

Some Properties of Fuzzy Filters in Lattice Implication Algebras

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Abstract

In this paper, several equivalent conditions for fuzzy filters are given in lattice implication algebra. The relation between fuzzy filters and fuzzy implicative filters is discussed. It is also proved that fuzzy implicative filters are fuzzy filters in lattice implication algebra.

Keywords: Fuzzy filters, Fuzzy implicative filters, Lattice implication algebra.

1. Introduction

In order to research the logical system whose propositional value was given in a lattice, Xu proposed the concept of lattice implication algebras, and discussed their properties in [1]. Xu and Qin introduced the notions of filters and implicative filters in lattice implication algebra, and investigated their properties in [2]. In [3], Xu applied the concept of fuzzy sets to lattice implication algebra and proposed the notions of fuzzy filters and fuzzy implicative filters. In [4], [5], the notions of implicative filters, positive implication and associative filter were studied. In [6], [7], fuzzy filters, fuzzy positive implication and fuzzy associative filters were presented, and their elementary properties were discussed. Filter theory play an important role in studying the structure of algebras.

In this paper several equivalent conditions for fuzzy filters are proved in lattice implication algebra. At the same time we discuss the relation of fuzzy filters and fuzzy implicative filters.

2. Preliminaries

Definition 2.1 [1] (**Quasi-lattice implication algebra**) Let (L, \vee, \wedge, O, I) be a bounded lattice with an order-reversing involution', I and O the greatest and the smallest element of L respectively, and

$$\rightarrow: L \times L \rightarrow L$$

be a mapping. $(L, \vee, \wedge, ', \rightarrow, O, I)$ is called a quasi-lattice implication algebra if the following conditions

hold for any $x, y, z \in L$,

$$(I_1) x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z);$$

$$(I_2) x \rightarrow x = I;$$

$$(I_3) x \rightarrow y = y' \rightarrow x';$$

$$(I_4) x \rightarrow y = y \rightarrow x = I \text{ implies } x = y;$$

$$(I_5) (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x.$$

Definition 2.2 [1] (**Lattice implication algebra**) A quasi-lattice implication algebra is called a lattice implication algebra, if (I_1) and (I_2) hold for any $x, y, z \in L$,

$$(I_1) (x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z);$$

$$(I_2) (x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z).$$

Theorem 2.1[1] Let L be a quasi-lattice implication algebra, then for any $x, y, z \in L$,

$$(1) \text{ If } I \rightarrow x = I, \text{ then } x = I;$$

$$(2) I \rightarrow x = x \text{ and } x \rightarrow O = x';$$

$$(3) O \rightarrow x = I \text{ and } x \rightarrow I = I;$$

$$(4) (x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = I.$$

Theorem 2.2[1] Let L be a lattice implication algebra, then for any $x, y \in L$, $x \leq y$ if and only if $x \rightarrow y = I$.

Theorem 2.3[1] Let L be a lattice implication algebra, then for any $x, y, z \in L$,

$$(1) (x \rightarrow z) \rightarrow (y \rightarrow z) = y \rightarrow (x \vee z)$$

$$= (z \rightarrow x) \rightarrow (y \rightarrow x);$$

$$(2) (z \rightarrow x) \rightarrow (z \rightarrow y) = (x \wedge z) \rightarrow y$$

$$= (x \rightarrow z) \rightarrow (x \rightarrow y).$$

Theorem 2.4[1] Let L be a lattice implication algebra, then for any $x, y, z \in L$,

$$(1) z \rightarrow (y \rightarrow x) \geq (z \rightarrow y) \rightarrow (z \rightarrow x);$$

$$(2) z \leq y \rightarrow x \text{ if and only if } y \leq z \rightarrow x.$$

Theorem 2.5[1] Let L be lattice implication

algebra, then the following statements are equivalent:

- (1) For any $x, y, z \in L$, $x \rightarrow (y \rightarrow z) = (x \wedge y) \rightarrow z$;
- (2) For any $x, y \in L$, $x \rightarrow (x \rightarrow y) = x \rightarrow y$;
- (3) For any $x, y, z \in L$,
 $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = I$.

In lattice implication algebra L , we define binary operations \otimes and \oplus as follows:
for any $x, y \in L$,

$$x \otimes y = (x \rightarrow y')';$$

$$x \oplus y = x' \rightarrow y.$$

We can get the following theorem:

Theorem 2.5[1] Let L be a lattice implication algebra, then for any $x, y, z \in L$,

- (1) $x \otimes y = y \otimes x$, $x \oplus y = y \oplus x$;
- (2) $x \rightarrow (x \otimes y) = x' \vee y = (x \oplus y) \rightarrow y$;
- (3) $(x \rightarrow y) \otimes x = x \wedge y$;
- (4) $x \rightarrow (y \rightarrow z) = (x \otimes y) \rightarrow z$;
- (5) $x \rightarrow (y \rightarrow z)$ if and only if $x \otimes y \leq z$;

Definition 2.3[2] A non-empty subset F of lattice implication algebra L is called a filter of L if it satisfies:

- (F1) $I \in F$,
- (F2) $(\forall x \in F)(\forall y \in L)(x \rightarrow y \in F \Rightarrow y \in F)$.

