Tetravalent Half-arc-transitive Graphs of Order Pqr

Xiuyun Wang
School of Mathematic Science
University of Jinan
Jinan, Shandong, China
1519168700@qq.com

Abstract—A graph is half-arc-transitive if its automorphism group acts transitively on its vertex set, edge set, but not arc set. Let n be a product of three primes. The problem on classification of the half-arc-transitive graphs of order n has been considered in [J Algebraic Combin 1(1992) 275-282, Discrete Math 310(2010) 1721-1724, European J Combin 28(2007) 726-733], and it was solved for the cases where n is a prime cube or twice a product of two primes. In this paper, we give the classification of the tetravalent half-arc-transitive graphs of order pqr, where p, q, r are distinct odd primes.

***Keywords-** cayley graph; vertex-transitive graph; half-arc-transitive graph; simple group;quotient graph*

I. INTRODUCTION

All graphs considered in this paper are finite, connected, undirected and simple, but with an implicit orientation of the edges when appropriate. Given a graph X, denote by V(X), E(X), A(X) and Aut(X) the vertex set, edge set, arc set and automorphism group of X, respectively. A graph X is said to be vertex-transitive, edge-transitive and arc-transitive(symmetric) if Aut(X) acts transitively on V(X), E(X) and A(X), respectively. The graph X is said to be half-arc-transitive provided that it is vertex- and edge- but not arc-transitive. More generally, by a half-arc-transitive action of a subgroup G of Aut(X) on X we shall mean a vertex- and edge-, but not arc-transitive action of G on X. In this case we say that the graph X is G-half-arc-transitive.

In 1947, Tutte[1] initiated the investigation of half-arc-transitive graphs by showing that a vertex- and edge-transitive graph with odd valency must be arc-transitive, and few years later, Bouwer[2] gave a construction of 2k-valent half-arc-transitive graph for every $k \geq 2$. Following these two classical articles, half-arc-transitive graphs have been exten-sively studied from different perspectives over decades by many authors. (for example, see [3,4,5,6]).

In fact, constructing and characterizing half-arc-transitive graphs with small valencies is currently an active topic in algebraic graph theory. In view of the fact that 4 is the smallest admissible valency for a half-arc-transitive graph, special attention has rightly been given to the study of tetravalent half-arc-transitive graphs. In particular, constructing and classifying the tetravalent half-arc-transitive graphs is currently one of active topics in algebraic graph theory (for example, see [7-9] and [10-13]).

II. PRELIMINARY RESULTS

For the purpose of this paper, we introduce a result due to Marusic.

Let $m\geq 3$ be an integer, $n\geq 3$ an odd integer and let $r\in Z_n^*$ satisfy $r^m = \pm 1$. The graph X(r; m, n) is defined to have vertex set $V=\{u_i^j \mid i\in Z_m, j\in Z_n\}$ and edge set $E=\{\{u_i^j, u_{i+1}^{j+r^i}\} \mid i\in Z_m, j\in Z_n\}$.

A. *Proposition 2.1*

[14, Theorem 3.4] A connected tetravalent graph X is a tightly attached half-arc-transitive graph of odd radius n if and only if $X \cong X(r;m,n)$, where $m\geq 3$, and $r\in Z_n^*$ satisfying $r^m = \pm 1$, and moreover none of the following conditions is fulfilled:

(1) $r^2 = \pm 1$;

(2) (r;m,n) = (2;3,7);

(3) (r;m,n) = (r;6,7k), where $k\geq 1$ is odd, $(7,k) = 1$, $r^6 = 1$, and there exists a unique solution $q\in \{r,-r, r^{-1}, -r^{-1}\}$ of the equation $x^2+x-2=0$ such that $7(q-1)=0$ and $q\equiv 5 \pmod 7$.

Now we state two simple observations about half-arc-transitive graphs.

B. *Proposition 2.2*

[13, Proposition 2.6] Let X be a connected half-arc-transitive graph of valency 2n. Let A=Aut(X) and let A_u be the stabilizer of $u\in V(X)$ in A. Then each prime divisor of $|A_u|$ is a divisor of n!.

C. *Proposition 2.3*

[9, Propositions 2.1 and 2.2] Let X = Cay(G, S) be half-arc-transitive. Then S contains no involutions, and there is no $\alpha\in Aut(G, S)$ such that $s^{\alpha}=s^{-1}$ for some $s\in S$. In particular, there are no half-arc-transitive Cayley graphs on abelian group.

