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Tauberian Theorems In Quantum Calculus

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Abstract

In this paper we attempt to establish some tauberian theorems in quantum calculus. This constitutes the beginning of the study of the q-analogue of analytic theory of numbers which is the aim of a forthcoming paper.

1 Introduction , notation and preliminaries

We begin with recalling some historical notions that we need to know before studying their q-analogues. For further information see the nice book of D.V. Widder [9]. The paper will give q-analogues of most of the results in sections $2-4$ of Chapter 8 of this book.

We say that a series

$$\sum_{k=0}^{\infty} a_k \tag{1.1}$$

is summable (A), in the Abel sense, to the value S if the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ converges for $|x| < 1$ and $\lim_{x\to 1^-} f(x) = S$.

Abel [1826], proved that if the series (1.1) converges to S then it is summable (A) to the value S. Nevertheless when we take $a_k = (-1)^k$ it is easy to show that the converse statement is not true.

Tauber [1897] proved that by the additional condition

$$na_n = o(1) \tag{1.2}$$

the converse holds.

Later J. E. Littlewood [1910], proved that if $\sum a_n$ is summable (A) to S and

$$na_n = O(1) \tag{1.3}$$

then $\sum a_n$ converges to S.

Nonnegativity of the coefficients a_n implies that, if $\sum a_n$ is summable (A) to S then $\sum a_n$ converges to S.

We recall that a series (1.1) is said to be summable (C), in the Cesaro sense, to S if

$$\lim_{n\to\infty} \frac{s_0 + s_1 + \cdots + s_n}{n+1} = S \tag{1.4}$$

where we have put $s_k = \sum_{i=0}^{k} a_i$.

For this type of convergence the converse statement is not true. But G. H. Hardy [1910] showed that the converse is valid under the tauberian condition (1.3).

The tauberian theorems given by Widder [9]are textbook versions of results obtained earlier by others. They are analogues for integrals of the theorems for series discussed above.

In this paper, our aim is to study a tauberian theorems in Quantum Calculus and we establish some results which they will be used as in analytic number theory.

To this end and in order to make the paper more self contained we begin with giving some usual notions used in the q-theory. Throughout this paper, we will fix $q \in]0,1[$ that $\frac{Log(1-q)}{Logq} \in \mathbb{Z}$ and we adapt the notations of Gasper-Rahman's book [4].

For $a \in \mathbb{C}$, the q-shifted factorials are defined by

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n = 1,2,\ldots; \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{1.5}$$

We also denote

$$(a_1, a_2, \ldots, a_p; q)_n = (a_1;q)_n (a_2;q)_n \ldots (a_p;q)_n, \qquad n = 0,1,2,3,\ldots\infty, \tag{1.6}$$

$$[x]_q = \frac{1-q^x}{1-q}, \quad x \in \mathbb{C} \quad \text{and} \quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}, \qquad n \in \mathbb{N}. \tag{1.7}$$

The q-derivative $D_q f$ of a function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad \text{if} \quad x \neq 0, \tag{1.8}$$

$(D_q f)(0) = f'(0)$ provided $f'(0)$ exists. If f is differentiable then $(D_q f)(x)$ tends to $f'(x)$ as q tends to 1.

The q-Jackson integrals from 0 to a and from 0 to ∞ are defined by (see [7])

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n, \tag{1.9}$$

$$\int_0^\infty f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n, \tag{1.10}$$

provided the sums converge absolutely.
The q-Jackson integral in a generic interval $[a, b]$ is given by (see [7])

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx. \tag{1.11}$$

The improper integral is defined in the following way (see [2])

$$\int_0^{\frac{\infty}{A}} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A}. \tag{1.12}$$

We remark that for $n \in \mathbb{Z}$, we have

$$\int_0^{\frac{\infty}{q^n}} f(x)d_qx = \int_0^{\infty} f(x)d_qx. \tag{1.13}$$

A q-analogue of the integration by parts formula is given by

$$\int_0^a f(x)(D_qg(x))d_qx = f(a)g(a) - \lim_{M\to\infty} f(aq^M)g(aq^M) - \int_0^a (D_qf(x))g(x)d_qx. \tag{1.14}$$

Jackson [7] defined a q-analogue of the Gamma function by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad x \neq 0, -1, -2, \dots. \tag{1.15}$$

It is well known that it satisfies

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x), \quad \Gamma_q(1) = 1 \quad \text{and} \quad \lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x), \quad \Re(x) > 0. \tag{1.16}$$

