Linguistic Truth-Valued Concept Lattice Based on Lattice-Valued Logic

Li Yang Yang Xu

Intelligent Control Development Center, Southwest Jiaotong University, Chengdu 610031, P.R. China

Abstract

The theory of concept lattice is approached from the view of formal logic. In this paper, we present a new relationship between concept lattice and lattice-valued logic for dealing with uncertainty linguistic information conveniently. We give the definition of $\delta -$ degree linguistic truth-valued concept lattice and linguistic many-valued context according to the concrete linguistic information and establish linguistic truth-valued context with the help of lattice implication algebra. On the basis of this, we discuss the completeness of the set of $\delta -$ degree concepts in accordance with the defined Galois connection and do further research on the properties of $\delta -$ degree linguistic truth-valued concept lattice.

Keywords: Concept lattice, Linguistic truth-valued logic, Galois connection, $\delta -$ degree concept

1. Introduction

Concept lattice introduced by Rudolf Wille [1, 2] has been a topic of interest for about two decades. As a powerful formal tool for data organization and data analysis, concept lattice is very important both in theories and actual applications, and has been widely and successfully used in many fields. Bivalent-logic-based concept lattice originates from single-valued context, and with the development of concept lattice (reference [4]), many researchers have discussed extended concept lattice originated from many-valued context. However, there exist two shortcomings in extended concept lattice: (1) only with the help of bivalent logic, many-valued context must be transformed into single-valued context, that is to say, transforming extended concept lattice into classical concept lattice for dealing with relevant problems; (2) the type of many-valued context mainly involves pure numerical values, which strain the variety of information, especially linguistic information. As everyone knows, in intelligent information age, linguistic information process is absolutely necessary. Therefore, this paper proposes the theory of lattice-valued-logic-based concept lattice in order to resolve comparable and incomparable linguistic information on the basis of reference [5]. In Section 2, we give an overview of bivalent-logic-based concept lattice and lattice implication algebra. In Section 3, we introduce the notions of linguistic many-valued context and linguistic truth-valued context as an extension and development of classical formal context and thus present a transformation method between them. Mathematical properties of lattice-valued-logic-based concept lattice are discussed in Section 4, where we show that the set of α degree concepts in a given universe forms a complete lattice. Concluding remarks are presented in Section 5.

2. Preliminaries

2.1 Bivalent-logic-based concept lattice

In Bivalent-logic-based concept lattice theory, the single-valued context is defined as a set structure (G, M, I) consisting of sets G and M and a binary relation $I \subseteq G \times M$. The elements of G and M are called objects and attributes, respectively, and the relationship gIm is read: the object g has the attribute m.

For a set of objects $A \subseteq G$, A^* is defined as the set of features shared by all the objects in A, that is, $A^* = \{m \in M \mid gIm \ \forall g \in A\}$. Similarly, for $B \subseteq M$, B^* is defined as the set of objects that posses all the features in B, that is, $B^* = \{g \in G \mid gIm \ \forall m \in B\}$. Thus we establishes a Galois connection between the power sets of G and M [7, 8]. A formal concept of the context (G, M, I) is defined as a pair (A, B) with $A \subseteq G$, $B \subseteq M$ and $A^* = B$, $B^* = A$. The set A is called the extent and B the intent of the concept (A, B), and the interested readers can be referred to [1, 2].

In classical concept lattice, for every attribute, we only care whether it has value or not. However, in the process of data analysis, there are more situations about the relation between objects and attributes which can not easily use the relation "object has or doesn't has some attributes" for describing.

