

## Properties of Lower and Upper Approximation Operators under Various Kinds of Relations

Wei-Feng DU<sup>a</sup>, Teng ZI<sup>b</sup>

School of Mathematics, Physics and Information Engineering, Jiaxing University, Jiaxing China

<sup>a</sup>23031520@qq.com, <sup>b</sup>105787283@qq.com

**Keywords:** Rough Sets, Lower and Upper Approximation Operators, Binary Relation, Equivalence Relation.

**Abstract.** One of the direction of rough sets theory is to extend the equivalence relation, using more general binary relation such as the partial order relation, tolerance relation or similarity relations instead of the strict equivalence relation. So the scope of application of rough sets theory could be extended. But in the use of these more general relation instead of equivalence relation, some good properties of the original Pawlak approximation space  $(U, R)$  may no longer be satisfied. In this paper, various properties of two pairs of lower and upper approximation operators and the relationship between them are acquired through analysis and proof. The two pairs of lower and upper approximation operator will have good properties when the binary relation  $R$  only satisfies reflexivity, and the two pairs of approximation operators are the same under the equivalence relation. Theoretical analysis shows that, in the process of extending classical rough set theory to generalized rough set theory, reflexivity is a minimum conditions to be satisfied, under this condition, the lower and upper approximation operators meeting the corresponding properties can be chosen to adapt to the requirement of practical application.

### Introduction

Intelligent information processing is a hot problem in the research of theory and application in information science. With the development of computer science and technology, especially the development of computer network, the amount of information expands rapidly and the requirements on the information analysis tools, are increasingly high. People want to get the potential knowledge from the data automatically. Especially in the past 20 years, knowledge discovery (rule extraction, data mining, machine learning) has been widely used in artificial intelligence science, and different methods of knowledge discovery emerge as the times require. Rough set theory proposed in 1982 by Poland mathematician professor Pawlak is a mathematical tool to deal with imprecise, inconsistent, incomplete information analysis[1]. The initial prototype of rough set theory comes from simple information model, its basic idea is to form concepts and rules from the classification of relational database. The universe could be classified through equivalence relation, and knowledge discovery could be realized by the approximation of the target concept. As a kind of data analysis theory, rough set theory is another mathematical tool after the probability theory, fuzzy set and evidence theory to handle uncertainty problem. Because of the novel idea, unique method and simple calculation, rough set theory has become an important intelligent information processing technology[2,3]. The theory has been widely applied for machine learning, knowledge discovery, data mining and decision support and analysis.

At present, the research on rough set theory has achieved fruitful results, but the theory is still in development, as the founder of rough set theory Pawlak think: there are still some problems to be solved in theory. Part of the problem is, the classical Pawlak rough set model based on equivalence relation, equivalence relations and partition is one one correspondence, the strict requirements limit the application of rough set theory. Therefore, based on the more general binary relations, such as partial order relation, tolerance relation and similarity relations instead of strict equivalence relation, or based on the more general concept than partition, such as neighborhood, covering[4,5] as basic

elements, and then approximation operators will be extend and the classical rough set will be extended to all kinds of generalized rough set. This is a focus of rough set theory research[6].

When using the more general relation instead of equivalence relation, some good properties on the original Pawlak approximation space  $(U, R)$  may no longer satisfy. Therefore, to discuss the properties meeting in various more general relations from the theory has a vital significance.

## Basic Concepts

### Two Pairs of Operators

Let  $R$  be binary relation on  $U$ , set

$$[x]_R = \{y \in U | (x, y) \in R\}, \forall x \in U$$

The literature[7] defines two pairs of operators,  $\underline{R}(X)$  and  $\overline{R}(X)$  in formula (1), (2),  $\underline{R}'(X)$  and  $\overline{R}'(X)$  in formula (3), (4):

