Pseudo-BL Algebras and PD-posets

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Abstract

Pseudo-difference posets (for short, PD-posets) have been introduced as a new quantum logic structure. In this paper, the connections between PD-posets and pseudo-BL algebras are investigated.

Keywords: PD-poset, pseudo-BL algebra.

1. Introduction

In order to establish mathematical foundations of quantum mechanics, in 1936, Birkhoff and von Neumann [1] proposed a quantum logic, i.e., an algebraic system describing a propositional system of quantum mechanics. Since then the quantum logic has been extensively investigated by many researchers. In 2001, Dvurecenskij and Vetterlein [3] introduced a quantum structure called the pseudo-effect algebras. Recently, Ma et al. [11], from another point of view, introduced a equivalent quantum structure called PD-posets as a noncommutative extension of D-posets [6].

The notion of BL algebras were introduced by Hajek [5] as algebraic structure for his well-known Basic Logic. In order to generalize BL-algebras in a noncommutative form, Di Nola [2] introduced the notion of pseudo-BL algebras.

In [7], we investigated the connections between PD-posets and pseudo-BCK algebras. In this paper, as a continuation, we investigate the connections between PD-posets and pseudo-BL algebras. We show that a pseudo-BL algebra is a PD-poset iff $x^{-\sim} = x^{\sim -} = x$. Using this result, we obtain some related corollaries.

2. Preliminaries

A *PD-poset* [11] is a poset $(PD; \leq, \ominus_l, \ominus_r, 0, 1)$ with a maximum element 1 and a minimum element 0 and two partial binary operations $\ominus_l$ and $\ominus_r$ satisfying the following axioms:

(PD1) $b \ominus_l a$ is defined iff $b \ominus_r a$ is defined iff $a \leq b$.

(PD2) $b \ominus_l a \leq b, b \ominus_r a \leq b$.

(PD3) $a \leq b$ implies $b \ominus_l (b \ominus_r a) = a$ and $b \ominus_r (b \ominus_l a) = a$.

(PD4) $a \leq b \leq c$ implies $c \ominus_l b \leq c \ominus_l a$ and $c \ominus_r b \leq c \ominus_r a$.

(PD5) $a \leq b \leq c$ implies $(c \ominus_l a) \ominus_r (c \ominus_l b) = b \ominus_l a$ and $(c \ominus_r a) \ominus_l (c \ominus_r b) = b \ominus_r a$.

(PD6) If $1 \ominus_r ((1 \ominus_l b) \ominus_l a)$ is defined, then there exist $d, e \in PD$ such that

$$1\ominus_r((1\ominus_l b)\ominus_l a) = 1\ominus_r((1\ominus_l a)\ominus_l d) = 1\ominus_r((1\ominus_l e)\ominus_l b).$$

If $1 \ominus_l ((1 \ominus_r b) \ominus_r a)$ is defined, then there exist $f, g \in PD$ such that

$$1\ominus_l((1\ominus_r b)\ominus_r a) = 1\ominus_l((1\ominus_r a)\ominus_r f) = 1\ominus_l((1\ominus_r g)\ominus_r b).$$

In our previous paper [8], we have showed that (PD6) in the definition of PD-posets is not independent, since it follows from (PD1)-(PD5). That is we have following Lemma:

Lemma 2.1 [8]. A poset $(PD; \leq, \ominus_l, \ominus_r, 0, 1)$ with a maximum element 1 and a minimum element 0, and two partial binary operations $\ominus_l$ and $\ominus_r$ is a PD-poset iff the following hold:

(PD1) $b \ominus_l a$ is defined iff $b \ominus_r a$ is defined iff $a \leq b$.

(PD2) $b \ominus_l a \leq b, b \ominus_r a \leq b$.

(PD3) $a \leq b$ implies $b \ominus_l (b \ominus_r a) = a$ and $b \ominus_r (b \ominus_l a) = a$.

(PD4) $a \leq b \leq c$ implies $c \ominus_l b \leq c \ominus_l a$ and $c \ominus_r b \leq c \ominus_r a$.

(PD5) $a \leq b \leq c$ implies $(c \ominus_l a) \ominus_r (c \ominus_l b) = b \ominus_l a$ and $(c \ominus_r a) \ominus_l (c \ominus_r b) = b \ominus_r a$.