2.2 Lattice implication algebra

Definition 1[3] Let $(L,\wedge,\vee,O,I)$ be a bounded lattice with an order-reversing involution $'$, I and O the greatest and the smallest element of L respectively, and $\rightarrow: L\times L\rightarrow L$ a mapping. If the following conditions hold for any x, y, z$\in L$:

(I_1) $x\rightarrow(y\rightarrow z)=y\rightarrow(x\rightarrow z)$

(I_2) $x\rightarrow x=I$

(I_3) $x\rightarrow y=y'\rightarrow x'$

(I_4) $x\rightarrow y=y\rightarrow x=I$ implies $x=y$

(I_5) $(x\rightarrow y)\rightarrow y=(y\rightarrow x)\rightarrow x$

(l_1) $(x\vee y)\rightarrow z=(x\rightarrow z)\wedge(y\rightarrow z)$

(l_2) $(x\wedge y)\rightarrow z=(x\rightarrow z)\vee(y\rightarrow z)$

then $(L,\wedge,\vee,',\rightarrow,O,I)$ is called a lattice implication algebra.

Definition 2[3] Let $(L_\alpha,\vee_\alpha,\wedge_\alpha,'_{(\alpha)},\rightarrow_\alpha,O_\alpha,I_\alpha)$ $(\alpha\in J)$ be a family lattice implication algebras, where J is an index set. Define

$$A=\prod\{L_\alpha \mid \alpha\in J\}$$
$$=\left\{f \,\middle|\, f:J\rightarrow\bigcup_{\alpha\in J}L_\alpha, s.t, \forall\alpha\in J, f(\alpha)\in L_\alpha\right\}$$

For any $f,g\in A,\alpha\in J$, binary operations $\vee,\wedge,\rightarrow$ and an unary operation ´ on A are defined as follows:

$$(f\vee g)(\alpha)=f(\alpha)\vee_\alpha g(\alpha),$$
$$(f\wedge g)(\alpha)=f(\alpha)\wedge_\alpha g(\alpha),$$
$$f'(\alpha)=(f(\alpha))'^{(\alpha)},$$
$$(f\rightarrow g)(\alpha)=f(\alpha)\rightarrow_\alpha g(\alpha),$$

and $O(\alpha)=O_\alpha, I(\alpha)=I_\alpha$. It can be proved that these operations are well defined and A is lattice implication algebra. It is called the direct product or lattice implication product algebra L_α $(\alpha\in J)$. Specially, if $|J|$=2, then $A=L_1\times L_2$ is a lattice implication product algebra of L_1 and L_2.

3. Linguistic truth-valued context

In many actual applications, for example, evaluating the synthesize quality of students and the property of cars and so on, people are fond of directly using such linguistic forms as "best", "better" or "bad", etc.

Definition 3 A linguistic many-valued context (G,M,W,I), where G is the set of objects, M is the set of attributes, $W=\bigcup W_m$ is the set of linguistic values and W_m is a domain of attribute m, I is a relation between G and M, i.e., $I:G\times M\rightarrow W$, such that for any $g\in G$, $m\in M$, there is at most one linguistic value $w\in W$ satisfying $I(g,m)=w$.

For example, a linguistic many-valued context (G,M,W,I) will be shown in table 1, which G ={ g_1, g_2, g_3, g_4}, M ={ m_1, m_2, m_3, m_4}, W_{m_j} ={ bad, rather good, very good, best} ∪ { less, little, much, more} ∪ {a little small, bigger, biggest} ∪ {weightiest, weighty, light, lighter}.

	m_1	m_2	m_3	m_4
g_1	best	less	a little small	weightiest
g_2	very good	much	smallest	weighty
g_3	rather good	more	bigger	light
g_4	bad	little	biggest	lighter

Table 1: A linguistic many-valued context.

According to different linguistic information, the corresponding linguistic many-valued context will be constructed. But the established linguistic many-valued context will not provide possibility for seeking formal concepts and structure analysis of concept lattice because of the complexity of linguistic information. Therefore, it is necessary to research some characteristics of the linguistic many-valued context and find solutions for this problem.

In linguistic many-valued context (G,M,W,I), for physical dimensional attributes, we should combine the set of attributes with the linguistic assessment variables appropriately.