In [6], the authors proved that

$$\forall A > 0, \quad \Gamma_q(s) = K(A,s)\int_0^{\frac{\infty}{A(1-q)}} x^{s-1}e_q^{-x}d_qx, \tag{1.17}$$

where

$$K(A,s) = \frac{A^s}{1+A}\frac{(-\frac{1}{A},q)_\infty(-A,q)_\infty}{(-\frac{1}{A}q^s,q)_\infty(-Aq^{1-s},q)_\infty}, \tag{1.18}$$

and

$$e_q^x = \frac{1}{((1-q)x;q)_\infty}. \tag{1.19}$$

In particular, for $A = \dfrac{1}{1-q}$, we have

$$\Gamma_q(s) = K_q(s)\int_0^{\infty} x^{s-1}e_q^{-x}d_qx, \tag{1.20}$$

where

$$K_q(t) = K(\frac{1}{1-q}, t) \tag{1.21}$$

The q-cosine function is given by (see [8])

$$\cos(x; q2) = \sum_{0}^{\infty} \frac{(-1)^n q^{n(n-1)} x^{2n}}{[2n]_q!} \tag{1.22}$$

We denote by

$$\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\} \cup \{0\}, \quad \mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\} \quad \text{and} \quad \widetilde{\mathbb{R}}_{q,+} = \mathbb{R}_{q,+} \cup \{0\}. \tag{1.23}$$

A function f is said q-integrable on $[0, \infty[$ if the series $\sum_{n\in\mathbb{Z}} q^n f(q^n)$ converges absolutely.
We write $L1(\mathbb{R}_{q,+})$ the set of all functions that are q-integrable on $[0, \infty[$.

The q-Mellin transform of a suitable function f on $\mathbb{R}_{q,+}$ is given by (see [3])

$$M_q(f)(s) = \int_0^{\infty} t^{s-1} f(t) d_q t. \tag{1.24}$$

We denote by $\langle \alpha_{q,f}, \beta_{q,f} \rangle$ the largest open vertical strip, called fundamental strip, such that the integral(1.24) converges for s in that strip.
The inversion formula for the q-Mellin transform is given by[3]

$$\forall x \in \mathbb{R}_{q,+}, \quad f(x) = \frac{Log(q)}{2i\pi(1-q)} \int_{c-\frac{i\pi}{Log(q)}}^{c+\frac{i\pi}{Log(q)}} M_q(f)(s) x^{-s} ds, \tag{1.25}$$

where $c \in]\alpha_{q,f}, \beta_{q,f}[$.
The q-Mellin convolution product of suitable functions f and g is defined by [3]

$$f *_M g(x) = \int_0^{\infty} f(y) g\left(\frac{x}{y}\right) \frac{d_q y}{y}, \qquad x \in \mathbb{R}_{q,+}. \tag{1.26}$$

One has the following relation

$$M_q[f *_M g] = M_q(f) M_q(g). \tag{1.27}$$

2 Tauberian theorems

In the remainder $a(t)$ is a function always satisfying the condition

$$\forall R \in \mathbb{R}_{q,+}, \quad \int_0^R \mid a(t) \mid d_q t < \infty. \tag{2.1}$$

Theorem 1. *Let $a(t)$ be a function defined on $(0, +\infty)$ and put*

$$f(x) = \int_0^\infty e_q^{-xt} a(t) d_q t, \qquad 0 < x < \infty. \tag{2.2}$$

If

$$a(t) \sim At^\gamma, \quad t \to \infty, \qquad \gamma > -1 \tag{2.3}$$

then

$$f(x) \sim \frac{A\Gamma_q(\gamma+1)}{x^{\gamma+1} K_q(\gamma+1)}, \quad \text{as} \quad x \to 0^+, \quad x \in \mathbb{R}_{q,+}, \tag{2.4}$$

where the function $K_q(s)$ is given by (1.21).

Proof. By (1.17)we have

$$\int_0^\infty e_q^{-xt} t^\gamma d_q t = \frac{\Gamma_q(\gamma+1)}{x^{\gamma+1} K_q(\gamma+1)}, \qquad x \in \mathbb{R}_{q,+}. \tag{2.5}$$