Let $(PD; \leq, \ominus_l, \ominus_r, 0, 1)$ be a PD-poset and $a \in PD$. Denote by $a^- = 1 \ominus_l a$ and $a^\sim = 1 \ominus_r a$.

3. Connections with pseudo-BL algebras

A *pseudo-BL algebra* (or call a *right-pseudo-BL algebra*) [2] is a structure $(A; \vee, \wedge, \oplus, \rightarrow, \hookrightarrow, 0, 1)$ of type (2,2,2,2,2,0,0), which satisfies the following axioms, for all $x, y, z \in A$:

(RC1) $(A; \vee, \wedge, 0, 1)$ is a bounded lattice,

(RC2) $(A; \oplus, 0)$ is a monoid ($\oplus$ is associative and $x \oplus 0 = 0 \oplus x = x$),

(RC3) $z \leq x \oplus y$ iff $y \rightarrow z \leq x$ iff $x \hookrightarrow z \leq y$,

(RC4) $x \vee y = (x \rightarrow y) \oplus x = x \oplus (x \hookrightarrow y)$,

(RC5) $(x \rightarrow y) \wedge (y \rightarrow x) = (x \hookrightarrow y) \wedge (y \hookrightarrow x) = 0$.

In a pseudo-BL algebra $(A; \vee, \wedge, \oplus, \rightarrow, \hookrightarrow, 0, 1)$, define two negations: $x^- = x \rightarrow 1, x^\sim = x \hookrightarrow 1$ for all $x \in A$.

Lemma 3.1 [4]. In a pseudo-BL algebra $(A; \vee, \wedge, \oplus, \rightarrow, \hookrightarrow, 0, 1)$, the following hold:

(i) $x \leq y$ iff $y \rightarrow x = 0$ iff $y \hookrightarrow x = 0$.

(ii) $(x \oplus y)^- = x \rightarrow y^-, (x \oplus y)^\sim = y \hookrightarrow x^\sim$.

(iii) $x^{-\sim} \leq x, x^{\sim -} \leq x, x^{-\sim -} = x^-, x^{\sim -\sim} = x^\sim$.

(iv) $x \leq y \Rightarrow y^- \leq x^-, y^\sim \leq x^\sim$.

(v) $(x \wedge y)^- = x^- \vee y^-, (x \wedge y)^\sim = x^\sim \vee y^\sim, (x \vee y)^- = x^- \wedge y^-, (x \vee y)^\sim = x^\sim \wedge y^\sim$.

Next, we add some properties of pseudo-BL algebras which are needed in the sequel.

Lemma 3.2. In a pseudo-BL algebra $(A; \vee, \wedge, \oplus, \rightarrow, \hookrightarrow, 0, 1)$, the following hold:

(i) $x \hookrightarrow (x \oplus y) \leq y \leq x \oplus (x \hookrightarrow y), y \hookrightarrow (y \oplus x) \leq x \leq x \oplus (x \hookrightarrow y)$.

(ii) $y \rightarrow (x \oplus y) \leq x \leq (x \rightarrow y) \oplus x, x \rightarrow (y \oplus x) \leq y \leq (x \rightarrow y) \oplus x$.

(iii) If $x \leq y$, then $x \oplus z \leq y \oplus z, z \oplus x \leq z \oplus y$.

(iv) If $x \leq y$, then $z \hookrightarrow x \leq z \hookrightarrow y, z \rightarrow x \leq z \rightarrow y$.

(v) If $x \leq y$, then $y \hookrightarrow z \leq x \hookrightarrow z, y \rightarrow z \leq x \rightarrow z$.

(vi) $0 \rightarrow x = 0 \hookrightarrow x = x$.

Proof. (i) By (RC4), $x, y \leq x \vee y = x \oplus (x \hookrightarrow y)$. By (RC3), $x \hookrightarrow (x \oplus y) \leq y$ iff $x \oplus y \leq x \oplus y$ and $y \hookrightarrow (y \oplus x) \leq x$ iff $y \oplus x \leq y \oplus x$.

(ii) Similarly.

(iii) By (ii), $z \rightarrow (x \oplus z) \leq x \leq y$, and by (RC3), we have $x \oplus z \leq y \oplus z$. By (i), $z \hookrightarrow (z \oplus x) \leq x \leq y$, and by (RC3), we have $z \oplus x \leq z \oplus y$.