Definition 4 For the set of attributes M in linguistic many-valued context (G, M, W, I), $M^{[V,\overline{V}]}$ is called the set of attributes with linguistic assessment variables, if it satisfies the following conditions:

(1) the set of linguistic assessment variables

$[V,\overline{V}]$ □ $\{[v_1,\overline{v}_1],[v_2,\overline{v}_2],\cdots,[v_n,\overline{v}_n]\}$ and

V={v_1,v_2,…,v_n};

(2) there exists a defined operation ´ : $v'_j=\overline{v}_j$, $\overline{v}'_j=v_j$ $(j=1,2,\cdots,n)$ s.t., $\overline{V}=\{\overline{v}_1,\overline{v}_2,\cdots,\overline{v}_n\}$, where every $[v,\overline{v}]$ is a linguistic assessment variable-pair with the practical meanings on the linguistic

truth-valued set [True, False].

For the transformation from M to $M^{[V,\bar{V}]}$, the set of attribute values W should also be made suitably transformation according to the practical situation.

Definition 5 The lattice implication algebra $L_{n\times 2}$ whose implication operator is well defined is called a linguistic truth-valued lattice implication algebra generated by L_n and L_2, if L_n and L_2 satisfy the following conditions:

(1) the set of modifiers L_n ={a_1, a_2, . . . , a_n}is called the lattice implication algebra with modifiers denoted by L_n which is defined on the chain $a_1 < a_2 < \cdots < a_n$ if its implication is Lukasiewicz implication;

(2) the set of meta truth values MT={True (Tr for short), false(Fa for short)}={b_1, b_2} is called a meta linguistic truth-valued lattice implication algebra (of course a Boolean algebra) defined on the chain Fa<Tr, if its operation "′ " is defined as: Tr′ =Fa and Fa′ =Tr, the operation "→" is defined as

$$\to: MT\times MT\to MT$$
$$x\to y = x'\vee y$$

Definition 6 A linguistic truth-valued context $\left(G, M^{[V,\bar{V}]}, L_{n\times 2}, \hat{I}\right)$, where G is the set of objects, $M^{[V,\bar{V}]}$ is the set of attributes with linguistic assessment variables, $L_{n\times 2}$ is a linguistic truth-valued lattice implication algebra, $\hat{I}$ is a relation between G and $M^{[V,\bar{V}]}$, i.e, $\hat{I}: G\times M^{[V,\bar{V}]}\to L_{n\times 2}$, such that for any $g\in G$, $m^{[v,\bar{v}]}\in M^{[V,\bar{V}]}$, there is at most one linguistic truth value $\left(a_i, b_j\right)\in L_{n\times 2}$ ($a_i\in L_n$, $b_j\in L_2$) satisfying $\hat{I}\left(g,m\right)=\left(a_i,b_j\right)$.

We can sum up the main points of the above definitions as follows:

Step 1: analyzing the characteristics of the linguistic information and selecting a linguistic truth-valued implication algebra $L_{n\times 2}=L_n\times L_2$;

Step 2: providing the set $M^{[V,\bar{V}]}$ relevant to the set M in linguistic many-valued context (G,M,W,I) by definition 4;

Step 3: establishing the transformation method according to the relation between the sets W and $L_{n\times 2}$;

Step 4: obtaining a concrete linguistic truth-valued context by definition 6.

According to reference [6], we will select a suitable and concrete linguistic truth-valued lattice implication algebra $L_{5\times 2}=L_5\times L_2$, whose Hasse diagram through proper regulation shown as Fig 1 to continue the above example. In $L_{5\times 2}=L_5\times L_2$, L_5={Slightly (Sl for short}, Rather (Ra), Exactly (Ex), Very (Ve), Absolutely (Ab)}={a_1, a_2, . . . , a_5} which is called as the set of modifiers, and let I=(a_5, b_2), A=(a_4, b_2), B=(a_3, b_2), C=(a_2, b_2), D=(a_1, b_2), E=(a_1, b_1), F=(a_2, b_1), G=(a_3, b_1), H=(a_4, b_1), O= (a_5, b_1)

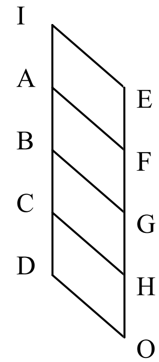

Fig. 1: Hasse Diagram of $L_{5\times 2}$.

