

Hexavalent Half-arc-transitive Graphs of Order $6p$

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Abstract—A graph is half-arc-transitive if its automorphism group acts transitively on its vertex set and edge set, but not arc set. Y-Q. Feng et al. gave the classification of tetravalent half-arc-transitive graph of order $6p$. In this paper, we proved that hexavalent half-arc-transitive graph of order $6p$ has order 42.

Keywords- Cayley graph; half-arc-transitive graph; transitive graph; Heawood graph; quotient graph

I. INTRODUCTION

Throughout this paper graphs are assumed to be finite, simple and undirected, but with an implicit orientation of the edges when appropriate. For a graph X , let $V(X)$, $E(X)$, $A(X)$ and $\text{Aut}(X)$ be the vertex set, the edge set, the arc set and the auto orphism group of X , respectively. Let D_{2n} be the dihedral group of order $2n$, and Z_n the cyclic group of order n as well as the ring of integers modulo n . Denote by \mathbb{Z}_n^* the multiplicative group of Z_n consisting of numbers coprime to n , and for a prime p , denote by \mathbb{Z}_p^m the elementary abelian group $Z_p \times Z_p \times \cdots \times Z_p$ (m times). For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the Cayley graph $\text{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. A graph X is isomorphic to a Cayley graph on G if and only if its automorphism group $\text{Aut}(X)$ has a subgroup isomorphic to G , acting regularly on vertices [1, Lemma 16.3].

A graph X is said to be vertex-transitive, edge-transitive or arc-transitive if $\text{Aut}(X)$ acts transitively on $V(X)$, $E(X)$, or $A(X)$, respectively. A graph is said to be half-arc-transitive provided that it is vertex-transitive and edge-transitive, but not arc-transitive. More generally, by a half-arc-transitive action of a subgroup G of $\text{Aut}(X)$ on a graph X we shall mean a vertex-transitive and edge-transitive, but not arc-transitive action of G on X . In this case, we shall say that the graph X is G -half-arc-transitive.

The investigation of half-arc-transitive graphs was initiated by Tutte[2] and he proved that a vertex- and edge-transitive graph with odd valency must be arc-transitive. In 1970 Bouwer[3] constructed a $2k$ -valent half-arc-transitive graph for every $k \geq 2$ and later more such graphs were constructed. In fact, constructing and characterizing half-arc-transitive graphs with small valencies is currently an active topic in algebraic graph theory (see[4, 5]). It was shown in [6] gave the classification of tetravalent half-arc-transitive graphs of

order $2pq$. In this paper, we proved that hexavalent half-arc-transitive graph of order $6p$ has order 42.

II. PRELIMINARY RESULTS

Now we state a simple observation about half-arc-transitive graphs (see [7])

A. Proposition 2.1

There are no half-arc-transitive graphs with fewer than 27 vertices.

The following proposition is straightforward (see [8, Propositions 2.1 and 2.2]).

B. Proposition 2.2

Let $X = \text{Cay}(G, S)$ be a half-arc-transitive graph. Then, there is no involution in S , and no $\alpha \in \text{Aut}(G, S)$ such that $s^\alpha = s^{-1}$ for some $s \in S$. In particular, there are no half-arc-transitive Cayley graphs on abelian groups. Li et al. [9] considered primitive half-arc-transitive graphs.

C. Proposition 2.3

[9, Theorem 1.4] There are no vertex-primitive half-arc-transitive graphs of valency less than 10.

The following proposition can be extracted from Theorem 2.4 and Table 1 in [10].

D. Proposition 2.4

Let X be a connected edge-transitive graph of order $2p$ for a prime p . Then X is symmetric. Assume $p \geq 7$. If X has valency 3 then one of the following holds:

(1) $X \cong G(2 \cdot 7, 3)$, the Heawood graph of order 14 and $\text{Aut}(G(2 \cdot 7, 3)) = \text{PGL}(2, 7)$;

(2) $X \cong G(2p, 3)$, $p \geq 13$ and $3 \mid (p-1)$. In this case, $\text{Aut}(G(2p, 3)) \cong (Z_p \times Z_3) \rtimes Z_2$; If X has valency 6 then one of the following holds:

(3) $X \cong B(\text{PG}(2, 5))$, $p=31$ and $\text{Aut}(B(\text{PG}(2, 5))) = \text{P}\Gamma\text{L}(3, 5) \rtimes Z_2$;

(4) $X \cong B^*(H(11))$, $p = 11$ and $\text{Aut}(B^*(H(11))) = \text{PSL}(2, 11) \rtimes Z_2$;

(5) $X \cong G(2p, 6)$ and $6 \mid (p-1)$. In this case, $\text{Aut}(G(2 \cdot 7, 6)) \cong S_7 \times Z_2$ and $\text{Aut}(G(2p, 6)) \cong (Z_p \times Z_6) \rtimes Z_2$ for $p \geq 13$.

