

# Modelling and Controller Design of Planar Inverted Pendulum System

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**Abstract**— The inverted pendulum is a classical control problem, which is an absolutely unstable system with the characteristics of higher-order, unstable, multivariable, nonlinear and strong coupling. There can be varieties of methods to be applied to control inverted pendulum system, such as PID control, optimal control, robust control, sliding mode control, fuzzy control, neural network control, human-imitating intelligent control. In this paper, firstly by using the principle of Newtonian mechanics, the mathematical model of the inverted pendulum system was established and the differential equations and state space representation were derived. Secondly, a controller was designed by using the Linear Quadratic Regulator (LQR) cost equation. Thirdly, the PID controller was designed in which the tuning of the PID parameters was focused. Finally, the simulation results show the inverted pendulum could be balanced with the both controllers.

**Keywords**- inverted pendulum; LQR; PID controller; stabilization; modelling

## INTRODUCTION

The inverted pendulum [1] is a classical control problem, which involves developing a system to balance a pendulum [2-10]. The control of it can effectively reflect many important questions, such as nonlinearity, controllability and robustness, etc.. The control of inverted pendulum is to make the cart and swinging rod achieve desired equilibrium position as soon as possible. In the meanwhile they won't have too strong oscillation range, too fast speed and angular velocity. When the inverted pendulum system achieves the desired position, it can overcome a range of disturbance and keep balance. As a control device, Inverted Pendulum has intuitive image and simple structure, As a control object, it is a higher-order times nonlinear, multivariable, strong coupling and unstable system, only by adopting effective methods to make it stable, so the inverted pendulum have become unflagging research topic in control theory. In this paper, modeling and controller design of planar inverted

pendulum was investigated. In the controller design process, LQR and PID controller were compared.

## MATHEMATICAL MODEL

### A. Structure and Parameters of Inverted Pendulum

There are strong couplings between the small car and swinging rod in the inverted pendulum system. However, the motions on the X and Y direction are separately controlled by two different motor in the experiment in which the mutual influence on both directions can be ignored, i.e. the system is decoupled. So the planar inverted pendulum can be decomposed into two linear inverted pendulums in the different direction. In the controller design of planar inverted pendulum, the controller on X, Y direction should be designed. According to similarity system in two directions, the design and the parameters of the two controllers are same. In this paper, only the X, Z plane motion model and control methods are introduced. The first-order inverted pendulum system can be seen as a cart that goes along the smooth guide sporting and a well quality swinging rod, as shown in Fig. 1.

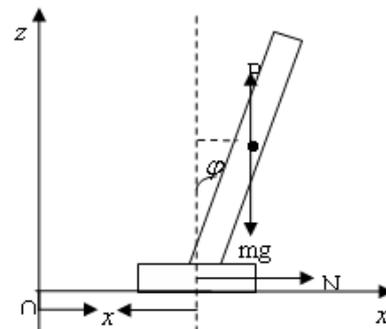


Figure 1. Linear Inverted Pendulum Model

Experimental parameters of the inverted pendulum are showed in Table I:

TABLE I. PARAMETERS OF THE INVERTED PENDULUM

Parameters	Physical significance	Numerical value
M	Cart mass	1.096Kg
m	Swing rod mass	0.109Kg
F	Driving force of the car	
f	Cart sliding coefficient of friction	0.1N/m/sec
$\varphi$	Joint angle between swing rod and vertical downward	
x	Horizontal displacement of the car	
I	rotational inertia of Swing rod mass	0.0034Kg · m <sup>2</sup>
l	Distance between swing rod centroid and center of rotation	0.25m

### B. Modelling of the System

By the vertical force analysis of the pendulum, we can obtain:

$$P - mg = m \frac{d^2}{dt^2} l(1 - \cos \varphi) \quad (1)$$

$$N = m \frac{d^2}{dt^2} (x - l \sin \varphi) \quad (2)$$

From the above two equations we can derive:

$$P = mg + ml(\ddot{\varphi} \cos \varphi + \dot{\varphi}^2 \sin \varphi) \quad (3)$$

$$N = m(\ddot{x} - l\ddot{\varphi} \sin \varphi + l\dot{\varphi}^2 \cos \varphi) \quad (4)$$

