \ge v_2(\{i\},\Phi) - v_1([S]_{h_l},[T]_{h_l}) + v_1([S]_{h_l},[T]_{h_l} \cup \{i\}) \ge 0 \quad (7)$$

By Expression (6)and (7),

$$v_2([S]_{h_l} \cup \{i\},[T]_{h_l}) - v_2([S]_{h_l},[T]_{h_l}) \\ = v_1([S]_{h_l},[T]_{h_l}) - v_1([S]_{h_l},[T]_{h_l} \cup \{i\}) .$$

Similarly, if

$v_1(\Phi,\{i\}) + v_2([S]_h,[T]_h) - v_2([S]_h \cup \{i\},[T]_h) \ge 0$, then

$$v_2([S]_{h_l} \cup \{i\},[T]_{h_l}) - v_2([S]_{h_l},[T]_{h_l}) \\ = v_1([S]_{h_l},[T]_{h_l}) - v_1([S]_{h_l},[T]_{h_l} \cup \{i\}) .$$

Consequently,

$$v_2([S]_{h_l} \cup \{i\},[T]_{h_l}) - v_2([S]_{h_l},[T]_{h_l}) \\ = v_1([S]_{h_l},[T]_{h_l}) - v_1([S]_{h_l},[T]_{h_l} \cup \{i\}) .$$

Using the Axiom $b3$, we have $\phi_{i,\Phi}^{v_2} = \phi_{\Phi,i}^{v_1}$. Thus,

$$\varphi_{i,\Phi}^{C_{v_2}}(U) = \sum_{l=1}^{q(U)} \phi_{i,\Phi}^{v_2}([U]_{h_l}) \cdot (h_l - h_{l-1}) \\ = \sum_{l=1}^{q(U)} \phi_{i,\Phi}^{v_1}([U]_{h_l}) \cdot (h_l - h_{l-1}) = \varphi_{\Phi,i}^{C_{v_1}}(U) .$$

Axiom c4: Let $U \in F(N)$ and $C_{v_1}, C_{v_2} \in G_F(N)$. It is clear that $C_{v_1} + C_{v_2} \in G_F(N)$ from the definition $G_F(N)$. Using *Axiom b4*, we have

$$\varphi_{i,\Phi}^{C_{v_1}+C_{v_2}} = \sum_{l=1}^{Q(U)} \phi_{i,\Phi}^{v_1+v_2}([U]_{h_l}) \cdot (h_l - h_{l-1}) \\ = \sum_{l=1}^{Q(U)} \left\{ \phi_{i,\Phi}^{v_1}([U]_{h_l}) + \phi_{i,\Phi}^{v_2}([U]_{h_l}) \right\} \cdot (h_l - h_{l-1}) = \varphi_{i,\Phi}^{C_{v_1}} + \varphi_{i,\Phi}^{C_{v_2}}$$

Proposition 2 If $C_v \in G_F(N)$ is superadditive, Shalpley value of C_v is a payoff function on $G_F(N)$.

Proof: If game $C_v \in G_F(N)$ is superadditive, then for $\forall (S,T) \in \aleph(N), S' \in N \setminus S \cup T$,

$C_v(S \cup S', T) \ge C_v(S,T) + C_v(S',\Phi)$. Thus, $v(S \cup \{i\}, N \setminus (S \cup \{i\})) - v(S, N \setminus (S \cup \{i\})) \ge v(\{i\},\Phi)$, $\forall (S,T) \in \wp(N \setminus \{i\})$ .From Theorem 2,

$$\varphi_{i,\Phi}^{C_v}(U) = \sum_{l=1}^{q(U)} \phi_{i,\Phi}^{v}([U]_{h_l}) \cdot (h_l - h_{l-1}) \\ \ge \sum_{l=1}^{q(U)} [v(\{i\},\Phi)] \cdot (h_l - h_{l-1}) = C_v(\{i\},\Phi).$$

Similarly, $\varphi_{\Phi,i}^{C_v} \le C_v(\Phi,\{i\})$. By *Axiom b1*,

$$\sum_{i \in N} \varphi_{i,\Phi}^{C_v} + \varphi_{\Phi,i}^{C_v} = C_{v(U)}(N,\Phi) - C_{v(U)}(\Phi,N) .$$

4. Illustrative example

Let $N = \{1,2,3\}$ be a set of investors and suppose that the capital of each $i \in N$ is m_i.There are two optional supply chains, i.e. v_1, v_2, which the three player choose to participate one of them or not participate. And the bi-cooperative game is formed by CPT type, i.e., $v = v_1 - v_2$. If the two cooperative game operate independently and compete with each other, their coalition income are as follows: $v_1(\{1\}) = v_1(\{2\}) = v_1(\{3\}) = 0.2$, $v_1(\{1,2\}) = v_1(\{2,3\}) = v_1(\{1,3\}) = 0.5$, $v_1(\{1,2,3\}) = 0.9$, $v_2(\{1\}) = v_2(\{2\}) = v_2(\{3\}) = 0.1$, $v_2(\{1,2\}) = v_2(\{2,3\}) = v_2(\{1,3\}) = 0.4$, $v_2(\{1,2,3\}) = 0.7$. A fuzzy coalition S defined by $S(1) = 0.2, S(2) = 0.4, S(3) = 0.5$, which means player 1,2,3 plan to take part in Supply-Chain cooperative game v by 20%,40%, 50% of his capital m_i, respectively.