The following propositions are some results about group theory. Check the orders of the non-abelian simple groups, we have the following proposition.

D. *Proposition 2.4*

[15, pp. 12-14, 135-136] Let G be a non-abelian simple group and let p>q>r be odd primes. If |G| has at most three prime divisors then G is isomorphic to one of the following:

A_5,A_6,PSL(2,7),PSL(2,8),PSL(2,17),PSL(3,3),PSU(3,3),PSU(4,2). If $|G| = 2^m pqr$ then $G\cong Sz(8)$, PSL(2,p), or PSL($2,2^t$) with an integer $t \geq 4$.

Let p be a prime and G = PSL(2,p^f). Assume that P is a Sylow p-subgroup of G, A and B are cyclic subgroups of G of order $(p^f-1)/(2_tp^f-1)$ and $(p^f+1)/(2_tp^f-1)$, respectively. It is well known that for any g∈G, P ∩P^g = 1 or P = P^g, A∩A^g = 1 or A=A^g, and B∩B^g =1 or B = B^g. Furthermore, $N_G(A) \cong D_{2|A|}$, $N_G(B) \cong D_{2|B|}$, $N_G(A)$ and $N_G(B)$ are maximal subgroups of G. Then we have the following proposition

E. *Proposition 2.5.*

Let p be a prime and G = PSL(2,p^f). Assume that P is a Sylow psubgroup of G and H is a maximal dihedral subgroup of G. Then for any element g ∈G,P = P^g or P∩P^g = 1, and H = H^g or |H ∩ H g | ≤ 2.

III. CLASSIFICATION

In this section, we determine the classification of tetravalent half-arc-transitive graphs of order pqr. The mainly ideas for the paper comes from two situation which named "Primitive" and "Non-Primitive". Fig .1 showed the idea for the method.

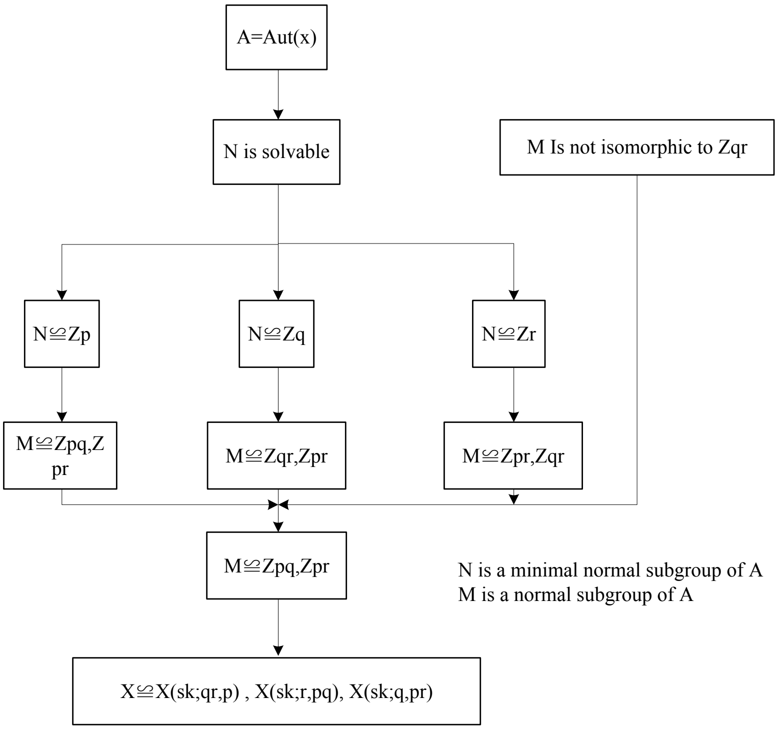

Figure 1. Method idea process

A. *Theorem 3.1*

Let 3≤r<q<p be distinct primes and let X be a connected tetravalent graph of order pqr. Then X is half-arc-transitive if and only if X≅ X(s^k;qr,p), X(s^k;r,pq) or X(s^k;q,pr).

B. *Proof*

Let X be a connected tetravalent half-arc-transitive graph of order pqr. Let A=Aut(X) and u∈V(X). By Proposition 2.2, the stabilizer A_u of u in A is a 2-group. Thus, $|A|=2^m pqr$ for some positive integer m. In particular, 2pqr||A|. Let N be a minimal normal subgroup of A, C = $C_A(N)$ and let M be a normal subgroup of A. Now, we prove the following claims.

