A New Method for Coupled Best Proximity Point Theorems in Partially Ordered Metric Spaces

Sumei Ai
Department of Mathematics
Cangzhou Normal University,
Cangzhou,China
e-mail: asm1034@sina.com.

Yongfu Su
Department of Mathematics
Tianjin Polytechnic University
Tianjin, China
e-mail::tjsuyongfu@163.com

Abstract—Several problems can be changed as equations of the form Tx = x, where T is a given self-mapping defined on a subset of a metric space, a normed linear space, a topological vector space or some suitable space. However, if T is a non-self mapping from A to B, then the aforementioned equation does not necessarily admit a solution. In this case, it is contemplated to find an approximate solution x in A such that the error d(x, Tx) is minimum, where d is the distance function. In view of the fact that d(x,Tx) is at least d(A,B), a best proximity point theorem guarantees the global minimization of d(x, Tx) bythe requirement that an approximate solution x satisfies the condition d(x, Tx) = d(A,B). Such optimal approximate solutions are called best proximity points of the mapping T. Interestingly, best proximity point theorems also serve as a natural generalization of fixed point theorems, for a besproximity point becomes a fixed point if the mapping under consideration is a self mapping. Research on the best proximity point is an important topic in the nonlinear functional analysis and applications. The aim of this paper is to obtain the coupled best proximity point theorems for generalized contraction in partially ordered metric spaces by P-operator technique. An example has also been given to support the usability of our results. Many recent results in this area have been improved.

Keywords- coupled best proximity point; generalized contraction; weak P-monotone property; fixed point.; contraction

I. INTRODUCTION AND PRELIMINARIES

Let A and B be two nonempty subsets of a metric space (X,d). We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x,y) = d(A,B), y \in B\}$$

$$B_0 = \{x \in B : d(x,y) = d(A,B), x \in A\}$$

where $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$.

In [1], the authors give sufficient conditions for when the sets A_0 and B_0 are nonempty. In [2], the authors prove that any pair (A,B) of nonempty, closed and convex subsets of a uniformly convex Banach space satisfes the $P-property$.

Definition 1.1 [3] Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \varnothing$ .Then the pair (A,B) is said to have the $P-property$ if and only if for any $x_1, x_2 \in A_0$, and $y_1, y_2 \in B_0$,

$$\begin{cases} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{cases} \Rightarrow d(x_1,x_2) = d(y_1,y_2).$$

Let A, B be two nonempty subsets of a complete metric space and consider a mapping $T : A \to B$. The best proximity point problem is whether we can find an element $x_0 \in A$ such that

$$d(x_0,Tx_0) = \min\{d(x,Tx) : x \in A\}.$$

Since $d(x,Tx) \geq d(A,B)$ for any $x \in A$, in fact, the optimal solution to this problem is the one for which the value $d(A,B)$ is attained.

In [4], the authors give a generalized result by considering a nonself map and they get the following theorem.

Theorem 1.2 [4] Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $T : A \to B$ be a Geraghty-contraction satisfying $T(A_0) \subseteq B_0$ .Suppose that the pair (A,B) has the $P-property$ .Then there exists a unique x^* in A such that $d(x^*,Tx^*) = d(A,B)$.

Let us recall the following definitions.

Definition 1.3 [4] Let $(X,\leq)$ be a partially ordered set and $F : X \times X \to X$. We say that F has the mixed monotone property if $F(x,y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \le x_2 \Rightarrow F(x_1, y) \le F(x_1, y);$$
$$y_1, y_2 \in X, y_1 \le y_2 \Rightarrow F(x, y_1) \ge F(x, y_2).$$
This definition coincides with the notion of a mixed monotone function on R^2 and $\le$ represents the usual total order in R.

Definition 1.4 [4] We call an element $(x, y) \in X \times X$ a coupled fixed point of the mapping F if
$$F(x, y) = x, F(y, x) = y.$$

T. Gnana Bhaskar, V. Lakshmikantham got the following theorems in 2006.

Theorem 1.5 [4] Let $(X, \le)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in (0,1]$ with
$$d(F(x, y), F(u, v)) \le \frac{k}{2}[d(x, u) + d(y, v)], \forall x \ge u, y \le v.$$
If there exists $x_0, y_0 \in X$ such that
$$x_0 \le F(x_0, y_0) \text{ and } y_0 \ge F(y_0, x_0)$$

Then, there exist $x, y \in X$ such that
$$x = F(x, y), y = F(y, x)$$

Theorem 1.6 [4]. Let $(X, \le)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that X has the following property:

(i) if a noncreasing sequence $\{x_n\} \to x$, then $x_n \le x, \forall n$;

(ii) if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \le y, \forall n.$

Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in (0,1]$ with
$$d(F(x, y), F(u, v)) \le \frac{k}{2}[d(x, u) + d(y, v)], \forall x \ge u, y \le v.$$
If there exists $x_0, y_0 \in X$ such that
$$x_0 \le F(x_0, y_0) \text{ and } y_0 \ge F(y_0, x_0)$$

Then, there exist $x, y \in X$ such that
$$x = F(x, y), y = F(y, x)$$

It can be proved that the coupled fixed point is in fact unique, provided that the product space $X \times X$ endowed with the partial order mentioned above enjoying the following property:

Every pair of elements has either a lower bound or an upper bound.