$$\underline{R}(X) = \{x \in U | [x]_R \subseteq X\} \quad (1)$$

$$\overline{R}(X) = \{x \in U | [x]_R \cap X \neq \emptyset\} \quad (2)$$

$$\underline{R}'(X) = \cup\{[x]_R | [x]_R \subseteq X\} \quad (3)$$

$$\overline{R}'(X) = \cup\{[x]_R | [x]_R \cap X \neq \emptyset\} \quad (4)$$

### Knowledge Base[8]

**Definition 1** Let  $U$  be a finite domain,  $\mathcal{F} \subseteq 2^U$ , if meet:

$$(1) \emptyset \in \mathcal{F}$$

$$(2) X, Y \in \mathcal{F} \Rightarrow X \cup Y \in \mathcal{F}$$

$$(3) X \in \mathcal{F} \Rightarrow \overline{X} \in \mathcal{F}$$

then  $\mathcal{F}$  is called the algebra.

**Definition 2** Let  $U$  be a finite domain,  $\mathcal{F} \subseteq 2^U$  is algebra, called  $(U, \mathcal{F})$  as the knowledge base.

**Definition 3** Let  $(U, \mathcal{F})$  is a knowledge base, said the

$$L: 2^U \rightarrow \mathcal{F}$$

as the necessity operator, if meet:

$$(L0) L(X) \subseteq X$$

$$(L1) L(U) \subseteq U$$

$$(L2) L(X \cap Y) = L(X) \cap L(Y)$$

where  $X, Y \in 2^U$ .

**Definition 4** Let  $(U, \mathcal{F})$  be a knowledge base, said the

$$H: 2^U \rightarrow \mathcal{F}$$

as the possibility operator, if meet:

$$(H0) X \subseteq H(X)$$

$$(H1) H(\emptyset) = \emptyset$$

$$(H2) H(X \cup Y) = H(X) \cup H(Y)$$

where  $X, Y \in 2^U$ .

**Definition 5** Let  $(U, \mathcal{F})$  is a knowledge base, where  $L$  and  $H$  were the necessity operator and the possibility operator, called  $L$  and  $H$  are dual, if

$$H(\overline{X}) = \overline{L(X)}$$

### Properties of the Operators under Binary Relation

If  $R$  is general binary relation on  $U$ , then the definition of formula (1) is not very strict, this can be illustrated by example 1.

**Example 1**  $U = \{x_1, x_2, x_3, x_4, x_5\}$ ,  $R = \{(x_1, x_2), (x_2, x_2), (x_2, x_3), (x_3, x_3), (x_3, x_5)\}$ ,  $X = \{x_2, x_3, x_4\}$

Then  $[x_1]_R = \{x_2\}$ ,  $[x_2]_R = \{x_2, x_3\}$ ,  $[x_3]_R = \{x_3, x_5\}$ ,  $[x_4]_R = \emptyset$ ,  $[x_5]_R = \emptyset$ .

Thus  $\underline{R}(X) = \{x_1, x_2, x_4, x_5\}$ ,  $\overline{R}(X) = \{x_1, x_2, x_3\}$ .

Here  $\underline{R}(X) \not\subseteq \overline{R}(X)$ .

This is contrary to intuition, therefore, we make a little modification to give a more rigorous definition:

$$\underline{R}(X) = \{x \in U | \emptyset \neq [x]_R \subseteq X\} \quad (1')$$

According to the definition of formula (1'),  $\underline{R}(X)$  may be recalculated as:  $\underline{R}(X) = \{x_1, x_2\}$

Here  $\underline{R}(X) \subseteq \overline{R}(X)$ .

In fact, by  $\emptyset \neq [x]_R \subseteq X \Rightarrow [x]_R \cap X \neq \emptyset$ ,  $\underline{R}(X) \subseteq \overline{R}(X)$  can be launched.

Then, should formula (3) also be amended accordingly? This can also be explained by example 1:

$$\underline{R}'(X) = [x_1]_R \cup [x_2]_R \cup [x_4]_R \cup [x_5]_R = \{x_2\} \cup \{x_2, x_3\} \cup \emptyset \cup \emptyset = \{x_2, x_3\}$$

If it is amended like formula (1),  $\underline{R}'(X) = [x_1]_R \cup [x_2]_R = \{x_2\} \cup \{x_2, x_3\} = \{x_2, x_3\}$ .