Now we give a well-known result.

E. Proposition 2.5

Let X be a connected arc-transitive cubic graph of order $4p$, where p is a prime. Then X is one of the following: Q_3 , the 3-dimensional cube; D_{20} , the dodecahedron; C_{28} , the Coxeter graph; and $GP(10,3)$, the generalized Peterson graph.

The following proposition can be extracted from [11] and [12].

F. Proposition 2.6

Let X be a connected hexavalent edge-transitive graph of order $3p$, where p is a prime. If X is half-arc-transitive, then $X \cong M(d,3,p)$ where $(d,p) \neq (2,7)$ or $(3,19)$ with $d|(p-1)/3$. If X is symmetric then one of the following holds:

- (1) $X \cong T_6^C$, the graph of order 30 and $\text{Aut}(T_6^C) = S_6$;
- (2) $X \cong L_2(19)^6_{57}$, $p = 19$ and with $\text{Aut}(X) \cong \text{PSL}(2,19)$.
- (3) $X \cong G(3p,3)$, $3 | p-1$ and $\text{Aut}(X) = (Z_p : Z_3) : S_3$;
- (4) $X \cong G(p,2)[3K_1]$.

Now we state two simple observations about half-arc-transitive graphs.

G. Proposition 2.7

[13, Proposition 2.6] Let X be a connected half-arc-transitive graph of valency $2n$. Let $A = \text{Aut}(X)$ and let A_u be the stabilizer of $u \in V(X)$ in A . Then each prime divisor of $|A_u|$ is a divisor of $n!$. In particular, if X has valency 6 then A_u is a $\{2,3\}$ -group.

H. Lemma 2.8

Let X be a connected edge-transitive graph of order $2n$ and $A = \text{Aut}(X)$. If A has an abelian normal subgroup N of order n , then X is a Cayley graph. Furthermore, If N is cyclic, and then X is non-half-arc-transitive.

I. Proof

Suppose that N be an abelian normal sub group of A . Then X is bipartite graph with the two orbits of N as its two bipartite sets. It is easy to see that N acts regularly on each partite set of X . Thus, one may identify $R(N) = \{R(n) | n \in N\}$ and $L(N) = \{L(n) | n \in N\}$ with the two partite sets of X . The actions of $n \in N$ on $R(N)$ and on $L(N)$ are just the right multiplication by n , that is $R(g)^n = R(gn)$ and $L(g)^n = L(gn)$ for any $g \in N$. Let $L(n_1)$, $L(n_2)$, $L(n_3)$ and $L(n_4)$ be the vertices adjacent to $R(1)$. Then $L(n_1n)$, $L(n_2n)$, $L(n_3n)$ and $L(n_4n)$ be the vertices adjacent to $R(n)$ for each $n \in N$. Since N is abelian, $R(n_1^{-1}n)$, $R(n_2^{-1}n)$, $R(n_3^{-1}n)$ and $R(n_4^{-1}n)$ are the vertices adjacent to $L(n)$ for each $n \in N$. Define a map α by $R(n) \rightarrow L(n^{-1})$ and $L(N) \rightarrow R(n^{-1})$. It is easy to show that $\alpha \in \text{Aut}(X)$. It follows that $\langle N, \alpha \rangle = 2n$ and $\langle N, \alpha \rangle$ acts regularly on $V(X)$. Thus, X is a Cayley graph. Furthermore, if N is cyclic, then $\langle N, \alpha \rangle = D_{2n}$. We assume that $X = \text{Cay}(D_{2n}, S)$ and $D_{2n} = \langle a, b | a^n = b^2 = 1, a^b = a^{-1} \rangle$. Note that X is a bipartite graph. It follows that S has no element of odd order. Thus, S contains involutions. By Proposition 2.2, X is not half-arc-transitive.

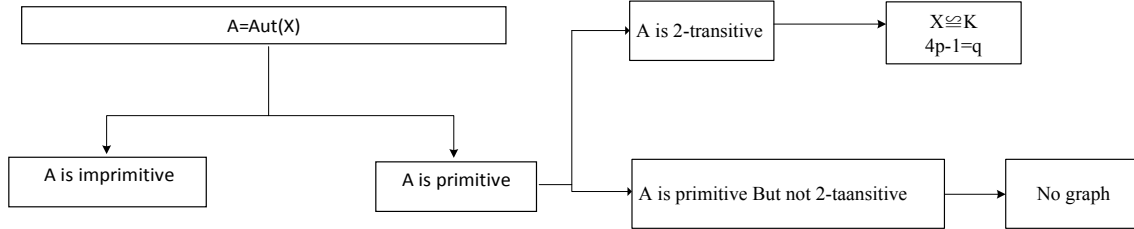


Figure1. Process flowchart of Theorem 3.1

III. MAIN RESULT

The following theorem is the main result of this paper. Fig. 1 showed the Proof flowchart of the theorem.