Then we will get the following table 2:

	$m_1^{[\text{good,bad}]}$	$m_2^{[\text{much,little}]}$	$m_3^{[\text{big,small}]}$	$m_4^{[\text{light,weighty}]}$
g_1	I	H	E	O
g_2	A	B	O	G
g_3	C	A	A	B
g_4	G	G	I	A

Table 2: An linguistic truth-valued context.

4. Linguistic truth-valued concept lattice

Definition 7 In a linguistic truth-valued context $\left(G, M^{[V,\bar{V}]}, L_{n\times 2}, \hat{I}\right)$, for a set of objects $A\subseteq G$, a set of attributes $B\subseteq M^{[V,\bar{V}]}$, $\delta\in L_n$. Denote

$$A^{\Delta}=\left\{m^{[v,\bar{v}]}\in M^{[V,\bar{V}]}\middle|\forall g\in A, \hat{I}\left(g,m^{[v,\bar{v}]}\right)\geq\delta\right\}$$
$$B^{\Delta}=\left\{g\in G\middle|\forall m^{[v,\bar{v}]}\in B, \hat{I}\left(g,m^{[v,\bar{v}]}\right)\geq\delta\right\}$$

Theorem 1 The operation (Δ,Δ) in the definition is a Galois connection between the power sets of G and $M^{[V,\bar{V}]}$.

Proof. $A\subseteq B^{\Delta}$

$$\Leftrightarrow\ \forall g\in A\subseteq G\ ,\ \forall m^{[v,\bar{v}]}\in B\ ,\ \hat{I}\left(g,m^{[v,\bar{v}]}\right)\geq\delta$$
$$\Leftrightarrow\ \forall m^{[v,\bar{v}]}\in B\ ,\ \forall g\in A\subseteq G\ ,\ \hat{I}\left(g,m^{[v,\bar{v}]}\right)\geq\delta$$
$$\Leftrightarrow\ B\subseteq A^{\Delta}$$

Theorem 2 Let $\left(G, M^{[V,\bar{V}]}, L_{n\times 2}, \hat{I}\right)$ be a linguistic truth-valued context, and $A_1, A_2 \subseteq G$, $B_1, B_2 \subseteq M^{[V,\bar{V}]}$, then

(1) $A \subseteq A^{\Delta\Delta}$, $B \subseteq B^{\Delta\Delta}$;

(2) $A_1 \subseteq A_2 \Rightarrow A_2^{\Delta} \subseteq A_1^{\Delta}$, $B_1 \subseteq B_2 \Rightarrow B_2^{\Delta} \subseteq B_1^{\Delta}$;

(3) $A^{\Delta} = A^{\Delta\Delta\Delta}$, $B^{\Delta} = B^{\Delta\Delta\Delta}$;

(4) $\left(A_1 \cup A_2\right)^{\Delta} = A_1^{\Delta} \cap A_2^{\Delta}$, $\left(B_1 \cup B_2\right)^{\Delta} = B_1^{\Delta} \cap B_2^{\Delta}$.

Remark For every different $\delta \in L_n$, "Δ" can be denoted as "Δ_δ".

Definition 8 Let $\left(G, M^{[V,\bar{V}]}, L_{n\times 2}, \hat{I}\right)$ be a linguistic truth-valued context, and $A \subseteq G$, $B \subseteq M^{[V,\bar{V}]}$, if $A = B^{\Delta}$, $B = A^{\Delta}$, then $(A,B)^{\delta}$ is called a α degree concept. The set A and set B are called extent and intent of $(A,B)^{\delta}$, respectively.