The moment equilibrium equation is as follows:

$$Pl \sin \varphi + Nl \cos \varphi = I\ddot{\varphi} \quad (5)$$

Substituting (3),(4) into (5),we can obtain:

$$mgl \sin \varphi + ml\ddot{x} \cos \varphi - ml^2 \dot{\varphi}^2 \sin \varphi = I\ddot{\varphi} \quad (6)$$

On the specific issues,  $\varphi \ll 1$ , using the following approximate treatment  $\cos \varphi = 1, \sin \varphi = \varphi, (\frac{d\varphi}{dt})^2 = 0$ , we can obtain the following equation from (6) by the linearization techniques

$$(I + ml^2)\ddot{\varphi} - mgl\varphi = ml\ddot{x} \quad (7)$$

Assumes that the system initial conditions are zero, the Laplace transform of (7) is as follows:

$$(I + ml^2)\phi(s)s^2 - mgl\phi(s) = mlX(s)s^2 \quad (8)$$

Due to the output of the system is the angle of swinging rod, we can be obtained the relation between the system input and output by (8):

$$X(s) = \left( \frac{I + ml^2}{ml} - \frac{g}{s^2} \right) \phi(s) \quad (9)$$

If we take the car movement distance as the system input, the transfer function is

$$\frac{\phi(s)}{X(s)} = \frac{mls^2}{(I + ml^2)s^2 - mgl} \quad (10)$$

From the experiment platform provided by the Googol Technology ( Shenzhen ) Ltd, the input signal is the car's acceleration. So the transfer function of the system is as follows:

$$\frac{\phi(s)}{V(s)} = \frac{ml}{(I + ml^2)s^2 - mgl} \quad (11)$$

Assume the state equations of the system is

$$\begin{aligned} \dot{X} &= AX + Bu \\ y &= CX + Du \end{aligned} \quad (12)$$

From (7), we can obtain

$$\ddot{\varphi} = \frac{mgl}{I + ml^2} \varphi + \frac{ml}{I + ml^2} \ddot{x} \quad (13)$$

Let  $X = [x \quad \dot{x} \quad \varphi \quad \dot{\varphi}]^T, u = \ddot{x}$ , the state equations of the system can be obtained.

$$\begin{aligned} \dot{X} &= \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\varphi} \\ \ddot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{mgl}{I + ml^2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \varphi \\ \dot{\varphi} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{ml}{I + ml^2} \end{bmatrix} u \\ y &= \begin{bmatrix} x \\ \varphi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \varphi \\ \dot{\varphi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \end{aligned} \quad (14)$$

Substituting the actual physical parameters into the model, the transfer function of the system is

$$\frac{\phi(s)}{V(s)} = \frac{0.02725}{0.0102125s^2 - 0.26705} \quad (15)$$

The state equations of the system is

$$\dot{x} = \begin{bmatrix} \ddot{x} \\ \dot{x} \\ \ddot{\varphi} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 26.1493 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \varphi \\ \dot{\varphi} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2.6683 \end{bmatrix} u \quad (16)$$

$$y = \begin{bmatrix} x \\ \varphi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \varphi \\ \dot{\varphi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

### LQR CONTROLLER DESIGN

Using the knowledge of the modern control theory, we can know the system is fully controllable.

Four variables in the system  $x, \dot{x}, \varphi, \dot{\varphi}$  represent the car of the displacement, acceleration of the car, the angle of swinging rod and the angle acceleration of the swinging rod, respectively. Assuming that the four state variables are measurable, using the program  $K = lqr(A, B, Q, R)$  of Matlab to calculate the feedback matrix  $K$ , in which the two parameters  $Q, R$  are used to balance the weights of input and state variables and can be chosen by us. In the matrix  $Q$ ,  $Q_{11}$  represents the weight of the car position and  $Q_{33}$  the weights of the angle of swinging rod. Choose the simplest case, let  $R = 1$ , according to the experimental simulation for many times, we obtain the following parameters  $Q_{11} = 1000$ ,  $Q_{33} = 1$  to makes the system with good dynamic performance and stability. The corresponding feedback matrix is  $K = [-31.6228 \ -20.3233 \ 73.8007 \ 14.4308]$ . The tracking simulation with LQR controller is shown as Fig .2.