Definition 2.4[3] A fuzzy set A of lattice implication algebra L is called a fuzzy filter of L if it satisfies:

- (F3) $(\forall x \in L)(A(I) \geq A(x))$,
- (F4) $(\forall x, y \in L)(A(y) \geq \min\{A(x), A(x \rightarrow y)\})$.

Theorem 2.6[3] Let L be a lattice implication algebra and A a fuzzy filter of L , then for any $x, y \in L$, $x \leq y$ implies $A(x) \leq A(y)$.

Definition 2.5[3] Let L be a lattice implication algebra, A is a non-empty fuzzy set of L . A is called a fuzzy implicative filter of L if it satisfies:

- (F5) $A(I) \geq A(x)$ for any $x \in L$;
- (F6) $A(x \rightarrow z) \geq \min\{A(x \rightarrow y), A(x \rightarrow (y \rightarrow z))\}$ for any $x, y, z \in L$.

3. Main properties

Theorem 3.1 Let A be a fuzzy set of L . A is a fuzzy filter of L if and only if it satisfies the following conditions: for any $x, y, z \in L$

- (1) $A(x) \leq A(I)$;
- (2) $A(x \rightarrow z) \geq \min\{A(x \rightarrow y), A(y \rightarrow z)\}$.

Proof. Assume that A is a fuzzy filter of L , then (1) is trivial and for any $x, y, z \in L$

$$A(x \rightarrow z) \geq \min\{A(y \rightarrow z), A((y \rightarrow z) \rightarrow (x \rightarrow z))\}.$$

By theorem 2.2 and $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z))$

$$= (x \rightarrow y) \rightarrow ((z \rightarrow y) \rightarrow (x \rightarrow y))$$

$$= (z \rightarrow y) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow y))$$

$$= (z \rightarrow y) \rightarrow I = I,$$

then $(x \rightarrow y) \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$.

Since theorem 2.6, hence

$$A((y \rightarrow z) \rightarrow (x \rightarrow z)) \geq A(x \rightarrow y).$$

It follows that

$$A(x \rightarrow z) \geq \min\{A(y \rightarrow z), A(x \rightarrow y)\}.$$

Conversely, suppose that A satisfies (1) and (2), for any $x \in L$, it follows that $A(x) \leq A(I)$. By (2), it follows that for any $x, y, z \in L$,

$$A(z) \geq \min\{A(y), A(y \rightarrow z)\}, \text{ where } x = I.$$

Hence A is a fuzzy filter of L . \square

Theorem 3.2 Let A be a fuzzy set of L . A is a fuzzy filter of L if and only if it satisfies the following conditions: for any $x, y, z \in L$,

- (1) $A(x) \leq A(I)$;
- (2) $A(z) \geq \min\{A(x), A(y), A(x \rightarrow (y \rightarrow z))\}$.

Proof. Suppose that A is a fuzzy filter of L , then (1) is trivial and for any $x, y, z \in L$

$$A(z) \geq \min\{A(x), A(x \rightarrow z)\},$$

$$A(x \rightarrow z) \geq \min\{A(y), A(y \rightarrow (x \rightarrow z))\}.$$

Hence

$$A(z) \geq \min\{A(x), A(y), A(y \rightarrow (x \rightarrow z))\},$$

i.e., $A(z) \geq \min\{A(x), A(y), A(x \rightarrow (y \rightarrow z))\}$.

Conversely, by (1) and (2), it follows that

$$A(x) \leq A(I) \text{ and for any } x \in L.$$

If we take $x = y$, then

$$A(z) \geq \min\{A(x), A(x \rightarrow (x \rightarrow z))\}.$$

From $(x \rightarrow z) \rightarrow (x \rightarrow (x \rightarrow z)) = I$ and theorem 2.2, we get

$$x \rightarrow z \leq x \rightarrow (x \rightarrow z),$$

$$A(x \rightarrow z) \leq A(x \rightarrow (x \rightarrow z)).$$

It follows that $A(z) \geq \min\{A(x), A(x \rightarrow z)\}$.