(iv) By (i), $x \leq y \leq z \oplus (z \hookrightarrow y)$, and by (RC3), we have $z \hookrightarrow x \leq z \hookrightarrow y$. By (ii), $x \leq y \leq (z \rightarrow y) \oplus z$, and by (RC3), we have $z \rightarrow x \leq z \rightarrow y$.

(v) If $x \leq y$, by (iii) and (i), $z \leq x \oplus (x \hookrightarrow z) \leq y \oplus (x \hookrightarrow z)$, and by (RC3), we have $y \hookrightarrow z \leq x \hookrightarrow z$. Similarly, if $x \leq y$, by (iii) and (ii), $z \leq (x \rightarrow z) \oplus x \leq (x \rightarrow z) \oplus y$, and by (RC3), we have $y \rightarrow z \leq x \rightarrow z$.

(vi) By (RC4) and (RC2), we have $x = 0 \vee x = (0 \rightarrow x) \oplus 0 = 0 \rightarrow x$ and $x = 0 \vee x = 0 \oplus (0 \hookrightarrow x) = 0 \hookrightarrow x$.

Lemma 3.3 [4]. Let $(A; \vee, \wedge, \oplus, \rightarrow, \hookrightarrow, 0, 1)$ be a pseudo-BL algebra with the property $x^{-\sim} = x^{\sim -} = x$ for all $x \in A$. Define

$$y \odot x = (x^- \oplus y^-)^\sim (= (x^\sim \oplus y^\sim)^- = x^\sim \rightarrow y = y^- \hookrightarrow x).$$

Then the following hold:

(i) $y \oplus x = (x^- \odot y^-)^\sim = (x^\sim \odot y^\sim)^-$.

(ii) $x \vee y = x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$.

(iii) $x \wedge y = x \odot (x^- \oplus y) = y \odot (y^- \oplus x) = (x \oplus y^\sim) \odot y = (y \oplus x^\sim) \odot x$.

Now, we investigate the connections between pseudo-BL algebras and PD-posets.

Theorem 3.4. Let $(A; \vee, \wedge, \oplus, \rightarrow, \hookrightarrow, 0, 1)$ be a pseudo-BL algebra. Define two partial binary operations $\ominus_l$ and $\ominus_r$ on A by: $y \ominus_l x$ is defined iff $y \ominus_r x$ is defined iff $x \leq y$, and in this case

$$y \ominus_l x = x \rightarrow y \quad \text{and} \quad y \ominus_r x = x \hookrightarrow y.$$

Then $(A; \leq, \ominus_l, \ominus_r, 0, 1)$ is a PD-poset iff $x^{-\sim} = x^{\sim -} = x$ for all $x \in A$.

Proof. Suppose that $(A; \leq, \ominus_l, \ominus_r, 0, 1)$ is a PD-poset. For all $x \in A$, by (PD3), we have

$$x^{-\sim} = (x \rightarrow 1) \hookrightarrow 1 = 1 \ominus_r (1 \ominus_l x) = x,$$
$$x^{\sim -} = (x \hookrightarrow 1) \rightarrow 1 = 1 \ominus_l (1 \ominus_r x) = x.$$

Conversely, suppose that $x^{-\sim} = x^{\sim -} = x$ for all $x \in A$.

(i) Since $0 \leq x$, by Lemma 3.2 (v) and (vi), we have $x \hookrightarrow y \leq 0 \hookrightarrow y$ and $x \rightarrow y \leq 0 \rightarrow y$, and so $x \hookrightarrow y \leq y$ and $x \rightarrow y \leq y$. Hence

$$y \ominus_l x = x \rightarrow y \leq y, y \ominus_r x = x \hookrightarrow y \leq y.$$

(ii) By Lemma 3.1 (ii) and Lemma 3.3, we have

$$x \hookrightarrow y = (y^- \oplus x)^\sim = x^\sim \odot y, x \rightarrow y = y \odot x^- = (x \oplus y^\sim)^-.$$