By considering the function $a(t) - At^\gamma$, we see that we can suppose that $A = 0$ and $a(t) = o(t^\gamma)$ as $t \to \infty$.
Let $\varepsilon > 0$, we have $a(t) = o(t^\gamma)$ as $t \to \infty$, it follows that there exists $p_0 \in \mathbb{N}$ such that for all $p \leq -p_0$, we have

$$\mid a(q^p) \mid < \varepsilon q^{p\gamma}.$$

So, for all $x \in \mathbb{R}_{q,+}$,

$$\begin{aligned}
\mid f(x) \mid &= \mid (1-q) \sum_{n=-\infty}^{\infty} e_q^{-xq^n} a(q^n) q^n \mid \\
&\leq (1-q) \sum_{n=-\infty}^{-p_0} e_q^{-xq^n} \mid a(q^n) \mid q^n + (1-q) \sum_{n=-p_0+1}^{\infty} e_q^{-xq^n} \mid a(q^n) \mid q^n \\
&\leq \varepsilon (1-q) \sum_{n=-\infty}^{-p_0} e_q^{-xq^n} q^{n\gamma} q^n + (1-q) \sum_{n=-p_0+1}^{\infty} \mid a(q^n) \mid q^n \\
&\leq \varepsilon \frac{\Gamma_q(\gamma+1)}{x^{\gamma+1} K_q(\gamma+1)} + (1-q) \sum_{n=-p_0+1}^{\infty} \mid a(q^n) \mid q^n.
\end{aligned}$$

Thus

$$x^{\gamma+1} \mid f(x) \mid \leq \varepsilon \frac{\Gamma_q(\gamma+1)}{K_q(\gamma+1)} + x^{\gamma+1}(1-q) \sum_{n=-p_0+1}^{\infty} \mid a(q^n) \mid q^n. \tag{2.6}$$

From the relation (2.1), we have

$$\sum_{n=-p_0+1}^{\infty} \mid a(q^n) \mid q^n < \infty. \tag{2.7}$$

Therefore

$$\lim_{x\to 0} x^{\gamma+1}(1-q)\sum_{n=-p_0+1}^{\infty} \mid a(q^n) \mid q^n = 0. \tag{2.8}$$

Which implies that

$$f(x) = o(\frac{1}{x^{\gamma+1}}), \quad \text{as} \quad x \to 0^+. \tag{2.9}$$

■

Definition 1. 1. We say that the q-integral

$$\int_0^\infty a(t)d_qt \tag{2.10}$$

is summable (A) to the value S if the q-Laplace integral

$$f(x) = \int_0^\infty e_q^{-xt}a(t)d_qt \tag{2.11}$$

converges for $x > 0$ and $\lim_{x\to 0^+} f(x) = S$.

2. We say that the q-integral (2.10) is summable (C) if

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x A(t)d_qt = S, \tag{2.12}$$

where we have put

$$A(x) = \int_0^x a(t)d_qt.$$

The following result is the regularity theorem.

Theorem 2. *Let $a(t)$ be a function defined on $(0, +\infty)$ and for $0 < x < \infty$, we put*

$$f(x) = \int_0^\infty e_q^{-xt}a(t)d_qt. \tag{2.13}$$

If

$$\int_0^\infty a(t)d_qt = S,$$

then

$$\lim_{x\to 0^+} f(x) = S. \tag{2.14}$$

Proof. For $x \geq 0$ and $t \in \mathbb{R}_{q,+}$, we have

$$|e_q^{-xt}a(t)| \leq |a(t)|. \tag{2.15}$$

According to the Lebesgue theorem, we obtain

$$\lim_{x\to 0^+}\int_0^\infty e_q^{-xt}a(t)d_qt = \int_0^\infty a(t)d_qt. \tag{2.16}$$

■

We deduce that the convergence of (2.10) implies its summability to the same value. Conversely, we have the following result:

Theorem 3. *Suppose that*

1. $f(x) = \int_0^\infty e_q^{-xt} a(t) d_q t, \qquad 0 < x < \infty;$
2. $\lim_{x \to 0^+} f(x) = S;$
3. $a(t) \geq 0.$

Then

$$\int_0^\infty a(t) d_q t = S. \tag{2.17}$$

Proof. According to the monotone convergence theorem from Lebesgue integration theory, we have

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \int_0^\infty e_q^{-xt} a(t) d_q t \tag{2.18}$$

$$= \int_0^\infty \lim_{x \to 0^+} e_q^{-xt} a(t) d_q t = \int_0^\infty a(t) d_q t. \tag{2.19}$$