Hence A is a fuzzy filter of L . \square

Theorem 3.3 Let A be a fuzzy set of L . A is a fuzzy filter of L if and only if it satisfies the following conditions: for any $x, y, z \in L$,

$$(1) A(x) \leq A(I) ;$$

$$(2) A(z \rightarrow x) \geq \min\{A((z \rightarrow y) \rightarrow x), A(y)\} .$$

Proof. Assume that A is a fuzzy filter of L , then (1) holds and for any $x, y, z \in L$

$$A(z \rightarrow x) \geq \min\{A(y \rightarrow (z \rightarrow x)), A(y)\} .$$

Notice that for any $x, y, z \in L$,

$$\begin{aligned} & ((z \rightarrow y) \rightarrow x) \rightarrow ((z \rightarrow y) \rightarrow (z \rightarrow x)) \\ &= ((z \rightarrow y) \rightarrow x) \rightarrow (z \rightarrow ((z \rightarrow y) \rightarrow x)) \\ &= z \rightarrow (((z \rightarrow y) \rightarrow x) \rightarrow ((z \rightarrow y) \rightarrow x)) \\ &= I . \end{aligned}$$

$$\text{Hence } (z \rightarrow y) \rightarrow x \leq (z \rightarrow y) \rightarrow (z \rightarrow x) .$$

By theorem 2.6 we have

$$A((z \rightarrow y) \rightarrow x) \leq A((z \rightarrow y) \rightarrow (z \rightarrow x)) .$$

By theorem 2.4, we notice that

$$z \rightarrow (y \rightarrow x) \geq (z \rightarrow y) \rightarrow (z \rightarrow x) .$$

It follows that

$$A(z \rightarrow (y \rightarrow x)) \geq A((z \rightarrow y) \rightarrow (z \rightarrow x))$$

Hence

$$A(z \rightarrow (y \rightarrow x)) \geq A((z \rightarrow y) \rightarrow x) ,$$

$$\text{i.e., } A(y \rightarrow (z \rightarrow x)) \geq A((z \rightarrow y) \rightarrow x) .$$

$$\text{Therefore } A(z \rightarrow x) \geq \min\{A((z \rightarrow y) \rightarrow x), A(y)\} .$$

Conversely, if A satisfies (1) and (2), then the proof of (1) is obvious. By (2), it follows that

$$A(x) \geq \min\{A(y \rightarrow x), A(y)\} , \text{ where } z = I .$$

Hence A is a fuzzy filter of L . \square

Theorem 3.4 Let L be a lattice implication algebra and A be a fuzzy set of L , if A is a fuzzy implicative filter, the following statements are satisfied and equivalent:

$$(1) A \text{ is a fuzzy filter and for any } x, y \in L ,$$

$$A(x \rightarrow y) \geq A(x \rightarrow (x \rightarrow y)) ;$$

$$(2) A \text{ is a fuzzy filter and for any } x, y, z \in L ,$$

$$A((x \rightarrow y) \rightarrow (x \rightarrow z)) \geq A(x \rightarrow (y \rightarrow z)) ;$$

$$(3) A(x) \leq A(I) \text{ and for any } x, y, z \in L ,$$

$$A(x \rightarrow y) \geq \min\{A(z \rightarrow (x \rightarrow (x \rightarrow y))), A(z)\} .$$

Proof. First, we prove that (1) holds. In fact, for any $x, y, z \in L$, we know $A(x) \leq A(I)$ and

$$A(x \rightarrow z) \geq \min\{A(x \rightarrow y), A(x \rightarrow (y \rightarrow z))\} .$$

It follows that

$$A(z) \geq \min\{A(y), A(y \rightarrow z)\} , \text{ where } x = I .$$

Hence A is a fuzzy filter.

By $A(x \rightarrow z) \geq A(x \rightarrow (x \rightarrow z))$, where $y = x$, hence (1) holds.

Assume (1) holds, we can prove (2). Since for any

$$x, y \in L, A(x \rightarrow y) \geq A(x \rightarrow (x \rightarrow y)) , \text{ then}$$

$$\begin{aligned} & A((x \rightarrow y) \rightarrow (x \rightarrow z)) \\ &= A(x \rightarrow ((x \rightarrow y) \rightarrow z)) \\ &\geq A(x \rightarrow (x \rightarrow ((x \rightarrow y) \rightarrow z))) , \end{aligned}$$

$$\text{and } x \rightarrow (x \rightarrow ((x \rightarrow y) \rightarrow z))$$

$$= x \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z))$$

$$= x \rightarrow ((y \rightarrow x) \rightarrow (y \rightarrow z))$$

$$\geq x \rightarrow (y \rightarrow z) .$$

By theorem 2.6, it follows that

$$A(x \rightarrow (x \rightarrow ((x \rightarrow y) \rightarrow z))) \geq A(x \rightarrow (y \rightarrow z)) ,$$

$$\text{i.e., } A((x \rightarrow y) \rightarrow (x \rightarrow z)) \geq A(x \rightarrow (y \rightarrow z)) .$$

Therefore, (2) holds.

