The Research of Ruin Problem in Levy Process

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Abstract—Levy process, as one of the hot issues in the modern ruin theory, attracts more and more scholars in recent years. In particular, the risk model in this paper, as the promotion of the classical model, is different from the classical Cramer-Lundberg risk model. According to the definition of the expected discounted penalty function, we give its expression, which is under the joint distribution of the ruin time τ, the deficit at ruin $-X_\tau$, the surplus immediately prior to ruin $X_{\tau-}$ and the last minimum surplus level before ruin $\underline{X}_{\tau-}$. Using the method of asymptotic equivalence for the ruin probability, we derive the asymptotic equation of ruin probability. In the same way, we can obtain the asymptotic estimate of the tail condition of its supremum related to Levy processes as the Levy measure belongs to subexponential tail. The results are considerably meaningful for the calculate of the ruin probability.

Keywords- Ruin probabilities; Risk model; Levy process; Subexponential distribution; The expected discounted penalty function.

I. INTRODUCTION

Levy process is essentially a stochastic process with stationary and independent increments. The basic theory was developed, principally by Paul Levy in the 1930s. In the past 15 years there has been a renaissance of interest and a plethora of books, articles and conferences. Intuitively speaking, Levy process can be seen as a continuous time random walk, it is characterized by independent and stationary increments. The main Levy process are Brown motion, simple and complex Poisson process, Subordinators, and so on. Levy process is one of the most simple general class, it has a continuous path, with random jump occurs at random time of any size. It is a half martingale and type a natural subset of the process in a stochastic dynamic system. Levy process is also a very good noise model and it has good properties by the industry and academia and it widely used in the world.

One of the good properties of Levy processes is infinite divisibility, i.e. Infinite divisibility is the underlying probabilistic idea which a Levy process embodies dynamically.

Let μ be be a probability measure on R^d. Define $\mu^{*n} = \mu * \cdots * \mu$ (n times). We say that has a convolution n th root if there exists a probability measure for which $\mu^{\frac{1}{n}}$ for which

$$(\mu^{\frac{1}{n}})^{*n} = \mu$$

μ is infinitely divisible if it has a convolution n th root for all $n \in N$. In this case $\mu^{\frac{1}{n}}$ is unique.

In this paper, we introduce a new risk model which is on the basis of the classical Cramer-Lundberg model, and it can be written as

$$U_t = u + X_t, t \geq 0, \quad (1)$$

where $u \geq 0$ is the initial surplus, X_t is a Levy process ($X_0=0$), representing the net aggregate cash inflow of an insurance company. As is customary, the symbols E_x and P_x will denote the expectation and the probability measure related to the process started at x, and if the process is started from zero we will use simple notations E and P. Notice that since X_t is a Levy process, the model in (1) contains all elements of a traditional risk model. Indeed, the constant rate premium is included as the drift of X_t, the so-called perturbation comes in as the Brownian component of X_t and the pure aggregate claims is present as the jump part of X_t, which could be set as a compound Poisson or an infinite activity process. With this in mind, we assume the process X_t to have a positive drift such that $E[X_1]>0$ in order to avoid the possibility that X becomes negative almost surely. This condition is often expressed in terms of a safety loading. For instance, notice that we can recuperate the classical Cramer-Lundberg model if $Y_t = ct - S_t$, where $c=(1+\theta)E[S_1]$, S is a compound Poisson process modeling aggregate claims. The drift c, with a positive safety loading $\theta>0$, is the collected premium rate.

One of the advantages of considering a general Levy risk model like (1) is that we can use the tools and methods of the fluctuation theory of Levy processes to bring new insight into the ruin problem. Indeed, the model in (1) contains the classical Cramer-Lundberg model as a particular case allowing for a more comprehensive understanding of the ruin problem. Moreover, the process in (1) models the aggregate claims as one single object unlike the Cramer-Lundberg case where the aggregate claims is seen as having two sources of randomness.

Xuemiao Hao and Qihe Tang (2009)[1] had studied the asymptotic ruin probabilities of the Levy insurance model under periodic taxation. when the Levy measure under periodic taxation and reinsurance are subexponential tails, convolution-equivalent tails and exponential-like tails, we obtain the asymptotic expression of ruin probability in insurance company. Zbigniew and Bert zwart [2] give the expression of the supremum of the renewal process with incremental when the tail behavior of distrubution is subexponential. In this paper ,We continue to study the character of Levy measure when the tail behavior of distrubution is subexponential on the basis of [1] and [2], then we give precise of the ruin probability and the asymptotic estimates of the tail the behavior of the distributionf the supremum of Levy process. More Levy risk model studied in [3, 4, 5, 6].