Hence, if $x \leq y$, by Lemma 3.3 (iii), we have

$$\begin{aligned} y \ominus_l (y \ominus_r x) &= (x \hookrightarrow y) \rightarrow y \\ &= (y^- \oplus x)^\sim \rightarrow y = y \odot (y^- \oplus x) \\ &= x \wedge y = x, \end{aligned}$$

and

$$\begin{aligned} y \ominus_r (y \ominus_l x) &= (x \rightarrow y) \hookrightarrow y \\ &= (x \oplus y^\sim)^- \hookrightarrow y = (x \oplus y^\sim) \odot y \\ &= x \wedge y = x. \end{aligned}$$

(iii) If $x \leq y \leq z$, by Lemma 3.2 (v), we have

$$z \ominus_l y = y \to z \leq x \to z = z \ominus_l x,$$
$$z \ominus_r y = y \hookrightarrow z \leq x \hookrightarrow z = z \ominus_r x.$$

(iv) From (ii), $x \hookrightarrow y = x^{\sim} \odot y, x \to y = y \odot x^{-}$. If $x \leq y \leq z$, by Lemma 3.3, we have

$$(z \ominus_l x) \ominus_r (z \ominus_l y) = (y \to z) \hookrightarrow (x \to z)$$
$$= (z \odot y^{-}) \hookrightarrow (z \odot x^{-}) = (z \odot y^{-})^{\sim} \odot (z \odot x^{-})$$
$$= (y \oplus z^{\sim}) \odot (z \odot x^{-}) = ((y \oplus z^{\sim}) \odot z) \odot x^{-}$$
$$= (y \wedge z) \odot x^{-} = y \odot x^{-}$$
$$= x \to y = y \ominus_l x,$$

and

$$(z \ominus_r x) \ominus_l (z \ominus_r y) = (y \hookrightarrow z) \to (x \hookrightarrow z)$$
$$= (y^{\sim} \odot z) \to (x^{\sim} \odot z) = (x^{\sim} \odot z) \odot (y^{\sim} \odot z)^{-}$$
$$= (x^{\sim} \odot z) \odot (z^{-} \oplus y) = x^{\sim} \odot (z \odot (z^{-} \oplus y))$$
$$= x^{\sim} \odot (z \wedge y) = x^{\sim} \odot y$$
$$= x \hookrightarrow y = y \ominus_r x.$$

Hence $(A; \leq, \ominus_l, \ominus_r, 0, 1)$ is a PD-poset as Lemma 2.1. The proof is complete.

In a bounded poset, a chain with upper bound 1 and lower bound 0 is called a *standard chain*. Note that a bounded poset is the union of standard chains.

Lemma 3.5 [7]. In a PD-poset, every standard chain is a lattice-ordered, bounded pseudo-BCK algebra.

Lemma 3.6 [3]. A lattice-ordered, bounded pseudo-BCK algebra coincides with a pseudo-BL algebra with the property $x^{-\sim} = x^{\sim -} = x$, and coincides with a pseudo-MV algebra.

Combining Theorem 3.4, Lemmas 3.5 and 3.6, we have the following:

Corollary 3.7. In a PD-poset, every standard chain is a pseudo-BL algebra with the property $x^{-\sim} = x^{\sim -} = x$ and a pseudo-MV algebra, respectively.

Lemma 3.8 [7]. If a PD-poset $(PD; \leq, \ominus_l, \ominus_r, 0, 1)$ is a lower semilattice, define two binary operations $\star$ and $\circ$ on PD by

$$x \star y = x \ominus_l (x \wedge y),$$
$$x \circ y = x \ominus_r (x \wedge y).$$

Then $(PD; \leq, \star, \circ, 0, 1)$ is a lattice-ordered, bounded pseudo-BCK algebra iff $(x \star y) \circ z = (x \circ z) \star y$ holds for all $x, y, z \in PD$.

Combining Theorem 3.4, Lemma 3.6 and 3.8, we have the following:

Corollary 3.9. If a PD-poset $(PD; \leq, \ominus_l, \ominus_r, 0, 1)$ is a lower semilattice, let $\star$ and $\circ$ be two binary operations on PD defined by $x \star y = x \ominus_l (x \wedge y), x \circ y = x \ominus_r (x \wedge y)$. Then the following are equivalent:

(i) $(x \star y) \circ z = (x \circ z) \star y$ holds for all $x, y, z \in PD$.

(ii) PD is a pseudo-BL algebra with the property $x^{-\sim} = x^{\sim -} = x$.

(iii) PD is a pseudo-MV algebra.

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