In the calculation of the union, less  $\emptyset$  may participate in the operation, but this does not affect the final value.  $\underline{R}'(X) \subseteq \overline{R}'(X)$  still keeps, so it is not necessary to amend. Thus the two pairs of operators should be  $\underline{R}(X)$  and  $\overline{R}(X)$  shown in formula (1') and (2),  $\underline{R}'(X)$  and  $\overline{R}'(X)$  shown in formula (3) and (4) under general binary relation.

### Properties of the Operators under General Binary Relation

**Theorem 1** Let  $R$  be general binary relation on  $U$ , where  $X \subseteq U$ , thus

$$\underline{R}'(X) \subseteq X$$

**Proof**  $\forall y \in \underline{R}'(X)$ , by formula (3), will  $\exists [x]_R$ , such that  $y \in [x]_R$ , and  $[x]_R \subseteq X$ , so  $y \in X$ , which permit  $\underline{R}'(X) \subseteq X$ .

And  $\underline{R}(X) \subseteq X$ ,  $X \subseteq \overline{R}(X)$ ,  $X \subseteq \overline{R}'(X)$  is not necessarily true, it can be illustrated by calculating the data of example 1 according to the formula:

$$\underline{R}(X) = \{x_1, x_2\}, \overline{R}(X) = \{x_1, x_2, x_3\}, \underline{R}'(X) = \{x_2, x_3\}, \overline{R}'(X) = \{x_2, x_3, x_5\}$$

## Properties of Operators under Reflexive Relation

**Lemma 1** Let  $R$  be reflexive relation on  $U$ ,  $\forall x$ , have

$$x \in [x]_R \neq \emptyset$$

**Proof** Because  $R$  satisfies reflexivity, so  $\forall x$ ,  $(x, x) \in R$ , thus  $x \in [x]_R$  and  $[x]_R \neq \emptyset$ .

Due to under the reflexive relation,  $[x]_R \neq \emptyset$ , then formula (1) may not be modified, thus the two pairs of operators should still be  $\underline{R}(X)$  and  $\overline{R}(X)$  shown in formula (1) and (2),  $\underline{R}'(X)$  and  $\overline{R}'(X)$  shown in formula (3) and (4).

**Theorem 2** Let  $R$  be reflexive relation on  $U$ , where  $X \subseteq U$ , thus

$$\underline{R}(X) \subseteq X \subseteq \overline{R}(X), \underline{R}'(X) \subseteq X \subseteq \overline{R}'(X)$$

**Proof**

$\forall x \in \underline{R}(X)$ , by formula (1) have  $[x]_R \subseteq X$ , by lemma 1 have  $x \in [x]_R$ , so  $x \in X$ , therefore, which permit  $\underline{R}(X) \subseteq X$ .

$\forall x \in X$ , by lemma 1 have  $x \in [x]_R$ , so  $x \in [x]_R \cap X \neq \emptyset$ , by formula (2) have  $x \in \overline{R}(X)$ , therefore, which permit  $X \subseteq \overline{R}(X)$ .

$\forall x \in X$ , by lemma 1 have  $x \in [x]_R$ , so  $x \in [x]_R \cap X \neq \emptyset$ , by formula (4) have  $[x]_R \subseteq \overline{R}'(X)$ , so  $x \in \overline{R}'(X)$ , therefore, which permit  $X \subseteq \overline{R}'(X)$ .

And by theorem 1 have  $\underline{R}'(X) \subseteq X$ , so the theorem is proved.