One of the main objects of interest in ruin theory is the ruin time τ, representing the first passage time of U_t below zero when $U_0 = u$, the relation of type:

$$\tau = \inf\{t>0 : U_t < 0\},$$

where we set $\tau=\infty$, if $U_t \ge 0$ for all $t \ge 0$.

The expression of ruin probability in model (1) is

$$\psi(u) = P(\tau<\infty | U_0 = u), u \ge 0 .$$

We define the running infimum and the running supremum of a given Levy process X_t

$$\underline{X}_t = \inf_{0\le s\le t} X_s \text{ and } \overline{X}_t = \sup_{0\le s\le t} X_s ,$$

Consider the model (1), the following quantities characterize the first downward passage of U_t below zero and contain information on the ruin event that could be relevant in risk management applications:

- the ruin time, τ
- the deficit at ruin, $-X_\tau$
- the surplus immediately prior to ruin, $X_{\tau-}$
- the last minimum surplus level before ruin, $\underline{X}_{\tau-}$

All of these quantities encapsulate relevant knowledge about the ruin event. Even more valuable information can be found in the distribution of the deficit at ruin. H.U. Gerber and E.S.W. Shiu (1998)[7] studied the ruin event in the compound Poisson case by analyzing the joint law of the ruin time τ, the deficit at ruin $-X_\tau$, the surplus immediately prior to ruin $X_{\tau-}$ and the last minimum surplus level before ruin $\underline{X}_{\tau-}$ of the expected discounted penalty function. E. Biffis and A. E. Kyprianou (2010)[8] studied the expected discounted penalty function by analyzing the joint law of the ruin time τ, the deficit at ruin $-X_\tau$, the surplus immediately prior to ruin $X_{\tau-}$. Follow this ideal, we set out to study the following two generalized expected discounted penalty function for the model (1).

Definition 1 The expected discounted penalty function for the model (1)

(i) Finite time

$$\Phi_t(x,\delta) = E_x[e^{-\delta\tau}\omega(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) I_{\{\tau<t\}} | U_0 = u],$$

(ii) Infinite time

$$\Phi(x,\delta) = E_x[e^{-\delta\tau}\omega(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) I_{\{\tau<\infty\}} | U_0 = u].$$

where $\delta(\delta>0)$ is the force of interest, $\omega(\cdot,\cdot,\cdot)$ is the bounded measure function on R_+^3, satisfied $\omega(0,0,0)=\omega_0>0$. $\Phi_t(x,\delta)\to\Phi(x,\delta)$ if $t\to\infty$. $\omega(x_1,x_2,x_3)\equiv 1$ and $\delta=0$. In the presence of the expected discounted penalty function represent the ruin probability in finite time and the ultimate ruin probability. Notice that the value at (y,z,u) of infinite-time and finite-time versions of the joint distribution of the deficit at ruin, the surplus before ruin and the last minimum before ruin can be written as $F_x(y,z,u)$ and $F_{x,t}(y,z,u)$.

Indeed, when $\delta=0$ and ω is an indicator function

$$\omega(x_1,x_2,x_3) = I_{[0,y)}(x_1) I_{[0,z)}(x_2) I_{[0,u)}(x_3),$$

we have $\Phi(x,\delta)=F_x(y,z,u)$ and $\Phi_t(x,\delta)=F_{x,t}(y,z,u)$.

The rest of this paper consists of two sections. After listing some preliminaries on Levy processes and several important distribution classes in section 2, we present our main results in section 3 , that is the asymptotic equation of ruin probability and the asymptotic estimate of the tail condition of its supremum related to Levy processes as the Levy measure belongs to subexponential tail.

II PRELIMINARIES

Let $X=(X_t,t\ge 0)$ be a stochastic process defined on a probability space (Ω,F,P). We say that it has independent increments if for each $n\in N$ and each $0\le t_1<t_2<\cdots<t_{n+1}<\infty$, the random variables $(X(t_{j+1})-X(t_j),1\le j\le n)$ are independent and they have stationary increments if each

$$X(t_{j+1})-X(t_j)\overset{d}{=}X(t_{j+1}-t_j)-X(0).$$

Define 2 We say that X is a Levy process if

(i) Each $X(0)=0$ $(a.s.)$,

(ii) X has independent and stationary increments,

(iii) X has cadlad paths, that is, the trajectories are right-continuous with left limits,