C. *Claim I*

A has a solvable minimal normal subgroup. Suppose that all minimal normal subgroups of A are non-solvable. Then N =T^k where T is a non-abelian simple group. Note that |A| = 2mpqr. Thus, k=1 and N is a non-abelian simple group. By Proposition 2.4, N≅A_5, PSL(2,7), Sz(8), PSL(2,p) with p≥11 or PSL(2,2^t) with t≥4. Since C∩N is a normal subgroup of N and N>C, we have C∩N = 1. First suppose that N = A_5 or PSL(2,7). Then C is solvable. By the assumption, we have C = 1. By N/C-theorem, A≅ A/C ≤ Aut(N), a contradiction. Thus, N≅Sz(8), PSL(2,p) or PSL(2,2^t). Furthermore, N is transitive on V(X). It follows that X≅Cos(N,H,HSH) where H = N_α is a Sylow 2-subgroup of N. Since X has valency 4, there exists an element s∈S such that $|H|/|H\cap H^s|=2$. Note that the intersection of any two distinct Sylow 2-subgroups of Sz(8) and PSL(2,2^t) is trivial. Hence, N ≇ Sz(8) or PSL(2,2^t). It follows that N=PSL(2,p) with p≥11. Note that 4||H| and |N|=p(p+1)(p−1)/2. By Proposition 2.5, $|H \cap H^s| \le 2$, implying that |H| = 4. Thus, p(p+1)(p−1)/2= 4pqr, that is (p + 1)(p − 1) = 8qr. Note that 3 ≤ r < q < p and 3 | p^2 − 1. Then r = 3. A simple calculation shows that p = 11 or 13. Then, N = PSL(2,11) or PSL(2,13). Since C∩N = 1, we have that C is 2-group, implying that C=1. Hence, A=PSL(2,11), PGL(2,11), PSL(2,13) or PGL(2,13). Suppose that A = PGL(2,11) or PGL(2,13). Then |H| = 8. By magma, there no exists an element s∈A such that $|H \cap H^s| = 4$, HsH≠Hs^{-1}H and <H，s>=A, a contradiction. Suppose that A=PSL(2,11) or PSL(2,13). Then|H| = 4.

Then there exists an element $s \in A$ such that $|H \cap H^s| = 2$, $<H, s>=A$ and $HsH \neq Hs^{-1}H$. However, by magma, there exists an element $g \in PGL(2,11)$ or $PGL(2,13)$ such that $(HsH)g = Hs^{-1}H$. It follows that X is symmetric, a contradiction. Thus, we may assume that N is solvable. From the "Primitive" situation we could get the complete graph. Fig .2 showed the algorithms for "Non-Primitive" situation. We can get the graphs from four different *lengths*.

