

On the Rough Approximation of Non-Convex Set

Fangan Deng¹ Tao Zhou¹ Yang Xu²

¹Department of Mathematics, Shanxi University of Technology, Hanzhong 723001, China

²Intelligent Control Development Center, Southwest Jiaotong University, Chengdu 610031, China

Abstract

In this article, we have established the concept of the upper and the lower approximation of non-convex set. Meantime, the properties of rough approximation of non-convex set have been investigated. As we know, every non-convex set can be approached by the upper and lower approximation convex set with respect to a given direct. Finally, the relationship between the shadow of u and u -direction of the rough approximation sets have been given.

Keywords: Convex set, Non-convex set, Rough approximation

1. Introduction

Let V be a real linear vector space. Following we will introduce some basic definitions.

Definition 1 Let $S \subset V$. For every $\lambda \in (0, 1)$, if

$$\lambda x^1 + (1 - \lambda)x^2 \in S, \quad (1)$$

where $x^1, x^2 \in S$, then S is said to be convex in V .

Definition 2 Let $S \subset R^m$ be a non-empty set, $u \in R^m$ be a non-zero vector. Then

(1) If $\lambda \in (0, 1)$, and for every $x^1, x^2 \in S$, there exist the real number $\gamma \geq 0$ such that

$$\lambda x^1 + (1 - \lambda)x^2 + \gamma u \in S, \quad (2)$$

then S is called a convex set with respect to u -direction;

(2) The Set

$$S(u) = \{y - \lambda u | y \in S, \lambda \geq 0\} \quad (3)$$

is called the u -shadow of S . The following diagram can best expressed (see fig. 1).

Remark 1 Any non-empty convex set $S \subset R^m$ is convex set with respect to u -direction, for any non-zero vector $u \in R^m$. Besides, for any non-empty set S , we obtain $S \subset S(u)$.

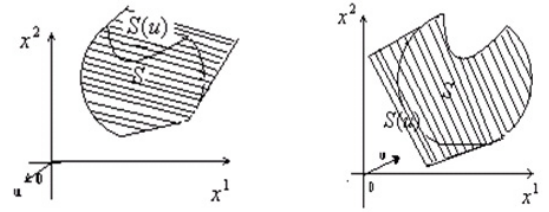


Fig. 1: u -shadow of S .

2. Preliminaries

As well known, convex sets are of good character, but not every set is convex. So how to express the non-convex set with a pair of rough approximation convex set is the main emphasis of our paper. For the sake of convenience, we will establish the following concept.

Definition 3 Let $S \subset V$, $\forall x^1, x^2 \in S$. A direction can be established by the segment $\lambda x^1 + \lambda x^2$, which denoted by p .

According to Definition 3, the non convex set S can be divided into affirmative convex set and possibly convex set, with the aid of the line paralleled to p . If S is a finite field, then there must exist a unique segment S_L . With this segment, a maximal convex subset can be confirmed, which is the maximal subset among the affirmative convex set, and we call it the generalized lower approximation set of S with respect to p -direction, denoted by \underline{S}_p . The following form can better express

$$\underline{S}_p = \cup \{Y \in V \setminus S_L \mid Y \subset S\}. \quad (4)$$

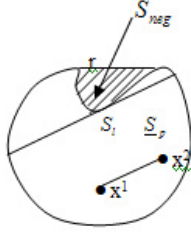


Fig. 2: Sketch map.

Clearly, $\forall x^1, x^2 \in \underline{S}_p$, we obtain $\lambda x^1 + (1 - \lambda)x^2 \in \underline{S}_p$.

Definition 4 In definition 4, every direction p is equivalent to a equivalence relation. In S , every segment paralleled to p belongs to the same equivalence class.