■

Remarks.1- The hypothesis (1) of Theorem 3 is one of the two q-Laplace transforms introduced in §9 by W. Hahn [5], and further elaborated by W. H. Abdi [1].
2- The hypothesis (3) of Theorem 3 is essential and cannot be omitted. To confirm we consider the function $a(t)$ defined on $\mathbb{R}_{q,+}$ by

$$a(q^n) = \begin{cases} 0 & \text{if} \quad n \geq 0 \\ \frac{(-1)^n}{q^n} & \text{if} \quad n < 0 \end{cases}, \tag{2.20}$$

and put $f(x) = \int_0^\infty e_q^{-x^2 t} a(t) d_q t, \qquad x \in \mathbb{R}_{q,+}.$
We have

$$f(x) = \int_1^\infty e_q^{-x^2 t} a(t) d_q t. \tag{2.21}$$

By q-integration by parts, we obtain

$$f(x) = x2 \int_1^\infty e_q^{-x^2 t} A(qt) d_q t, \tag{2.22}$$

where

$$A(t) = \int_0^t a(u) d_q u. \tag{2.23}$$

Then,

$$f(x) = x2(1-q)\sum_{-\infty}^{-1} e_q^{-x^2q^n} A(q^{n+1})q^n. \tag{2.24}$$

On the other hand, for $t = q^p, \quad p < 0$

$$\begin{aligned} A(t) &= \int_0^t a(u)d_qu = \int_1^t a(u)d_qu \\ &= (1-q)\sum_{n=p}^{-1} a(q^n)q^n = (1-q)\sum_{n=p}^{-1}(-1)^n. \end{aligned}$$

So,

$$A(q^{2p}) = 0, \quad \text{et} \quad \mathrm{A(q^{2p+1}) = -(1-q)}, \qquad \mathrm{p < 0.} \tag{2.25}$$

From the relation (2.24), we have

$$f(x) = -(1-q)^2 x2\sum_{-\infty}^{-1} e_q^{-x^2q^{2p}} q^{2p} = -(1-q)\int_x^\infty te_q^{-t2}d_qt. \tag{2.26}$$

We deduce that

$$\lim_{x\to 0^+} f(x) = -(1-q)\int_0^\infty te_q^{-t2}d_qt, \tag{2.27}$$

but the q-integral $\int_0^\infty a(t)d_qt$ diverges.

Theorem 4. *Let $a(t)$ be a function defined on $\mathbb{R}_{q,+}$ and*

$$f(x) = \int_0^{+\infty} e_q^{-xt}a(t)d_qt. \tag{2.28}$$

If

$$\int_0^x a(t)d_qt \sim Ax, \quad \textit{as} \quad x \to \infty, \tag{2.29}$$

then

$$f(x) \sim \frac{A}{x}, \quad \text{as} \quad x \to 0^+, \qquad x \in \mathbb{R}_{q,+}. \tag{2.30}$$

Proof. Put $A(x) = \int_0^x a(t)d_qt$, so $D_qA(x) = a(x)$ and $f(x) = \int_0^\infty e_q^{-xt}D_qA(t)d_qt$. By q-integration by parts, we obtain

$$f(x) = x\int_0^\infty e_q^{-xt}A(qt)d_qt.$$

From the relation (2.29) and the theorem (1) for $\gamma = 1$ and the fact that $\Gamma_q(2) = 1$ and $K_q(2) = q$ we have

$$f(x) \sim \frac{Aq\Gamma_q(2)}{xK_q(2)} = \frac{A}{x}, \quad \text{as} \quad x \to 0^+. \tag{2.31}$$

■

To obtain an adequate converse, we impose some additional condition that is the function $a(t)$ is bounded.

Theorem 5. *Let $a(t)$ be a bounded function on $\mathbb{R}_{q,+}$ and*

$$f(x) = \int_0^{+\infty} e_q^{-xt} a(t) d_q t. \tag{2.32}$$

If

$$f(x) \sim \frac{A}{x}, \qquad x \to 0^+ \tag{2.33}$$

then

$$\int_0^x a(t) d_q t \sim Ax, \qquad x \to +\infty. \tag{2.34}$$

We shall obtain this result in the next section as a special case of a much more general theorem.

3 A Basic Tauberian Theorem: uniqueness property.

Definition 2. A function g has the uniqueness property ($g \in U$) if the two assertions hold

1. $g \in L1(\mathbb{R}_{q,+})$

2. For $x \in \mathbb{R}_{q,+}$ and $a \in B$ (bounded), the equation $\int_0^{+\infty} g(\frac{t}{x}) a(t) d_q t = 0$ implies $a(t) = 0, \quad t \in \mathbb{R}_{q,+}$

Theorem 6. *Given a, g, and h three functions defined on $\mathbb{R}_{q,+}$ satisfying the following conditions:*

1. $g \in U$

2. $a \in B$

3. $h \in L1(\mathbb{R}_{q,+})$

4. $\frac{1}{x} \int_0^{+\infty} g(\frac{t}{x}) a(t) d_q t \longrightarrow A \int_0^{+\infty} g(t) d_q t,$ *as* $x \to +\infty.$