A very important question in this context is how a player i can predict the expected return of investing a share $S(i)$ of his capital m_i in fuzzy coalition S.

Firstly, we can estimate the coalition incomes for crisp bi-cooperative game v ,for example,

$v(N,\Phi) = v_1(\{1,2,3\}) - v_2(\Phi) = 0.9$;

$v(\Phi, N) = v_1(\Phi) - v_2(\{1,2,3\}) = -0.7$.

Then, according to (1) and (2), the crisp Shapley values for bi-cooperative game can be obtained.

$$\phi^{v}_{1,\Phi}(N) = \phi^{v}_{2,\Phi}(N) = \phi^{v}_{3,\Phi}(N) = \sum_{S \subseteq N \setminus i} (n-s-1)!s!/n! \times [v(S \cup \{i\}, N \setminus S \cup \{i\}), v(S, N \setminus S \cup \{i\})] = 0.3,$$

$$\phi^{v}_{\Phi,1}(N)\phi^{v}_{\Phi,2}(N) = \phi^{v}_{\Phi,3}(N) = \sum_{S \subseteq N \setminus i} (n-s-1)!s!/n! \times [v(S \cup \{i\}, N \setminus S \cup \{i\}), v(S, N \setminus S \cup \{i\})] = 0.23.$$

Similarly, we can also compute the crisp Shapley values other coalitions repeatedly.

Next, by Formula (3), we can compute coalition incomes for fuzzy bi-cooperative game C_v.

$$C_{v(U)}(N,\Phi) = 0.2 \cdot v(\{1,2,3\},\Phi) + 0.2 \cdot v(\{2,3\},\Phi) + 0.1 \cdot v(\{3\},\Phi) = 0.3$$

$$C_{v(U)}(\Phi, N) = 0.2 \cdot v(\Phi,\{1,2,3\}) + 0.2 \cdot v(\Phi,\{2,3\}) + 0.1 \cdot v(\Phi,\{3\}) = -0.23$$

Finally, using (4) and (5),we can compute the fuzzy Shapley value for every kind of coalitions.

$$\varphi^{C_v}_{1,\Phi}(S) = 0.2 \cdot \phi^{v}_{1,\Phi}(N) + (0.4-0.2) \cdot \phi^{v}_{1,\Phi}(\{2,3\}) + (0.5-0.4) \cdot \phi^{v}_{1,\Phi}(\{3\}) = 0.06 ,$$

$$\varphi^{C_v}_{\Phi,1}(S) = 0.2 \cdot \phi^{v}_{\Phi,1}(N) + (0.4-0.2) \cdot \phi^{v}_{\Phi,1}(\{2,3\}) + (0.5-0.4) \cdot \phi^{v}_{\Phi,1}(\{3\}) = 0.04666,$$

Similarly, $\varphi^{C_v}_{1,\Phi}(S) = 0.11$, $\varphi^{C_v}_{\Phi,2}(S) = 0.08666$,

$\varphi^{C_v}_{2,\Phi}(S) = 0.13, \varphi^{C_v}_{\Phi,3}(S) = 0.09666$.

Consequently, we can see that

$$\sum_{i=1}^{3} (\varphi^{C_v}_{i,\Phi} + \varphi^{C_v}_{\Phi,i}) = C_c(N,\Phi) - C_v(\Phi, N)$$

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References

[1] L. S. Shapley, A value for *n*-persons games. *Annals of Mathematics Studies*, 28: 307-318, 1953.

[2] R .J. Aumann and L. S. Shapley, Values of non-atomic games. *Princeton*: Princeton University Press, 1974.

[3] J. P. Aubin, *Mathematical methods of game and economic theory (revised edition).* Amsterdam: North-Holland, 1982.

[4] J. P. Aubin. Coeur et valeur des jeux flous à paiements latéraux. *Comptes Rendus Hebdomadaires des Séances de 1' Académie des Sciences*, 279-A: 891-894, 1974.

[5] M. Tsurumi ,T. Tanino and M. Inuiguchi, A Shapley function on a class of cooperative fuzzy games.*European Journal of Operational Research*, 129: 596-618, 2001.

[6] J.M.Bilbao, *Cooperative games on Combinatorial Structures.* Kluwer Academic Publishers, Dordrecht,2000.

[7] D.Felsenthal and M.Machover, Ternary voting games. *Internat, J.Game Theory* 26:335-351, 1997

[8] S.Greco, B.Matarazzo and R.Slowinski, Bipolar Sugeno and choquet intergrals. *Eurofuse Workshop on Informations Systems, Varenna, Italy*,9:191-196, 2002.

[9] Michel Grabisch and Christophe Labreuche,Bi-capacities-I:definition, Mobius transform and interaction. *Fuzzy sets and systems*,151:211-236, 2005.

[10] G.Choquet. Theory of capacities. *Ann.Inst. Fourier5*:131-295, 1953.

[11] M.Sugeno, *Theory of fuzzy intergrals and its applications.*Ph.D.Thesis,Tykyo Institute of Technology,1974.

[12] A..Tversky and D.Kahneman,Advance in prospect theory: cumulative representation of uncertainty. *J.Risk Uncertainty* 5: 297-323, 1992.

[13] Michel Grabisch and Christophe Labreuche, Bi-capacities-II:the Choquet intergral. *Fuzzy sets and systems*,151: 237-259, 2005.

[14] D. Butnariu., Stability and Shapley value for an n-persons fuzzy games. *Fuzzy Set and System*, 4: 63-72, 1980.