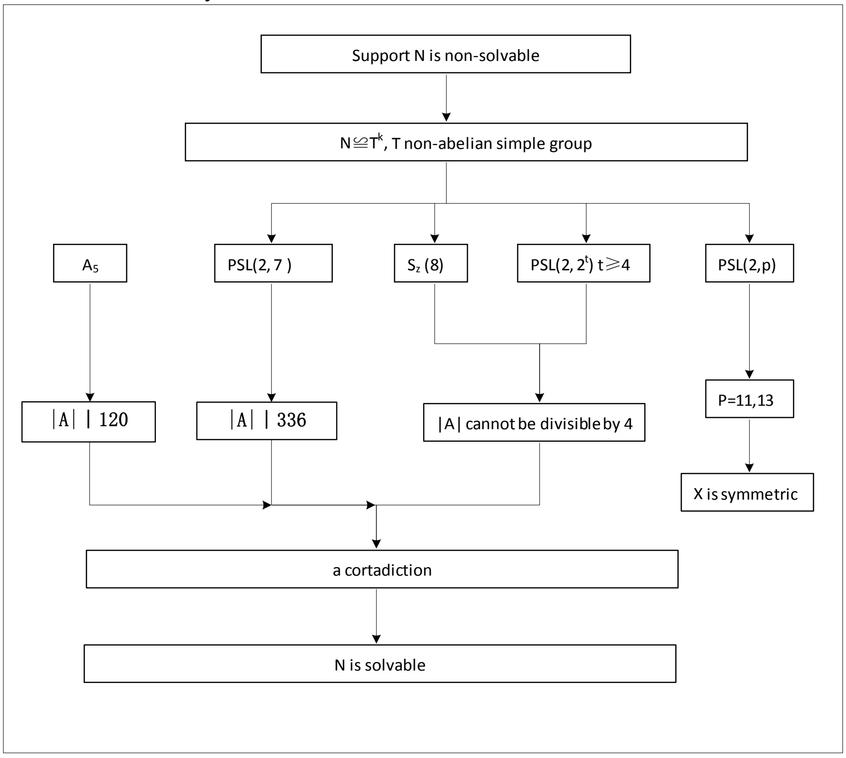

Figure 2. Non-Primitive algorithm process

D. *Claim II*

M is not isomorphic to Z_{qr}, Suppose that $M \cong Z_{qr}$. Let $C=C_A(M)$. Then $A/C \le Z_{q-1} \times Z_{r-1}$. Since $p > q > r$, we have $pqr \mid |C|$, that is $C > M$. Take a minimal normal subgroup B/M of A/M such that $B/M \le C/M$. Then $B/M \cong Z_p$ or Z_2^s . Note that A has no non-trivial normal 2-subgroup. Suppose that $B/M \cong Z_2^s$. Then the Sylow 2-subgroup of B is normal in A, a contradiction. Thus $B/M \cong Z_p$, that is $B \cong Z_{pqr}$. Then X is a Cayley graph on group B, by Proposition 2.3, it is impossible.

E. *Claim III*

If $M \cong Z_{pt}$ for t=q or r, then $X \cong X(s;q,pr)$ or $X(s;r,pq)$.
In this case, $A/C \le Z_{p-1} \times Z_{t-1}$. Consider the quotient graph X_M of X corresponding to the orbits of M. Then $|X_M|=qr/t$. If X_M has valency 4, then the stabilizer K_u of u in K fixes each neighborhood of u in X because K fixes each orbit of M. It follows that $K_u = 1$ and $K = M$. This implies that X_M is A/M-half-arc-transitive, furthermore, A/M is non-abelian. Thus $C > M$. If X_M has valency 2 and $K_u = 1$, then $A/M \cong D_{2qr/t}$. In this case we have $C > M$. Take a minimal normal subgroup B/M of A/M such that $B/M \le C/M$. Note that A has no non-trivial normal 2-subgroup. Then $B/M \cong Z_{qr/t}$. Thus $B/M \cong Z_{qr/t}$, that is $B \cong Z_{pqr}$. Then X is a Cayley graph on group B, a contradiction. Thus X_M has valency 2 and $K_u \neq 1$. It is easy to know that $X \cong X(s;q,pr)$ or $X(s;r,pq)$.

By Claim I, $N \cong Z_p$, Z_p , or Z_r . Let $C = C_A(N)$. First, we assume that $N \cong Z_r$. Then $A/C \le Z_{r-1}$. Note that $q > q > r$. That means that $pq \| |C|$. Take a minimal normal subgroup B/N of A/N such that $B/N \le C/N$. Then $B/N \cong Z_p$ or Z_q . It follows that $B \cong Z_{pr}$ or Z_{qr} and B is a normal subgroup of A . By Claim II and III, we have that $B \cong Z_{pr}$ and $X \cong X(s;q,pr)$. Second, we assume that $N \cong Z_q$. Similarly, we have $X \cong X(s;r,pq)$. Now we assume that $N \cong Z_p$. Similarly argument to Claim III, we have that $X \cong X(s;r,pq)$, $X(s;q,pr)$ or $X(s;qr,p)$.By Proposition 2.1, $X \cong X(s;r,pq)$, $X(s;q,pr)$ and $X(s;qr,p)$ are half-arc-transitive graphs.

F. *Remark*

In fact, by [6], we know these graphs are normal Cayley graphs on Frobenius group.

IV. CONCLUSION

In the paper, we give the classification of tetravalent half -arc transitive graph of order pqr, and proved that tetravalent half-arc transitive graph of order pqr must be tightly attached half-arc-transitive graph. In fact, we know that these graphs must be normal Cayley graphs on meta-cyclic group (see [6]), implying that these graphs must be meta-circulant. So far, the known half -arc transitive graphs are mostly meta-circulant, many scholars focus on judging whether half -arc transitive graphs is meta-circulant or not and how to form non-meta-circulant half-arc transitive graphs. Next, we want to give the classification of tetravalent half-arc transitive graphs of order square free and give some examples of non-meta-circulant half- arc-transitive graph.

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