It is known [5] that this condition is equivalent to:

Condition (*) : For every $(x, y), (x^*, y^*) \in X \times X$, there exists a $(z_1, z_2) \in X \times X$ that is comparable to $(x, y), (x^*, y^*)$.

Theorem 1.7 [4]. Adding condition (*) to the hypothesis of Theorem 1.8, then the uniqueness of the coupled fixed point of F can be obtained.

Theorem 1.8 [4]. In addition to the hypothesis of Theorem 1.8, suppose that every pair of elements of X has an upper bound or a lower bound in X. Then $x = y$.

Theorem 1.9 [4]. In addition to the hypothesis of Theorem 1.8 (resp. Theorem 1.9), suppose that x_0, y_0 in X are comparable. Then $x = y$.

We introduce the following definition.

Definition 1.10 Let A, B be subsets of a metric space X. An element $(x, y) \in A \times A$ is called a coupled best proximity point of $F : A \times A \to B$ if
$$d(F(x, y), x) = d(A, B), d(y, F(y, x)) = d(A, B).$$

The aim of this paper is to obtain the coupled best proximity point theorems for generalized contraction in partially ordered metric spaces by $P-$operator technique. An example has also been given to illustrate the theorems. Many recent results in this area have been improved.

II. MAIN RESULTS

Weak $P-$monotone property: Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \ne \varnothing$. Then the pair (A, B) is said to have the weak $P-$monotone property if and only if for any $x_1, x_2 \in A_0$, and $y_1, y_2 \in B_0$,
$$\begin{cases} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{cases} \Rightarrow d(x_1, x_2) \le d(y_1, y_2),$$
furthermore, $y_1 \ge y_2$ implies $x_1 \ge x_2$.

Now we are in a position to give our main results.

Theorem 2.1. Let X be a partially ordered set and (X, d) is a complete metric space. Let (A, B) be a pair of nonempty closed subsets of X such that $A_0 \ne \varnothing$. Let $F : A \times A \to B$ be a continuous mapping with $F : A_0 \times A_0 \subseteq B_0$ Suppose that F has mixed monotone property satisfying
$$d(F(x, y), F(u, v)) \le \frac{k}{2}[d(x, u) + d(y, v)], \forall x \ge u, y \le v, k \in (0,1]$$
Suppose that the pair (A, B) has the weak $P-$monotone

property. If there exist $x_0, y_0 \in A_0$ such that

$$d(x_0,\hat{x}_0)=d(A,B), d(y_0,\hat{y}_0)=d(A,B),$$
$$\hat{x}_0 \le F(x_0,y_0), \hat{y}_0 \ge F(y_0,x_0).$$

where $\hat{x}_0, \hat{y}_0 \in B_0$. Then there exists a $(x^*,y^*)\in A\times A$ such that

$$d(x^*,F(x^*,y^*))=d(A,B), d((y^*,x^*),y^*)=d(A,B).$$

Proof. We first prove that B_0 is closed. Let $\{y_n\}\subseteq B_0$ be a sequence such that $\{y_n\}\to y\in B$. It follows from the weak P-monotone property that

$$d(y_n,y_m)\to 0 \Rightarrow d(x_n,x_m)\to 0,$$

as $n,m\to\infty$, where $x_n,x_m\in A_0$, and

$$d(x_n,y_n)=d(A,B),\ d(x_m,y_m)=d(A,B).$$

Then $\{x_n\}$ is a Cauchy sequence so that $\{x_n\}$ converges strongly to a point $p\in A$. By the continuity of a metric d, we have $d(p,q)=d(A,B)$. That is, $q\in B_0$. Hence, B_0 is closed.

Let $\overline{A}_0$ be the closure of A_0 we claim that $F:\overline{A}_0\times\overline{A}_0\subseteq B_0$. In fact, if $x,y\in\overline{A}_0\setminus A_0$, then there exist sequences $\{x_n\},\{y_n\}\subseteq A_0$ such that $x_n\to x$, $y_n\to y$. By the continuity of F and the closeness of B_0, we have

$$F(x,y)=\lim_{n\to\infty}F(x_n,y_n)\in B_0.$$

That is, $F(\overline{A}_0\times\overline{A}_0)\subseteq B_0$.