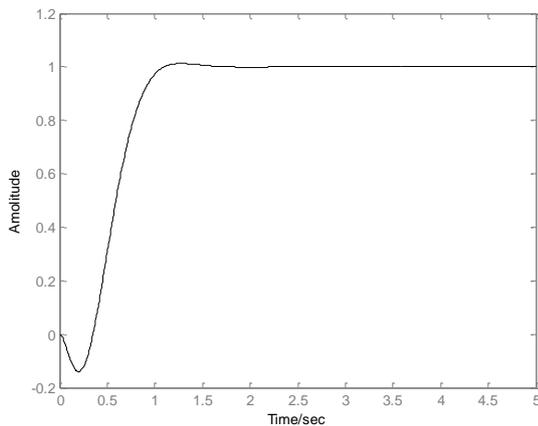


Figure 2. Tracking simulation with LQR controller

By the simulation results, we found that the stability of the control system is good, but dynamic performance is unsatisfactory. The inverted pendulum system is a servo system with higher request for dynamic performance, especially in the adjustment time in order to achieve the desired effect. Although the overshoot is small, the adjust time is about 1.5 s which is too long for swinging rod to restore the original state. All things considered, the control effect is better.

### PID CONTROLLER DESIGN

The main controllers used in engineering include proportional controller (P) controller, proportional-integral (PI) controller, proportional-derivative (PD) controller and proportional-integral-derivative (PID) controller. In this paper, we used the four different controllers to control the inverted pendulum system and analyzed the control effect of each controller.

#### A. P Controller

Let  $K_p = 100$ , the step response of the inverted pendulum with P controller was shown as Fig .3.

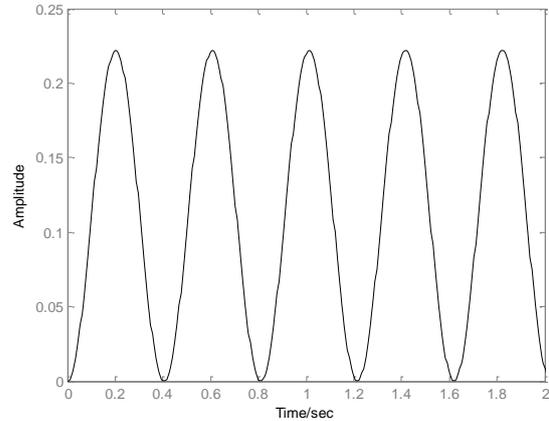


Figure 3. Step Response of the Inverted Pendulum with P Controller

From Fig .3, we can see that the proportional controller obviously can not reach the ideal effect. The proportion controller can improve the rapidity of system and reduce the steady-state error, but it can not make the inverted pendulum system has good dynamic performance and steady-state performance.

#### B. PI Controller

Let  $K_p = 100$ ,  $K_i = 0.1$ , the step response of the inverted pendulum with PI controller was shown as Fig .4.

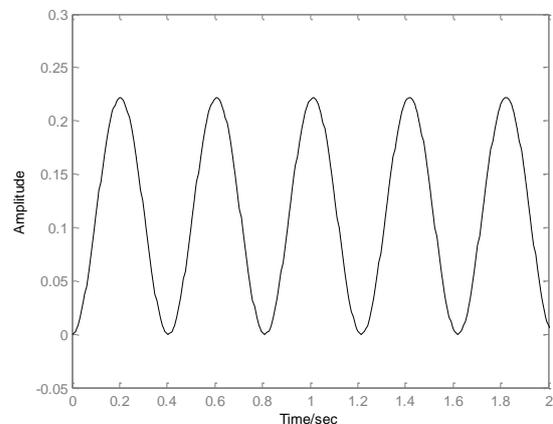


Figure 4. Step Response of the Inverted Pendulum with PI Controller

From Fig .4, we can see that the control performance the inverted pendulum system of PI controller is similar to P controller; both of them will produce continuous

oscillation. PI controller cannot meet the design requirements.