**Example 2** Keeping  $U$  and  $X$  of example 1 invariant, and changing  $R$  to reflexive closure of  $R$  of example 1, i.e.  $R = \{(x_1, x_1), (x_1, x_2), (x_2, x_2), (x_2, x_3), (x_3, x_3), (x_3, x_5), (x_4, x_4), (x_5, x_5)\}$

Then,  $[x_1]_R = \{x_1, x_2\}$ ,  $[x_2]_R = \{x_2, x_3\}$ ,  $[x_3]_R = \{x_3, x_5\}$ ,  $[x_4]_R = \{x_4\}$ ,  $[x_5]_R = \{x_5\}$

Thus,  $\underline{R}(X) = \{x_2, x_4\}$ ,  $\overline{R}(X) = \{x_1, x_2, x_3, x_4\}$ ,  $\underline{R}'(X) = \{x_2, x_3, x_4\}$ ,  $\overline{R}'(X) = \{x_1, x_2, x_3, x_4, x_5\}$

$\underline{R}(X) \subseteq X \subseteq \overline{R}(X)$ ,  $\underline{R}'(X) \subseteq X \subseteq \overline{R}'(X)$  hold.

**Theorem 3**  $\underline{R}$  is necessity operator.

**Proof** In order to prove  $\underline{R}$  be the necessity operator, only that  $\underline{R}$  satisfies the three properties of the necessity operator:

(1) By theorem 2  $\underline{R}$  meet  $L_0$ ;

(2)  $\forall x \in U$ ,  $[x]_R \subseteq U$ , by formula (1)  $x \in \underline{R}(U)$ , so  $U \subseteq \underline{R}(U)$ , and  $\underline{R}(U) \subseteq U$  must establish, which permit  $\underline{R}(U) = U$ ;

(3)  $\underline{R}(X \cap Y) = \{x \in U | [x]_R \subseteq X \cap Y\} = \{x \in U | [x]_R \subseteq X \wedge [x]_R \subseteq Y\}$   
 $= \{x \in U | [x]_R \subseteq X\} \cap \{x \in U | [x]_R \subseteq Y\} = \underline{R}(X) \cap \underline{R}(Y)$

**Theorem 4**  $\overline{R}$  is the possibility operator.

**Proof** In order to prove  $\overline{R}$  be the possibility operator, only that  $\overline{R}$  satisfies the three properties of the possibility operator:

(1) By theorem 2  $\overline{R}$  meet  $H_0$ ;

(2)  $\overline{R}(\emptyset) = \{x \in U | [x]_R \cap \emptyset \neq \emptyset\} = \emptyset$ ;

$$(3) \bar{R}(X \cup Y) = \{x \in U \mid [x]_R \cap (X \cup Y) \neq \emptyset\} = \{x \in U \mid [x]_R \cap X \neq \emptyset \vee [x]_R \cap Y \neq \emptyset\} \\ = \{x \in U \mid [x]_R \cap X \neq \emptyset\} \cup \{x \in U \mid [x]_R \cap Y \neq \emptyset\} = \bar{R}(X) \cup \bar{R}(Y)$$

**Theorem 5**  $\underline{R}$  and  $\bar{R}$  are dual.

$$\textbf{Proof } \bar{R}(\bar{X}) = \{x \in U \mid [x]_R \cap \bar{X} \neq \emptyset\} = \sim \{x \in U \mid [x]_R \cap \bar{X} = \emptyset\} = \sim \{x \in U \mid [x]_R \subseteq X\} = \overline{\underline{R}(X)},$$

so the theorem is proved.

**Theorem 6**  $\underline{R}'$  is necessity operator.

**Proof** In order to prove  $\underline{R}'$  be the necessity operator, only that  $\underline{R}'$  satisfies the three properties of the necessity operator:

(1) By theorem 2  $\underline{R}'$  meet  $L_0$ ;