Define an operator P_{A_0}: $F(\overline{A}_0\times\overline{A}_0)\to A_0$ by $P_{A_0}y=\{x\in A: d(x,y)=d(A,B)\}$. From the weak $P-$monotone property, we can know that P_{A_0} is single valued. By the definition of F and the weak $P-$monotone property, we have

$$d\big(P_{A_0}F(x,y),P_{A_0}F(u,v)\big)\le d\big(F(x,y),F(u,v)\big)$$
$$\le \frac{k}{2}\big[d(x,u)+d(y,v)\big]$$

for any $x\ge u, y\le v\in\overline{A}_0$. Let $x_n,y_n,x,y\in\overline{A}_0$, $x_n\to x$, $y_n\to y$. From the above inequality and F is continuous, we have

$$F(x_n,y_n)\to F(x,y)$$
$$\Leftrightarrow d(F(x_n,y_n),F(x,y))\to 0$$
$$\Rightarrow d(P_{A_0}F(x_n,y_n),P_{A_0}F(x,y))\to 0$$
$$\Rightarrow P_{A_0}F(x_n,y_n)\to P_{A_0}F(x,y) \text{ as } n\to\infty.$$

So $P_{A_0}F$ is continuous. Since F has the mixed monotone property and (A,B) has the weak $P-$monotone property, we can get

$$\begin{cases} d(P_{A_0}F(x,y),F(x,y))=d(A,B)\\ d(P_{A_0}F(u,y),F(u,y))=d(A,B)\\ F(x,y)\ge F(u,y)\subseteq B_0\end{cases}$$
$$\Rightarrow P_{A_0}F(x,y)\ge P_{A_0}F(u,y).$$

as the similar way, we can get

$$\begin{cases} d(P_{A_0}F(x,y),F(x,y))=d(A,B)\\ d(P_{A_0}F(x,v),F(x,v))=d(A,B)\\ F(x,y)\ge F(x,v)\subseteq B_0\end{cases}$$
$$\Rightarrow P_{A_0}F(x,y)\ge P_{A_0}F(x,v).$$

for any $x\ge u, y\le v\in\overline{A}_0$. This shows that $P_{A_0}F$ is mixed monotone. Because there exist $x_0,y_0\in A_0$ such that

$$d(x_0,\hat{x}_0)=d(A,B), d(y_0,\hat{y}_0)=d(A,B),$$
$$\hat{x}_0\le F(x_0,y_0), \hat{y}_0\ge F(y_0,x_0).$$

where $\hat{x}_0,\hat{y}_0\in B_0$. Then we can obtain

$$\begin{cases} d(P_{A_0}F(x_0,y_0),F(x_0,y_0))=d(A,B)\\ d(x_0,\hat{x}_0)=d(A,B)\\ \hat{x}_0\le F(x_0,y_0)\in B_0\end{cases}$$
$$\Rightarrow x_0\le P_{A_0}F(x_0,y_0).$$

In the same way, we have $y_0\le P_{A_0}F(y_0,x_0)$.

This shows that $P_{A_0}F:(\overline{A}_0\times\overline{A}_0)\to\overline{A}_0$ is a contraction satisfying all the conditions in Theorem 1.8. Therefore, $P_{A_0}F$ has a coupled fixed point (x^*,y^*). That is

$$P_{A_0}F(x^*,y^*)=x^*\in A_0, P_{A_0}F(y^*,x^*)=y^*\in A_0,$$

which implies that

$$d(x^*,F(x^*,y^*))=d(A,B), d(F(y^*,x^*),y^*)=d(A,B).$$

That is the desired result.

The previous result still hold for F not necessarily continuous. Instead, we only need to require an additional property on X. We discuss this in the following theorem.

Theorem 2.2 Let X be a partially ordered set and (X,d) is a complete metric space. Let (A,B) be a pair of nonempty closed subsets of X such that $A_0\ne\varnothing$. Let $F:A\times A\to B$ be a mapping with $F:A_0\times A_0\subseteq B_0$ Suppose that F has mixed monotone property satisfying

$$d\big(F(x,y),F(u,v)\big)\le\frac{k}{2}\big[d(x,u)+d(y,v)\big], \forall x\ge u, y\le v, k\in(0,1]$$

Assume that $\overline{A}_0$ has the following property:

(i) if a noncreasing sequence $\{x_n\}\to x$, then $x_n\le x,\forall n;$

(ii) if a nonincreasing sequence $\{y_n\}\to y$, then $y_n\le y,\forall n.$

Suppose that the pair (A,B) has the weak $P-$monotone property. If there exist $x_0,y_0\in A_0$ such that

$$d(x_0,\hat{x}_0)=d(A,B), d(y_0,\hat{y}_0)=d(A,B),$$
$$\hat{x}_0\le F(x_0,y_0), \hat{y}_0\ge F(y_0,x_0).$$

where $\hat{x}_0,\hat{y}_0\in B_0$. Then there exists a $(x^*,y^*)\in A\times A$

such that

$$d(x^*, F(x^*, y^*)) = d(A,B), d((y^*, x^*), y^*) = d(A,B).$$

Proof: The proof is the same as Theorem 2.1 without proving the continuity of $P_{A_0}F$. Then Theorem 2.2 can be got by using Theorem 1.9.

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