It is worthy pointing that S_p is not convex in general. If we add a region S_{neg} to S , and make $S \cup S_{neg}$ become the minimal convex set including $S - S_p$, then $S \cup S_{neg}$ is said to be a generalized upper approximation set of S with respect to p - direction, denoted by \bar{S}_p . The following form can better express

$$\bar{S}_p = \cup \{Y \in V \setminus S_L \mid Y \cap S \neq \emptyset\} = \underline{S}_p \cup S_{bn}, \quad (5)$$

where $S_{bn} = \{S - S_p\} \cup S_{neg}$. Clearly, $\underline{S}_p \subseteq S \subseteq \bar{S}_p$ (see fig.2). According to the above definition and Remark, we can also get the following result.

(1) Let $S \subset V$. If p is different, then we can get the same generalized upper approximation set \bar{S}_p and different lower approximation set \underline{S}_p ;

(2) If S is convex, then $\bar{S}_p = \underline{S}_p$.

Definition 5 [2] Let $S \subset V$. Then the intersection of all the convex sets including S in V is called convex hull, denoted by $co(S)$. The convex hull is the smallest convex set including S .

3. Main Results

Theorem 1 Let $S \subseteq V$. Suppose that p_1 and p_2 are two directions, then the following results hold.

- (1) If $p_1 \parallel p_2$, then $\underline{S}_{p_1} = \underline{S}_{p_2}$;
- (2) If $p_1 \nparallel p_2$, then $\underline{S}_{p_1} \cap \underline{S}_{p_2} \neq \emptyset$, and $\underline{S}_{p_1} \cap \underline{S}_{p_2}$ is also convex.

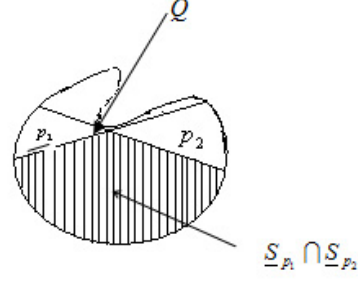


Fig. 3: $\underline{S}_{p_1} \cap \underline{S}_{p_2}$.

Proof.(1) By Definition 4, it can be easily to get the result.

(2) Let l_{p_1} and l_{p_2} be two polar line paralleled to p_1 and p_2 , respectively, and $l_{p_1} \cap l_{p_2} = Q$. Then from the definition of polar line, we obtain $Q \in \underline{S}_{p_1}$. Similarly, we also get $Q \in \underline{S}_{p_2}$, that is, $\underline{S}_{p_1} \cap \underline{S}_{p_2} \neq \emptyset$. $\forall y_1, y_2 \in \underline{S}_{p_1} \cap \underline{S}_{p_2}$, then $\forall \lambda \in (0, 1)$,

$$y_1, y_2 \in \underline{S}_{p_1} \Rightarrow \lambda y_1 + (1 - \lambda)y_2 \in \underline{S}_{p_1}. \quad (6)$$

Similarly, we also have $\forall \lambda \in (0, 1)$,

$$y_1, y_2 \in \underline{S}_{p_2} \Rightarrow \lambda y_1 + (1 - \lambda)y_2 \in \underline{S}_{p_2}. \quad (7)$$

If there exist $\lambda_1^*, \lambda_2^*, \lambda_1^* + \lambda_2^* \in (0, 1)$ satisfied

$$(\lambda_1^* + \lambda_2^*)y_1 + (1 - (\lambda_1^* + \lambda_2^*))y_2 \notin \underline{S}_{p_1} \cap \underline{S}_{p_2},$$

and take $\lambda_1^* + \lambda_2^* = \lambda^*$, then $\lambda^*y_1 + (1 - \lambda^*)y_2 \notin \underline{S}_{p_1} \cap \underline{S}_{p_2}$, that is,

$$\begin{aligned} \lambda^*y_1 + (1 - \lambda^*)y_2 &\in \underline{S}_{p_1} \text{ and} \\ \lambda^*y_1 + (1 - \lambda^*)y_2 &\in \underline{S}_{p_2}, \end{aligned} \quad (8)$$

which is a contradiction with the form (6) and (7). Hence $\forall \lambda_1, \lambda_2, \lambda_1 + \lambda_2 \in (0, 1)$,

$$(\lambda_1 + \lambda_2)y_1 + (1 - (\lambda_1 + \lambda_2))y_2 \in \underline{S}_{p_1} \cap \underline{S}_{p_2},$$

therefore $\underline{S}_{p_1} \cap \underline{S}_{p_2}$ is also convex.