Then

$$\frac{1}{x} \int_0^{+\infty} h(\frac{t}{x}) a(t) d_q t \to A \int_0^{+\infty} h(t) d_q t, \quad as \qquad x \to +\infty, \quad x \in \mathbb{R}_{q,+}.$$

Proof. Since we may consider the function $a(t)-A$ instead of $a(t)$, we can suppose $A=0$.
Put

$$H_1(y) = y\int_0^{+\infty} h(yt)a(t)d_qt$$

We shall to prove that

$$\lim_{y\to 0^+} H_1(y) = 0. \tag{3.1}$$

Assuming the contrary, there must exist $\delta > 0$ and a sequence of $\mathbb{R}_{q,+}$ $(x_n)_n$ tending to 0 such that $\mid H_1(x_n) \mid > \delta$.
Now, consider the sequence:

$$s_n(x) = H_1(x_n x).$$

Since,

$$\mid H_1(x) \mid \leq \int_0^{+\infty} \mid h(t)a(\frac{t}{x}) \mid d_qt \leq M\int_0^{+\infty} \mid h(t) \mid d_qt$$

then H_1 is a bounded function on $\mathbb{R}_{q,+}$.
Let $(b_m)_{m\in\mathbb{N}}$ the sequence defined by: $b_{2m} = q^m$, $\qquad b_{2m+1} = q^{-m-1}$.
From the bounded sequence $(s_n(b_m))$ we may pick, by the familiar diagonal process, a subsequence $s_{n_k}(b_m)$ such that

$$\lim_{k\to+\infty} s_{n_k}(b_m) = s(b_m), \qquad \forall\ m \in \mathbb{N}. \tag{3.2}$$

Thus

$$\lim_{k\to+\infty} s_{n_k}(x) = s(x), \qquad \forall\ x \in \mathbb{R}_{q,+}. \tag{3.3}$$

On the other hand, we have

$$G_1 *_M h_1 = H_1 *_M g_1 = g_1 *_M h_1 *_M a_1. \tag{3.4}$$

where $G_1(y) = y\int_0^{+\infty} g(yt)a(t)d_qt$, $\quad g_1(t) = tg(t)$, $\quad h_1(t) = th(t)$, $\quad a_1(t) = a(\frac{1}{t})$ and $f *_M g$ the q-Mellin convolution product of the functions f and g (see 1.26).
Using the Lebesgue theorem, we obtain

$$\begin{aligned}
\lim_{k\to+\infty}\int_0^{+\infty} g_1(t)s_{n_k}(\frac{y}{t})\frac{d_qt}{t} &= \lim_{k\to+\infty}\int_0^{+\infty} g_1(t)H_1(x_{n_k}\frac{y}{t})\frac{d_qt}{t}\\
&= \lim_{k\to+\infty}\int_0^{+\infty} h_1(t)G_1(x_{n_k}\frac{y}{t})\frac{d_qt}{t}\\
&= \int_0^{+\infty} g_1(t)s(\frac{y}{t})\frac{d_qt}{t}\\
&= \int_0^{+\infty} g(t)s(\frac{y}{t})d_qt.
\end{aligned}$$

From the hypothesis $\lim_{x\to 0^+} G_1(x) = 0$, we have

$$\frac{1}{x}\int_0^{+\infty} g(\frac{t}{x})s_1(t)d_qt = 0, \tag{3.5}$$

where $s_1(t) = s(\frac{1}{t})$.
Since $g \in U$ and s_1 is a bounded function on $\mathbb{R}_{q,+}$, then $s_1 = 0$.
But

$$s_1(1) = s(1) = s(q0) = \lim_{k\to+\infty} s_{n_k}(q0) = \lim_{k\to+\infty} H_1(x_{n_k}) \geq \delta. \tag{3.6}$$

The contradiction shows that $\lim_{y\to 0^+} H_1(y) = 0$.
Therefore,

$$\lim_{x\to+\infty} \frac{1}{x}\int_0^{+\infty} h(\frac{t}{x})a(t)d_qt = 0. \tag{3.7}$$

■

Lemma 1. *Let $a(t)$ be a bounded function such that*

$$\forall x \in \mathbb{R}_{q,+}, \qquad \int_0^{\infty} t^{\alpha-1}e_q^{-xt}a(t)d_qt = 0, \quad \alpha > 0 \tag{3.8}$$

then

$$\forall x \in \mathbb{R}_{q,+}, \qquad \int_0^{\infty} t^{\alpha-1}e_q^{-xt}\cos(\sqrt{t};q2)a(t)d_qt = 0, \tag{3.9}$$

where the q-cosine is given by (1.22).