Now suppose that (2) holds, we can prove (3). In fact, $A(x) \leq A(I)$ is obvious. For any $x, y, z \in L$, since

$$A((x \rightarrow y) \rightarrow (x \rightarrow z)) \geq A(x \rightarrow (y \rightarrow z)) ,$$

we get

$$A(x \rightarrow z) \geq A(x \rightarrow (x \rightarrow z)) , \text{ where } y = x .$$

Hence for any $x, y \in L$,

$$A(x \rightarrow y) \geq A(x \rightarrow (x \rightarrow y)) .$$

Since A is a fuzzy filter, then

$$A(x \rightarrow (x \rightarrow y)) \geq \min\{A(z \rightarrow (x \rightarrow (x \rightarrow y))), A(z)\} .$$

It follows that for any $x, y, z \in L$,

$$A(x \rightarrow y) \geq \min\{A(z \rightarrow (x \rightarrow (x \rightarrow y))), A(z)\} .$$

Hence (3) holds.

Assume that (3) holds, we prove (1). For any $x, y, z \in L$,

$$A(x \rightarrow y) \geq \min\{A(z \rightarrow (x \rightarrow (x \rightarrow y))), A(z)\} .$$

It follows that

$$A(y) \geq \min\{A(z \rightarrow y), A(z)\} \text{ where } x = I .$$

And $A(x) \leq A(I)$, hence A is a fuzzy filter. Since for any $x, y, z \in L$,

$$A(x \rightarrow y) \geq \min\{A(z \rightarrow (x \rightarrow (x \rightarrow y))), A(z)\},$$

then

$$A(x \rightarrow y) \geq A(x \rightarrow (x \rightarrow y)), \text{ where } z = I.$$

Hence (1) holds. \square

Theorem 3.5 Let A is a fuzzy filter of L . If $x \leq y \rightarrow z$ for any $x, y, z \in L$, then

$$A(z) \geq \min\{A(x), A(y)\}.$$

Proof. Suppose that A is a fuzzy filter of L , it follows that for any $x, y, z \in L$,

$$A(z) \geq \min\{A(y), A(y \rightarrow z)\}.$$

If $x \leq y \rightarrow z$, then

$$A(y \rightarrow z) \geq A(x).$$

Hence $A(z) \geq \min\{A(x), A(y)\}$. \square

Corollary 3.1 Let A is a fuzzy filter of L . If $(x \otimes y) \rightarrow z = I$ for any $x, y, z \in L$, then

$$A(z) \geq \min\{A(x), A(y)\}.$$

Theorem 3.6 Let A be a fuzzy set of L . A is a fuzzy filter of L if and only if it satisfies the following conditions: for any $x, y \in L$

(1) If $x \leq y$, then $A(x) \leq A(y)$;

(2) $A(x \otimes y) \geq \min\{A(x), A(y)\}$.

Proof. Suppose that A is a fuzzy filter of L , then (1) holds clearly and for any $x, y \in L$

$$A(x \otimes y) \geq \min\{A(x), A(x \rightarrow (x \otimes y))\}.$$

By $y \rightarrow (x \rightarrow (x \otimes y))$

$$= y \rightarrow (x' \vee y)$$

$$= (y \rightarrow x') \vee (y \rightarrow y)$$

$$= (y \rightarrow x') \vee I$$

$$= I$$

and theorem 2.2, then $x \rightarrow (x \otimes y) \geq y$.

By theorem 2.6, it follows that

$$A(x \rightarrow (x \otimes y)) \geq A(y).$$

Hence $A(x \otimes y) \geq \min\{A(x), A(y)\}$, i.e., (2) holds.

Conversely, if A is satisfied (1) and (2), then (F3) holds. Since

$$A(x \otimes y) \geq \min\{A(x), A(y)\},$$

then

$$\min\{A(x), A(x \rightarrow y)\} \leq A(x \otimes (x \rightarrow y)).$$

$$\text{By } (x \otimes (x \rightarrow y)) \rightarrow y$$

$$= (x \wedge y) \rightarrow y$$

$$= (x \rightarrow y) \vee (y \rightarrow y) = I$$

and theorem 2.2, then $y \geq x \otimes (x \rightarrow y)$. By (1)

$$A(x \otimes (x \rightarrow y)) \leq A(y),$$

it follows that

$$\min\{A(x), A(x \rightarrow y)\} \leq A(x \otimes (x \rightarrow y)) \leq A(y),$$

i.e., $A(y) \geq \min\{A(x), A(x \rightarrow y)\}$. Hence A is a fuzzy filter of L . \square

4. Conclusions

In this paper, we provide several equivalent conditions for fuzzy filters in lattice implication algebra. The relation between fuzzy filters and fuzzy implicative filters are studied, and we prove that fuzzy implicative filters are fuzzy filters in lattice implication algebra.

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