(2)  $\forall x \in U$ ,  $[x]_R \subseteq U$ , by formula (3)  $[x]_R \subseteq \underline{R}'(U)$ , and by lemma 1  $x \in [x]_R$ , so  $x \in \underline{R}'(U)$ , thus  $U \subseteq \underline{R}'(U)$ , and  $\underline{R}'(U) \subseteq U$  must establish, which permit  $\underline{R}'(U) = U$ ;

$$(3) \underline{R}'(X \cap Y) = \bigcup \{[x]_R \mid [x]_R \subseteq X \cap Y\} = \bigcup \{[x]_R \mid [x]_R \subseteq X \wedge [x]_R \subseteq Y\} \\ = \bigcup \{[x]_R \mid [x]_R \subseteq X\} \cap \bigcup \{[x]_R \mid [x]_R \subseteq Y\} = \underline{R}'(X) \cap \underline{R}'(Y).$$

**Theorem 7**  $\bar{R}'$  is the possibility operator.

**Proof** In order to prove  $\bar{R}'$  be the possibility operator, only that  $\bar{R}'$  satisfies the three properties of the possibility operator:

(1) By theorem 2  $\bar{R}'$  meet  $H_0$ ;

$$\bar{R}'(\emptyset) = \bigcup \{[x]_R \mid [x]_R \cap \emptyset \neq \emptyset\} = \emptyset;$$

$$\bar{R}'(X \cup Y) = \bigcup \{[x]_R \mid [x]_R \cap (X \cup Y) \neq \emptyset\} = \bigcup \{[x]_R \mid [x]_R \cap X \neq \emptyset \vee [x]_R \cap Y \neq \emptyset\} \\ = \bigcup \{[x]_R \mid [x]_R \cap X \neq \emptyset\} \cup \bigcup \{[x]_R \mid [x]_R \cap Y \neq \emptyset\} = \bar{R}'(X) \cup \bar{R}'(Y).$$

Thus, binary relation  $R$  only satisfying reflexivity can have really good property, and the partial order relation, tolerance relation and similarity relation are all reflective.

**Theorem 8**  $\underline{R}(X) \subseteq \underline{R}'(X)$ ,  $\bar{R}(X) \subseteq \bar{R}'(X)$ .

**Proof** By the definition of formula (1), (2) and (3), (4) to permit.

### Properties of Operators under Equivalence Relation

**Theorem 9** Let  $R$  be equivalence relation on  $U$ , have  $\underline{R}(X) = \underline{R}'(X)$ ,  $\bar{R}(X) = \bar{R}'(X)$ .

**Proof** By the definition of equivalence relation and formula (1), (2) and (3), (4) to permit.

**Lemma 2**  $(x, y) \in R \Leftrightarrow y \in [x]_R \Leftrightarrow x \in \bar{R}(\{y\})$ .

**Theorem 10** Let  $R$  be general binary relation on  $U$ ,  $\forall X \subseteq U$ , have

$$\underline{R}(X) = \underline{R}'(X), \bar{R}(X) = \bar{R}'(X) \quad (5)$$

Then  $R$  is equivalence relation on  $U$ .

**Proof**

(1) To prove reflexivity of  $R$

By theorem 1 have  $\underline{R}'(X) \subseteq X$ , so by formula (5)  $\underline{R}(X) \subseteq X$ , and by dual properties,  $X \subseteq \bar{R}(X)$  is easy to get, so  $\{x\} \subseteq \bar{R}(\{x\})$ , i.e.  $x \in \bar{R}(\{x\})$ , by lemma 2  $(x, x) \in R$ , so  $R$  is reflexive.

(2) To prove symmetry of  $R$

By the reflexivity of  $R$ ,  $\forall x, x \in [x]_R$ , so  $[x]_R \cap \{x\} \neq \emptyset$ , by formula (4)  $[x]_R \subseteq \bar{R}'(\{x\})$ , so by formula (5)  $[x]_R \subseteq \bar{R}(\{x\})$ ,  $\forall x, y \in U$ , if  $(x, y) \in R$ , by lemma 2  $y \in [x]_R$ , so  $y \in \bar{R}(\{x\})$ , again by lemma 2  $(y, x) \in R$ , so  $R$  is symmetric.