Proof. We have

$$\begin{aligned}\int_0^{\infty} t^{\alpha-1}e_q^{-xt}\cos(\sqrt{t};q2)a(t)d_qt &= (1-q)\sum_{n=-\infty}^{+\infty} q^{n\alpha}e_q^{-xq^n}\cos(q^{\frac{n}{2}};q2)a(q^n)\\ &= (1-q)\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k-1)}}{[2k]_q!}q^{(k+\alpha)n}e_q^{-xq^n}a(q^n).\end{aligned}$$

For all $x \in \mathbb{R}_{q,+}$, we have

$$|(-1)^k\frac{q^{k(k-1)}}{[2k]_q!}q^{(k+\alpha)n}e_q^{-xq^n}a(q^n)| \leq M\frac{q^{k(k-1)}}{[2k]_q!}q^{(k+\alpha)n}e_q^{-xq^n}. \tag{3.10}$$

On the other hand, we have

$$(1-q)\sum_{n=-\infty}^{+\infty} q^{(k+\alpha)n}e_q^{-xq^n} = \int_0^{\infty} t^{k+\alpha-1}e_q^{-xt}d_qt \tag{3.11}$$

$$= \frac{1}{x^{k+\alpha}}\frac{\Gamma_q(k+\alpha)}{K_q(k+\alpha)}. \tag{3.12}$$

By using the relation(1.16)and the fact that $K_q(s+1) = q^sK_q(s)$, we have

$$(1-q)\sum_{n=-\infty}^{+\infty} q^{(k+\alpha)n}e_q^{-xq^n} = \frac{1}{x^{k+\alpha}}\frac{[k+\alpha-1]_q\dots[\alpha]_q}{q^{k+\alpha-1}\dots q^{\alpha}}\frac{\Gamma_q(\alpha)}{K_q(\alpha)} \tag{3.13}$$

$$= \frac{1}{x^{k+\alpha}}\frac{[k+\alpha-1]_q\dots[\alpha]_q}{q^{\frac{k(k+2\alpha-1)}{2}}}\frac{\Gamma_q(\alpha)}{K_q(\alpha)}. \tag{3.14}$$

Thus

$$\frac{q^{k(k-1)}}{q^{\frac{k(k+2\alpha-1)}{2}}}\frac{1}{x^{k+\alpha}}\frac{[k+\alpha-1]_q\dots[\alpha]_q}{[2k]_q!}=o(q^{\frac{k2}{4}}),\quad as\quad k\to+\infty \tag{3.15}$$

which implies that the double series

$$\sum_{k=0}^{\infty}\sum_{n=-\infty}^{+\infty}|(-1)^k\frac{q^{k(k-1)}}{[2k]_q!}q^{(k+\alpha)n}e_q^{-xq^n}a(q^n)| \tag{3.16}$$

converges.
Therefore,

$$\sum_{n=-\infty}^{+\infty}\sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k-1)}}{[2k]_q!}q^{(k+\alpha)n}e_q^{-xq^n}a(q^n)=\sum_{k=0}^{\infty}\sum_{n=-\infty}^{+\infty}(-1)^k\frac{q^{k(k-1)}}{[2k]_q!}q^{(k+\alpha)n}e_q^{-xq^n}a(q^n), \tag{3.17}$$

and

$$\int_0^{\infty}t^{\alpha-1}e_q^{-xt}\cos(\sqrt{t};q2)a(t)d_qt=\sum_{k=0}^{\infty}(-1)^k\frac{q^{k(k-1)}}{[2k]_q!}\int_0^{\infty}e_q^{-xt}t^{k+\alpha-1}a(t)d_qt. \tag{3.18}$$

On the other hand, for all $x\in\mathbb{R}_{q,+}$

$$\int_0^{\infty}t^{\alpha-1}e_q^{-xt}a(t)d_qt=0, \tag{3.19}$$

then, for all $k\in\mathbb{N}$, $D_q^k\left(\int_0^{\infty}t^{\alpha-1}e_q^{-xt}a(t)d_qt\right)=\int_0^{\infty}(-1)^kt^{k+\alpha-1}e_q^{-xt}a(t)d_qt=0.$
Which completes the proof. ■

Proposition 1. The function $x^{\alpha-1}e_q^{-x}\in U$ on $\mathbb{R}_{q,+}$, $\alpha>0$.