Theorem 3 Let $\left(G, M^{[V,\bar{V}]}, L_{n\times 2}, \hat{I}\right)$ be a linguistic truth-valued context, denote

$L^{\delta}\left(G, M^{[V,\bar{V}]}, L_{n\times 2}, \hat{I}\right) = \left\{(A,B)^{\delta} \middle| A = B^{\Delta_\delta}, B = A^{\Delta_\delta}\right\}$, if $\left(A_1, B_1\right)^{\delta} \le \left(A_2, B_2\right)^{\delta} \Leftrightarrow A_1 \subseteq A_2$, then

$L^{\delta}\left(G, M^{[V,\bar{V}]}, L_{n\times 2}, \hat{I}\right)$ is a δ-degree linguistic truth-valued complete lattice, where the operations on

L^{δ} are

$$\underset{j\in J}{\wedge}\left(A_j, B_j\right)^{\delta} = \left(\bigcap_{j\in J} A_j, \left(\bigcup_{j\in J} B_j\right)^{\Delta_\delta\Delta_\delta}\right)^{\delta},$$

$$\underset{j\in J}{\vee}\left(A_j, B_j\right)^{\delta} = \left(\left(\bigcup_{j\in J} A_j\right)^{\Delta_\delta\Delta_\delta}, \bigcap_{j\in J} B_j\right)^{\delta}$$

Proof. $\forall j \in J$, $\left(A_j, B_j\right)^{\delta} \in L^{\delta}\left(G, M^{[V,\bar{V}]}, L_{n\times 2}, \hat{I}\right)$,

by theorem 2, $\bigcap_{j\in J} A_j \subseteq \left(\bigcap_{j\in J} A_j\right)^{\Delta_\delta\Delta_\delta}$, for $k \in J$,

$$\bigcap_{j\in J} A_j \subseteq A_k \Rightarrow \left(\bigcap_{j\in J} A_j\right)^{\Delta_\delta\Delta_\delta} \subseteq A_k^{\Delta_\delta\Delta_\delta} = A_k$$

$$\Rightarrow \left(\bigcap_{j\in J} A_j\right)^{\Delta_\delta\Delta_\delta} \subseteq \bigcap_{k} A_k \subseteq \bigcap_{j\in J} A_j$$, then

$$\left(\bigcap_{j\in J} A_j\right)^{\Delta_\delta\Delta_\delta} = \bigcap_{j\in J} A_j$$

$$\left(\bigcap_{j\in J} A_j\right)^{\Delta_\delta} = \left(\bigcap_{j\in J} A_j^{\Delta_\delta\Delta_\delta}\right)^{\Delta_\delta} = \left(\bigcup_{j\in J} A_j^{\Delta_\delta}\right)^{\Delta_\delta\Delta_\delta} = \left(\bigcup_{j\in J} B_j\right)^{\Delta_\delta\Delta_\delta}$$ so

$$\left(\bigcap_{j\in J} A_j, \left(\bigcup_{j\in J} B_j\right)^{\Delta_\delta\Delta_\delta}\right)^{\delta} = \left(\bigcap_{j\in J} A_j, \left(\bigcap_{j\in J} A_j\right)^{\Delta_\delta}\right)^{\delta}$$

$\in L^{\delta}\left(G, M^{[V,\bar{V}]}, L_{n\times 2}, \hat{I}\right)$, and $\left(\bigcap_{j\in J} A_j, \left(\bigcup_{j\in J} B_j\right)^{\Delta_\delta\Delta_\delta}\right)^{\delta}$ is an infimum of $\left\{\left(A_j, B_j\right)^{\delta} \mid j \in J\right\}$. Let $(A,B)^{\delta}$ be any infimum of this set, then $A \subseteq A_j \ (j \in J) \Rightarrow A \subseteq \bigcap_{j\in J} A_j$,

so, $(A,B)^{\delta} \le \left(\bigcap_{j\in J} A_j, \left(\bigcup_{j\in J} B_j\right)^{\Delta_\delta\Delta_\delta}\right)^{\delta}$, i.e.,

$$\underset{j\in J}{\wedge}\left(A_j, B_j\right)^{\delta} = \left(\bigcap_{j\in J} A_j, \left(\bigcup_{j\in J} B_j\right)^{\Delta_\delta\Delta_\delta}\right)^{\delta}.$$

$\underset{j\in J}{\vee}\left(A_j, B_j\right)^{\delta} = \left(\left(\bigcup_{j\in J} A_j\right)^{\Delta_\delta\Delta_\delta}, \bigcap_{j\in J} B_j\right)^{\delta}$ can be proved similarly.