Theorem 2 Let $S \subseteq V$. Then the following statements hold.

- (1) If S is convex, then $\underline{S}_p \subset co(S) = \bar{S}_p = S$, for any direction p ;
- (2) If S is not convex, then $\underline{S}_p \subset co(S) = \bar{S}_p$.

Proof. It is easy to get the result with the aid of Definition 3 and Definition 5

According to Theorem 2, for any p , S , we can see $\underline{S}_p \subseteq S \subseteq co(S) = \bar{S}_p$.

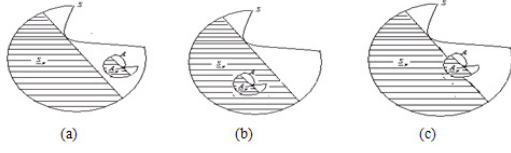


Fig. 4: Sketch map.

Theorem 3 Let $S_1, S_2 \subseteq V$. Then

- (1) $\overline{S}_p(S_1 \cap S_2) \subseteq \overline{S}_p \cap \overline{S}_p(S_2)$;
- (2) $\overline{S}_p(S_1 \cup S_2) \subseteq \overline{S}_p \cup \overline{S}_p(S_2)$.

Theorem 4 Let $S \subseteq V$ be a non-convex set. Given a non-empty set $A \subseteq S$. Then

- (1) If A is not convex, and $\underline{S}_p \cap A \neq \emptyset$, then $\underline{S}_p \cap \underline{A}_p \neq \emptyset$;
- (2) If A is not convex, and $A \subseteq \underline{S}_p$, then $\underline{A}_p \subseteq \underline{S}_p$, $\underline{A}_p \subseteq \overline{S}_p$.

Proof. (1) By the fact that $A \subseteq S$ and $\underline{A}_p \subseteq A$, then we have $(\underline{A}_p \cap \underline{S}_p \subseteq (A \cap \underline{S}_p))$. Since $A \cap \underline{S}_p = \emptyset$, $\underline{S}_p \cap \underline{A}_p \neq \emptyset$.

(2) Clearly, $\underline{A}_p \subseteq \underline{S}_p$ so following we will prove $\underline{A}_p \subseteq \overline{S}_p$. In fact, from the assumption that $A \subseteq \underline{S}_p$, we have $co(A \cap \underline{S}_p) = co(A) = co(A) \cap co(\underline{S}_p)$. Besides, because \underline{S}_p is convex, so $co(\underline{S}_p) = \underline{S}_p$, hence $\underline{A}_p = co(A) \subseteq \underline{S}_p$.

Meanwhile, according to the definition of u -shadow of S and the generalized rough approximation of u -direction, we can also get the following result.

Proposition 1 (1) If u -shadow of S is convex, then $\overline{S}_u \subseteq S(u)$;

(2) If u -shadow of S is not convex, then $\overline{S}_u \cap S(u) = S$;

(3) $\underline{S}_u \subseteq S(u)$.

Acknowledgments

This work is supported by the Natural Science Foundation of China(No. 70472072, 60474022), and the Doctor Foundation of Shaanxi University of Technology. Besides, the authors wish to thank the referee for the valuable comments and suggestions which helped to improve the presentation of this note.

References

- [1] B. Aupetit, Projections in real algebras, *Bull. London Math. Soc.*, 13 (1981) 412-414.
- [2] H. K. Du, X. Y. Yao, C. Y. Deng, *Invertibility of linear combinations of two idempotents*, *Proc. Amer. Math. Soc.*, 134 (2005) 1451-1457.
- [3] F.A. Deng, On the rough approximation of non-convex set, *J.of University of Science and Technolony of Suzhou(Natural Science)*, Vol.20, No.3 7-10.