A. Theorem 3.1

Let p be a prime and X be a hexavalent half-arc-transitive graph of order $6p$. Then X has order 42.

B. Proof

Suppose that X is a hexavalent half-arc-transitive graph of order $6p$. Let $A = \text{Aut}(X)$, $u \in V(X)$ and denote by A_u

the stabilizer of u in A . By Proposition 2.7, A_u is a $\{2,3\}$ -group and hence A is a $\{2, 3, p\}$ -group with $|A|$ not divisible by p^2 . The edge-transitivity of X implies that $18p || |A|$. By Proposition 2.1, $p \geq 5$, let N be a minimal normal subgroup and P a Sylow p -subgroup of A . Then $|P|=p$. Let B be abnormal subgroup of A . Let K be the kernel of A acting on the quotient graph X_B of X corresponding to the orbits of B . First we prove the following claims. Fig. 2 showed peocess flowchart of the imprimitive part.

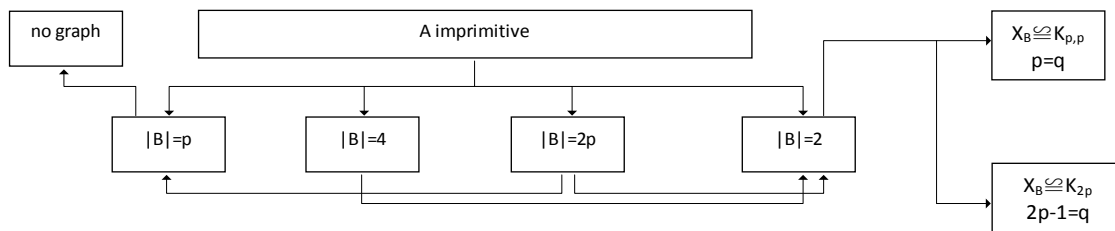


Figure2. process flowchart of the imprimitive

C. Claim I

B is not isomorphic to Z_6, Z_{2p}, Z_{3p} . Note that $|X| = 6p$. By Lemma 2.8, B is not isomorphic to Z_{3p} . Let $C = C_A(B)$. Suppose that $B \cong Z_6$. Then $A/C \leq Z_2$. Note that $p \geq 5$. It follows that $P \leq C$. Then $BP \cong Z_{6p} \leq A$ and BP acts regularly on $V(X)$. It follows that X is a Cayley graph on group BP , by Proposition 2.2, it is impossible. Suppose that $B = Z_{2p}$. Consider the quotient graph X_B . Then $|X_B| = 3$ and X_B has valency 2, that is, X_B is a 3-cycle, say $X_B = (B_0, B_1, B_2)$ with B_i and B_{i+1} adjacent for each $i \in Z_3$. The induced subgraph $T = \langle B_i, B_{i+1} \rangle$ of $B_i \cup B_{i+1}$ in X is an edge-transitive cubic graph of order $4p$. Furthermore, T is bipartite. By Proposition 2.5, it is impossible.

D. Claim II

If B is r -group, then $B \cong Z_r$, where $r = 2, 3$ or p . If B is p -group, then $B \cong Z_p$. Assume that B is a 2-group. Clearly, $B \leq K$ and since $|V(X)| = 6p$, orbits of B on $V(X)$ are of length 2. Then, $|X_B| = 3p$ and X_B has valency 2 or 6. If X_B has valency 2 then X has at most valency 4, a contradiction. Thus, X_B has valency 6. In this case, $K_u = 1$, $K = B \cong Z_2$. Now we assume that B is a 3-group. Then $|X_B| = 2p$ and X_B has valency 2, 3 or 6. Suppose that X_B has valency 2. Then $X \cong C_{2p}[3K_1]$ is symmetric, a contradiction. If X_B has valency 3, then K_u fixes every out-neighbor of u in the directed graph D , which implies $K_u = 1$. Thus, $B = K = Z_3$. If X_B has valency 6 then $K_v = 1$ and $B = K = Z_3$.