Theorem 4 Let $L^{\delta}\left(G, M^{[V,\bar{V}]}, L_{n\times 2}, \hat{I}\right)$ a δ-degree concept lattice, $\forall \beta \le \delta \in L_n$ and $A \subseteq G$, $B \subseteq M^{[V,\bar{V}]}$, then $A^{\Delta_\delta} \subseteq A^{\Delta_\beta}$, $B^{\Delta_\delta} \subseteq B^{\Delta_\beta}$.

Proof. By definition 7

$$A^{\Delta_\delta} = \left\{m^{[v,\bar{v}]} \in M^{[V,\bar{V}]} \middle| \forall g \in A, \hat{I}\left(g, m^{[v,\bar{v}]}\right) \ge \delta\right\}$$

$\forall \beta \le \delta \in L_n$ then we have

$$A^{\Delta_\delta} = \left\{m^{[v,\bar{v}]} \in M^{[V,\bar{V}]} \middle| \forall g \in A, \hat{I}\left(g, m^{[v,\bar{v}]}\right) \ge \delta \ge \beta\right\},$$

so $A^{\Delta_\delta} \subseteq A^{\Delta_\beta}$, $B^{\Delta_\delta} \subseteq B^{\Delta_\beta}$ can be proved similarly.

5. Conclusions

This paper bridges the gap between concept lattice and lattice-valued logic. For selecting a suitable mathematical tool to deal with both comparable and incomparable linguistic terms, we constructed lattice-valued-based concept lattice based on the definitions of linguistic many-valued context and linguistic truth-valued context, and investigated its properties under the Galois connection. In the course of the discussion, we explained the key of this approach by means of examples. Obviously, we can research the constructing algorithms of lattice-valued-based concept lattice, which will be solved in further discussing.

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References

[1] B. Ganter and R. Wille, *Formal Concept Analysis: Mathematical Foundations,* Springer, Berlin, Heidelberg, 1999.

[2] G. Birkhoff, *Lattice Theory*, in: American Mathematical Society Colloquium Publications XXV, Vol.25, American Mathematical Society, Providence, RI, 1967.

[3] Y. Xu, D. Ruan, K.Y. Oin and J. Liu, *Lattice-Valued Logic-An Alternative Approach to Treat Fuzziness and Incomparability*. Springer-Verlag, Berlin. 2003.

[4] W.X. Zhang, Y.Y. Yao and Y. Liang, *Rough Set and Concept Lattice*, Xi'an Jiaotong University Press, Xi'an. 2006.

[5] L. Yang and Z. M. Song, Formal concept analysis of many-valued context based on the interaction among objects, *2006 International Conference on Intelligent Systems and Knowledge Engineering (ISKE2006),* April 6-7, 2006, Shanghai, China.

[6] Y. Xu, S.W. Chen and J. Ma, Linguistic truth-valued lattice implication algebra and its properties, *Conference on Computational Engineering in Systems Applications(CESA)*, October 4-6, 2006, Beijing, China.

[7] R. Belohlavek, *A note on variable threshold concept lattices: Threshold-based operators are reducible to classical concept-forming operators,* Information Sciences 177:3186-3191, 2007.

[8] R. Belohlavek, J. Dvorak and J. Outrata, Fast factorization by similarity in formal concept analysis of data with fuzzy attributes, *Journal of Computer and System Sciences*, 73:1012-1022, 2007.