(iv) X is stochastically continuous, i.e. for all $a>0$ and for all $s\ge 0$,

$$\lim_{t\to s}P(|X(t)-X(s)|>a)=0.$$

Note that in the presence of (i)、(ii)、(iii) and (iv) is equivalent to the condition

$$\lim_{t\to 0}P(|X(t)|>a)=0.$$

For a Levy process $X=(X_t)_{t\ge 0}$, $(X_0=0)$ is defined on (Ω,F,P), the laplace exponent given by its characteristic function: $Ee^{-i\theta X_t}=e^{t\Phi(\theta)}$, $\theta\in\Theta$.

$$\Phi(\theta)=ia\theta+\frac{1}{2}\sigma^2\theta^2+\int_{(-\infty,\ \infty)}(1-e^{i\theta x}+i\theta xI_{(|x|\le 1)})\rho(dx)$$
(2)

$\Phi(\theta)$ is the laplace exponent, where $a\in(-\infty,\infty)$, $\sigma^2\ge 0$, Levy measuer $\rho\in(0,\infty)$ satisfied $\rho(\{0\})=0$ and $\int_0^1 x^2\rho(dx)<\infty$, $\int_1^\infty \rho(dx)<\infty$. The triplet (a,σ^2,ρ) (called Levy triplet) uniquely determines the distribution of the Levy process X.

Throughout this paper, for a Levy measure ρ and a distribution F on $(0,\infty)$, denote $\overline{\rho}(x)=\rho((x,\infty))$ and $\overline{F}(x)=1-F(x)$ for any $x\ge 0$. When $\overline{\rho}(1)>0$, introduce $\Pi(\cdot)=(\overline{\rho}(1))^{-1}\rho(\cdot)I_{(1,\infty)}$, which is a proper probability measure on $(1,\infty)$.

Hereafter, all limit relationships are according to $x\to\infty$
unless otherwise stated, and for two positive functions $a(\cdot)$ and $b(\cdot)$, write $a(\cdot)\sim b(\cdot)$ if $a(\cdot)/b(\cdot)\to 1$.

If $\overline{F}(x)>0$ for all $x\ge 0$ and relation

$$\lim_{x\to\infty}\frac{\overline{F}(x+y)}{\overline{F}(x)}=e^{-\alpha y},$$

holds, then for $\alpha\ge 0$, the distribution F defined on $(-\infty,\infty)$ belongs to $L(\alpha)$. Furthermore, a distribution F on $(0,\infty)$ is said to belong to the class $L(\alpha)$ for some $\alpha\ge 0$ if $F\in L(\alpha)$ and the limit

$$\lim_{x\to\infty}\frac{\overline{F^{2*}}(x)}{\overline{F}(x)}=2c,$$

exists and is finite, then for $\alpha\ge 0$, the distribution F defined on $(-\infty,\infty)$ belongs to $S(\alpha)$.

Where F^{2*} denotes the 2-fold distribution convolution of F, i.e., $F^{2*}(x)=\int_{0-}^{x}F(x-y)F(dy)$ for all $x\ge 0$. For later use, we write $F^{1*}(x)=F$ and $F^{n*}(x)=F^{(n-1)*}*F$ for $n=2,3,\ldots$. It is known that the constant c in the above formula is equal to $\int_{0-}^{x}e^{\alpha x}F(dx)$; In the literature, a distribution F in $L(\alpha)$ with $\alpha>0$ is usually said to have an exponential-like tail,and F in $S(\alpha)$ with $\alpha>0$ is said to have a convolution-equivalent tail.

Xuemiao Hao and Qihe Tang (2009)[1], the maximal net loss and the net loss of the company within the period $(n-1,n]$ are , respectively,

$$Y_n=\sup_{n-1\le t\le n}(L_t-L_{n-1}),\ Z_n=L_n-L_{n-1},$$

$$X_n=Z_n+\gamma Z_n^- -\delta Z_n^+=(1-\delta)Z_n^+-(1-\gamma)Z_n^-, \quad (3)$$

The random pairs (X_n,Y_n) can be written by

$$(X,Y)\overset{D}{=}((1-\delta)L_n^+-(1-\gamma)L_n^-,\sup_{0\le t\le 1}L_t),$$
(4)

here δ, γ satisfy $\delta\in[0,1)$ and $\gamma\in[0,1)$, such that

$$EX=(1-\delta)\mu_n^+-(1-\gamma)\mu_n^-<0,$$

where we shall assume that the Levy measure ρ of the Levy process L in the model has a tail $\overline{\rho}$ asymptotically equivalent to a convolution-equivalent tail. This is a natural assumption when studying the tail probability of Levy processes.

when δ and γ are both 0, that is to say, the taxation or reinsurance is not exist in Levy process, then all the results for model (1) under Levy process are still hold.