### C. PD Controller

Let  $K_p=100$ ,  $K_d=10$ , the step response of the inverted pendulum with PD controller was shown as Fig .5.

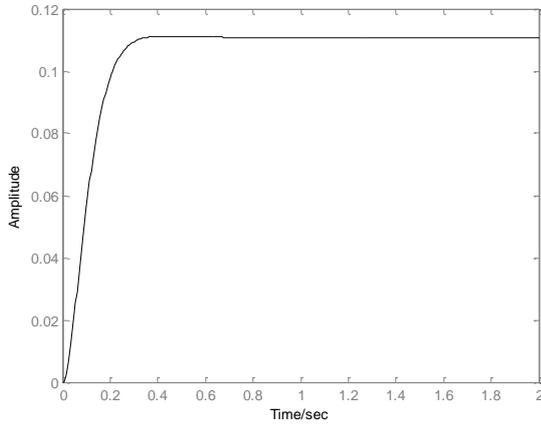


Figure 5. Step Response of the Inverted Pendulum with PD Controller

From Fig .5, we can see that the inverted pendulum system with PD controller has good dynamic performance and stability. First, the system realizes stable within 0.5 s which is better than the P controller and PI controller. Second, the selected parameters in the design ensure the system has good dynamic performance, the adjusting time is around 0.4s and the overshoot is negligible. The only shortage is the steady-state error. It can be concluded that the inverted pendulum system can not return to the vertical state with a disturbance by choosing PD controller.

### D. PID Controller

Let  $K_p=100$ ,  $K_i=1000$ ,  $K_d=10$ , the step response of the inverted pendulum with PID controller was shown as Fig .6.

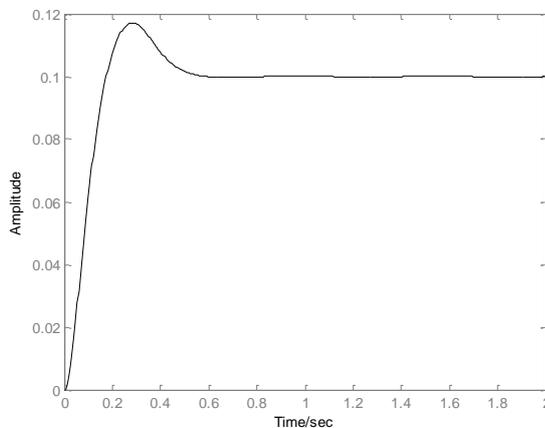


Figure 6. Step Response of the Inverted Pendulum with PID Controller

From Fig .6, we find that the inverted pendulum system under the control of the PID controller can achieve the better dynamic and steady-state performance. The system can realize stable at about 0.5 s, and there is no big

overshoot. We can adjust the parameters to get more satisfactory results. According to control principle, after many times parameter adjustment and experiment, we got a set of ideal data as follows:

$$K_p=1000, K_i=5000, K_d=40.$$

The corresponding step response of the inverted pendulum is shown as Fig .7.

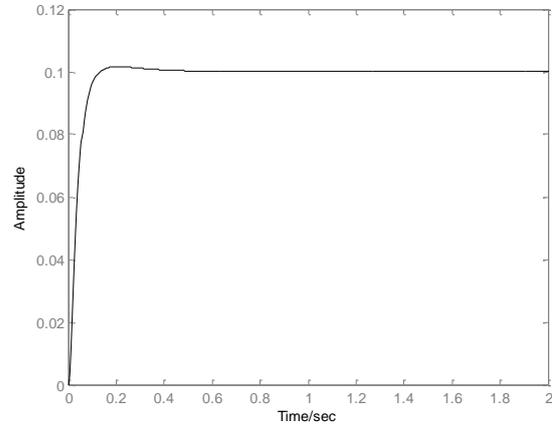


Figure 7. Step Response of the Inverted Pendulum with PID Controller

## CONCLUSIONS

In this paper, the planar inverted pendulum system was investigated. The mathematical model of the inverted pendulum system was established. The differential equations and state space representation were derived. The LQR and PID controller were designed to control the inverted pendulum system. It can be concluded that the reaction time is longer with LQR controller and the PID controller can achieve the ideal performance.

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