Proof. Let $a(t)$ be a bounded function such that

$$\forall x\in\mathbb{R}_{q,+},\qquad\int_0^{\infty}t^{\alpha-1}e_q^{-xt}a(t)d_qt=0. \tag{3.20}$$

According to lemma 1, we have

$$\forall x\in\mathbb{R}_{q,+},\qquad\int_0^{\infty}t^{\alpha-1}e_q^{-xt}\cos(\sqrt{t};q2)a(t)d_qt=0. \tag{3.21}$$

Thus

$$x^{-1}\int_0^{\infty}e_q^{\frac{-t}{x}}t^{\alpha-1}\cos(\sqrt{t};q2)a(t)d_qt=k*_M a_1(x)=0, \tag{3.22}$$

where $k(t)=t^{-1}e_q^{-\frac{1}{t}}$, $a_1(t)=t^{\alpha-1}\cos(\sqrt{t};q2)a(t)$ and $k*_M a_1(x)$ the q-Mellin convolution product of the functions k and a_1.
Therefore, from (1.27), we obtain

$$M_q(k*_M a_1)=M_q(k)M_q(a_1)=0. \tag{3.23}$$

Moreover,

$$M_q(k)(x) = M_q\left[\frac{1}{t}e_q^{\frac{-1}{t}}\right](x) = M_q\left[e_q^{-t}\right](1-x) = \frac{\Gamma_q(1-x)}{K_q(1-x)} \neq 0, \qquad 0 < \Re(x) < 1. \tag{3.24}$$

Then

$$M_q(a_1) = 0. \tag{3.25}$$

Using the q-Mellin's inversion formula (1.25), we obtain

$$a_1(t) = \cos(\sqrt{t}; q2)a(t) = 0. \tag{3.26}$$

So, there exists $t_0 \in \mathbb{R}_{q,+}$ such that $a(t_0) = 0$.
On the other hand let $y \in \mathbb{R}_{q,+}$, we have

$$\forall x \in \mathbb{R}_{q,+}, \qquad \int_0^\infty t^{\alpha-1} e_q^{-\frac{x}{y}t} a(t) d_qt = 0. \tag{3.27}$$

By making the change of variables $u = \frac{1}{y}t$, we obtain

$$\int_0^\infty t^{\alpha-1} e_q^{-xt} a(yt) d_qt = 0, \tag{3.28}$$

which implies that,

$$a(yt_0) = 0, \tag{3.29}$$

thus

$$a(t) = 0, \quad t \in \mathbb{R}_{q,+}. \tag{3.30}$$

■

By choosing in theorem 6, $g(x) = e_q^{-x} x^{\alpha-1} \quad \alpha > 0$ and $h(x) = x^{\beta-1}$ on $(0,1]$, $h(x) = 0$ on $]1, +\infty[$, $\beta > 0$, we have the following result:

Theorem 7. *Let $a(t)$ be a bounded function on $\mathbb{R}_{q,+}$ and let*

$$f(x) = \int_0^{+\infty} e_q^{-xt} t^{\alpha-1} a(t) d_qt, \qquad \alpha > 0.$$

Suppose

$$f(x) \sim \frac{A\Gamma_q(\alpha)}{x^\alpha K_q(\alpha)} \qquad \text{as} \qquad x \to 0^+,$$

then

$$\int_0^x t^{\beta-1} a(t) d_qt \sim A\frac{x^\beta}{[\beta]_q} \qquad \text{as} \qquad x \to +\infty.$$

In particular, for $\alpha = \beta = 1$, we obtain the theorem (5).

4 q-analogue of Tauber's theorem

Definition 3. A function f is said to be satisfies the property $(\wp)$ if for all $\varepsilon > 0$, there exists $K > 0$ such that

$$\forall x > K, \qquad |f(x) - f(qx)| < \varepsilon.$$

Proposition 2. Let $a(x)$ be a function defined on $(0, \infty)$ and satisfying the property $(\wp)$ such that,

$$\int_0^x a(t) d_q t \sim Ax, \qquad x \to +\infty, \tag{4.1}$$

then

$$\lim_{x \to +\infty} a(x) = A.$$

Proof. We can suppose that $A = 0$.
Let $\varepsilon > 0$, there exists $K > 0$ such that, $\forall x > K$

$$(1-q)x[a(qx) - \varepsilon] \leq (1-q)xa(x) \leq (1-q)x[a(qx) + \varepsilon].$$

Using the relation $\displaystyle\int_{qx}^x a(t) d_q t = (1-q)xa(x)$, we obtain

$$(1-q)x[a(qx) - \varepsilon] \leq \int_{qx}^x a(t) d_q t \leq (1-q)x[a(qx) + \varepsilon].$$