Theorem 1 of Palmowski and Zwart (2007)[2]: Let random pairs (X_n,Y_n), $n=1,2,\ldots$, be i.i.d. copies of a random pair (X,Y), denote $M=X\vee Y$. If $-\infty<EX<0$,

$E|M|<\infty$, and $\int_x^{\infty} P(M>y)dy$ is asymptotically equivalent to a subexponential tail, then

$$P(\sup_{n\geq 1}(\sum_{k=1}^{n-1} X_k + Y_n) > x) \sim \frac{1}{|EX|}\int_x^{\infty} P(M>y)dy. \quad (5)$$

III THE MAIN CONCLUSION AND PROOF

Corollary 1 Consider the model we discussed in (1), if Π and Π_I belong to class $S(\alpha)$ and $EX<0$ holds, then

$$\psi(u) = P(\tau<\infty | U_0 = u) \sim \frac{1}{|E(X_1)|}\int_x^{\infty} P(Y>y)dy.$$

Proof: A random pairs (X_n, Y_n), $n=1,2,\cdots$ is i.i.d. copies of a random pair (X,Y) given in (4). According to the theorem 1 Palmowski and Zwart (2007)[2] $M = X \vee Y$. Since $\Pi \in S$, It's clear that $Y \geq L_1^+ \geq X^+$ and $\Pi_I \in S$,

$$\int_x^{\infty} P(M>Y)dy = \int_x^{\infty} P(Y>y)dy,$$

Hence by (5)

$$P(\sup_{n\geq 1}(\sum_{k=1}^{n-1} X_k + Y_n) > x) \sim \frac{1}{|EX|}\int_x^{\infty} P(M>y)dy.$$

That is to say

$$\psi(u) = P(\tau<\infty | U_0 = u) = P(\sup_{n\geq 1}(\sum_{k=1}^{n-1} X_k + Y_n) > x) \sim \frac{1}{|E(X_1)|}\int_x^{\infty} P(Y>y)dy$$

Then the conclution proved.

Let $y>0$ be given and construct the random walk S_n^y, $n>0$. Set $S_0^y = 0$. For $k\geq 1$, set $X_k^y = X_k$ if $M \leq y$ and $X_k^y = M$ if $M>y$. Finally, let $S_n^y = X_1^y + \cdots + X_n^y$, $n\geq 1$. Informally, the increments of the random walk S_n^y, $n>0$, are those of S_n, $n>0$, except when a large value of M occurs.

Theorem 1 Consider the model we discussed in (1), let random pairs (X_n, Y_n), $n=1,2,\cdots$ is i.i.d. copies of a random pair (X,Y), denote $M = X\vee Y$, when $-\infty < EX < 0$, $E|M|<\infty$ and $\int_x^{\infty} P(M>y)dy$ is asymptotically equivalent to a subexponential tail, then we can obtain the asymptotic upper bound of the random walk

$$P\{\sup S_n^y > x\} \sim \frac{1}{-E\{X_1^y\}}\int_x^{\infty} P\{M>y\}dy.$$

Proof: Obviously, for any $y>0$, $n\geq 1$, we have that $S_n \leq S_n^y$. Moreover

$$\sup_{n\geq 1}[S_{n-1} + Y_n] \leq \sup_{n\geq 0} S_n^y + y,$$

For $x>y$, we have $P\{X_k^y > x\} = P\{M>x\}$, which implies that the integrated tail of X_k^y is subexponential. Thus, we can apply Veraverbeke theorem (see e.g. [9] or [10]), yielding

$$P\{\sup X_n^y > x\} \sim \frac{1}{-E\{X_1^y\}}\int_x^{\infty} P\{M>u\}du.$$

End.

According to the type we can obtain

$$\limsup_{x\to\infty}\frac{P\{M>x\}}{\int_x^{\infty} P\{M>u\}du} \leq \limsup_{x\to\infty}\frac{P\{\sup_{n\geq 1} S_n^y > x-y\}}{\int_x^{\infty} P\{M>u\}du} \leq \frac{1}{-E\{X_1^y\}}$$

By dominated convergence, we have that

$$-E\{X_1^y\} \to \mu \text{ as } y\to\infty.$$

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