(3) To prove transitivity of  $R$

$\forall X \subseteq U$ , if  $x \in \underline{R}'(X)$ , by formula (3), will  $\exists [y]_R$ , such that  $x \in [y]_R$ , and  $[y]_R \subseteq X$ . So  $[y]_R \subseteq \underline{R}'(X)$ , and  $x \in [y]_R$ , again by the formula (3)  $x \in \underline{R}'(\underline{R}'(X))$ , then can get  $\underline{R}'(X) \subseteq \underline{R}'(\underline{R}'(X))$ , so by formula (5)  $\underline{R}(X) \subseteq \underline{R}(\underline{R}(X))$ , by dual properties,  $\overline{R}(\overline{R}(X)) \subseteq \overline{R}(X)$  is easy to get.

$\forall x, y, z \in U$ , if  $(x, y) \in R$  and  $(y, z) \in R$ . By lemma 2  $y \in \overline{R}(\{z\})$ ,  $(x, y) \in R$  to  $y \in [x]_R$ , so  $y \in [x]_R \cap \overline{R}(\{z\}) \neq \emptyset$ , then  $x \in \overline{R}(\overline{R}(\{z\}))$ ,  $x \in \overline{R}(\{z\})$ , still by lemma 2  $(x, z) \in R$ , so  $R$  is transitive.

## Conclusions

In this paper, various properties of two pairs of lower and upper approximation operators and the relationship between them are acquired through analysis and proof. The two pairs of lower and upper approximation operator will have good properties when the binary relation  $R$  only satisfies reflexivity, and the two pairs of approximation operators are the same under the equivalence relation.

## Acknowledgements

This work was financially supported by the Open Project of Zhejiang Provincial Engineering Center on Media Data Cloud Processing and Analysis (Grant No: 2011C13008), the Zhejiang Provincial Technical Plan Project (Grant No. 2011C13008), and the Provincial Nature Science Foundation of Zhejiang (Grant No: LY12A01019, LY12F02019).

## References

- [1]Z. Pawlak. Rough sets. International Journal of Computer and Information Science[J]. 11(1982):341~356.
- [2]An A J, Stefanowski, Ramanna S, et al. Rough sets, fuzzy sets, data mining and granular computing[C]. RSFDGrC 2007,Toronto: Springer, 2007.
- [3]Wang Guoyin, Li Tianrui, Grzymala-Busse J, et al. Rough sets and knowledge technology[C]. RSKT2008,Berlin:Springer-Verlag, 2008.
- [4]Hu Qinghua, Yu Daren, Xie Zongxia. Numerical Attribute Reduction Based on Neighborhood Granulation and Rough Approximation[J]. Chinese Journal of Software, 2008, 19(3), 640~649.(in Chinese)
- [5]Zhu Feng, Wang Feiyue. Some Results on Covering Generalized Rough Sets[J]. Pattern Recognition & Artificial Intelligence, 2002, 15(1), 6~13.(in Chinese)
- [6]Wang Guoyin, Yao Yiyu, Yu Hong. A Survey on Rough Set Theory and Its Application[J]. Chinese Journal of Computers, 2009, 32(7), 1229~1246.(in Chinese)
- [7]Zhang Wenxiu,Wu Weizhi,Liang Jiye,Li Deyu,Rough Sets Theory and Approach[M],Beijing: Science Press, 2001.(in Chinese)
- [8]Zhang Wenxiu,Liang Yi,Wu Weizhi. Information System and Knowledge Discovery[M],Beijing: Science Press, 2003.(in Chinese)
- [9]Zhang Qinghua, Wang Guoyin, Xiao Yu. Approximation Sets of Rough Sets. Chinese Journal of Software,2012,23(7): 1745~1759.(in Chinese)
- [10]Zhai Xinglong, Li Xuwu. Continuous Attribute Reduction Based on Fuzzy Rough Dependence Degree. Computer Engineering and Applications, 2010, 46(17): 136~138.(in Chinese)