Thus, for all $x > K$,

$$(1-q)[a(qx) - \varepsilon] \leq \frac{1}{x} \int_{qx}^x a(t) d_q t \leq (1-q)[a(qx) + \varepsilon].$$

From the hypothesis (4.1), we obtain

$$\lim_{x \to +\infty} \frac{1}{x} \int_{qx}^x a(t) d_q t = 0,$$

thus,

$$-\varepsilon \leq \underline{\lim} a(x) \leq \overline{\lim} a(x) \leq \varepsilon.$$

Which completes the proof. ■

Theorem 8. *Let $a(t)$ be a function defined on $(0, \infty)$ and let $A(x) = \int_0^x a(t) d_q t$. suppose*

1. $\displaystyle\lim_{x \to +\infty} xa(x) = 0;$
2. $\displaystyle\int_0^x A(t) d_q t \sim Ax,$ *as* $\quad x \to \infty,$

then

$$\int_0^\infty a(t) d_q t = A. \tag{4.2}$$

Proof. We have $A(x)-A(qx)=\displaystyle\int_{qx}^{x}a(t)d_qt=(1-q)xa(x)$. From the hypothesis (1), we deduce that $A(x)$ satisfies the property $(\wp)$. We apply the proposition 2 to the function $A(x)$, which completes the proof.

■

Theorem 9. *Let $a(x)$ be a function defined on $(0,\infty)$and let $f(x)=\int_0^\infty e_q^{-xt}a(t)d_qt$. If*

1. *$xa(x)\in B$ (bounded) ;*
2. *$\lim\limits_{x\to 0^+} f(x)=A$,*

then

$$\int_0^x A(t)d_qt \sim Ax, \qquad x\to\infty. \tag{4.3}$$

Proof. Let $A(t)=\displaystyle\int_0^t a(u)d_qu$.
By q-integration by parts, we obtain

$$\int_0^\infty e_q^{-xt}a(t)d_qt = x\int_0^\infty e_q^{-xt}A(qt)d_qt. \tag{4.4}$$

From the hypothesis(2), we have

$$\int_0^\infty e_q^{-xt}A(qt)d_qt \sim \frac{A}{x}, \qquad x\to 0^+.$$

On the other hand, we have

$$A(x)-f(\frac{1}{x})=\int_0^x a(t)[1-e_q^{-\frac{t}{x}}]d_qt-\int_x^\infty e_q^{-\frac{t}{x}}a(t)d_qt, \qquad x\in\mathbb{R}_{q,+}. \tag{4.5}$$

It follows from the relation $1-e_q^{-x}=\displaystyle\int_0^x e_q^{-t}d_qt$, that $|1-e_q^{-x}|\le x$.
Therefore,

$$\begin{aligned} |A(x)-f(\frac{1}{x})| &\le M\int_0^x \frac{1}{x}d_qt+M\int_x^\infty \frac{e_q^{\frac{-t}{x}}}{t}d_qt && (4.6)\\ &\le M+M\int_1^\infty \frac{e_q^{-t}}{t}d_qt \le 2M && (4.7)\end{aligned}$$

From the hypothesis(2) and the inequality (4.7), we deduce that the functions f and $A(x)$ are a bounded ones.
We apply the theorem(5) to the function $A(qx)$, we obtain

$$\int_0^x A(qt)d_qt \sim Ax, \qquad x\to\infty. \tag{4.8}$$

Finally the changes of variables $u=qt$, gives

$$\int_0^x A(t)d_qt \sim Ax, \qquad x\to+\infty. \tag{4.9}$$

Which completes the proof. ■

Theorem 10. *Let $a(x)$ be a function defined on $(0, \infty)$, and let $f(x) = \int_0^\infty e_q^{-xt} a(t) d_q t$. If*

1. $xa(x) \in B$ *(bounded)* ;
2. $\lim\limits_{x \to +\infty} xa(x) = 0$;
3. $\lim\limits_{x \to 0^+} f(x) = A$,

then

$$\int_0^\infty a(t) d_q t = A$$

Proof. From the hypothesis(1) and the theorem (9), we have

$$\int_0^x A(t) d_q t \sim Ax, \qquad x \to +\infty. \tag{4.10}$$

The result follows from the hypothesis(2), the relation (4.10) and the theorem 8. ■

Remark. We can replace the hypothesis (1)of theorem 8 and the hypothesis (1) and (2) of theorem 10 by

$$\{x\} a(x) \in B, \tag{4.11}$$

where $\{x\} = \dfrac{q^x - q^{-x}}{q - q^{-1}}$, although in fact the resulting statements are weaker than these results.

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