E. Claim III

A has a solvable minimal normal subgroup. Suppose that all minimal normal subgroups of A are not solvable. Then $N \cong T^m$ where T is a nonabelian simple $\{2, 3, p\}$ -group. Since $|A|$ is not divisible by p^2 and $p \geq 5$, by [14, pp.12-14], we have that $m=1$ and $N=T$ is isomorphic to $A_5, A_6, L_2(7), L_2(8), L_3(3), U_3(3), L_2(17), U_4(2)$. Let $C = C_A(N)$ and K be the kernel of N acting on the orbits of N . Since $C \cap N$ is a normal subgroup of N , then C is a $\{2, 3\}$ -group. Thus, C is solvable, it follows that $C=1$. Then $A \cong A/C \leq \text{Aut}(N)$. Thus, N is not isomorphic to A_5 or $L_2(7)$ since $2 \cdot 3^2 \cdot p \nmid |A|$. Suppose that $N \cong L_2(8)$. Then $A = L_2(8)$ or $\text{Aut}(L_2(8))$, implying that $|N_v| = 2^2 \cdot 3$ or $2^3 \cdot 2$. However, by Atlas, N has no subgroup of order $2^2 \cdot 3$ or $2^3 \cdot 2$, a contradiction. For the case $N \cong U_3(3)$ or $U_4(2)$, we have the similarly contradiction. Suppose that $N \cong \text{PSL}(2, 17)$. Then $A = \text{PSL}(2, 17)$ or $\text{PGL}(2, 17)$. If $A = \text{PSL}(2, 17)$, then $|A_v| = 2^3 \cdot 3$. Then $A_v \cong S_4$ is a maximal subgroup of A , it follows that A acts primitively on $V(X)$. By Proposition 2.3, it is impossible. If $A = \text{PGL}(2, 17)$, then $|A_v| = 2^4 \cdot 3$, which is impossible because A has no subgroup of order $2^4 \cdot 3$.

Suppose $N \cong A_6$. Then $A \cong A_6$, or $A_6 < A \leq \text{Aut}(A_6)$. Note that $3^2 \nmid |N|$. If N is transitive, then N is half-arc-transitive. Thus, $|N_v| = 12$ and $X \cong \text{cos}(A_6, N_v \{g, g^{-1}\} A_v)$ such that $|N_v|/|N_v \cap N^g| = 3$ and $\langle N_v, g \rangle = A_6$ where $g \in A_6$. By Magma, it is impossible. Thus, N has two orbits, it follows that $N_v \cong S_4 < A_v$. Then N is primitive on each orbit since S_4 is a maximal subgroup of N . By [15], the

length of the orbits of N on each orbit is 1, 7, 7. It means that X cannot have valency 6, a contradiction.

Suppose $N \cong L_3(3)$. Set $H = A_v$. Then $A = L_3(3)$ or $\text{Aut}(L_3(3))$ and $|A_v| = 2^3 \cdot 3^2$ or $2^4 \cdot 3^2$. Suppose that $A = L_3(3)$. Then $X \cong \text{cos}(A, H \{g, g^{-1}\} H)$ where $|H| = 72$ and $g \in L_3(3)$ such that $|H|/|H \cap H^g| = 3$. It follows that $H \cap H^g$ is a subgroup of H with order 24, which is impossible since H has no subgroup of order 24. Now suppose that $A = \text{Aut}(L_3(3))$ and $X \cong \text{cos}(A, H \{g, g^{-1}\} H)$ where $|H| = 2^3 \cdot 3^2$ and $g \in \text{Aut}(L_3(3))$ such that $|H|/|H \cap H^g| = 3$. By ATLAS, A_v is a subgroup of $L_3(3)$. Thus, $H \cap H^g$ is a subgroup of H . By magma, it is impossible.

We have proved that A has at least one solvable minimal normal subgroup, say N . By Claim II, we have $N \cong Z_p, Z_2, Z_3$. Let $C = C_A(N)$. Suppose that $N \cong Z_p$. Then $A/C \leq Z_{p-1}$. Suppose that $C = N$. Then A is abelian, which is impossible. Thus, $C > N$. Let M/N be a minimal normal subgroup of A/N contained in C/N . Then M is a normal subgroup of A and M/N is an elementary abelian r -group for $r=2$ or 3 . Furthermore, $M = N \times R$, where R is a Sylow r -subgroup of M . Clearly, R is characteristic in M and so normal in A . By Claim II, $R \cong Z_2$ or Z_3 . It follows that $M \cong Z_{2p}$ or Z_{3p} , contrary to Claim I. Suppose that $N \cong Z_2$. By Claim II, we have X_N has valency 6, $K_u = 1$, $K = N \cong Z_2$ and $A/N \leq \text{Aut}(X_N)$. Then X_N is A/N -half-arc-transitive. Let M/N be a minimal normal subgroup of A/N . Suppose that M/N is solvable. By Claim II, N is a maximal normal 2-subgroup of A . Then M/N is an elementary abelian r -group for $r=3$ or p . Similarly, we have $M \cong Z_6$ or Z_{2p} , contrary to Claim I. Thus, M/N is unsolvable, it follows that A/N is unsolvable. Note that $A/N \leq \text{Aut}(X_N)$. By Proposition 2.6, $X_N \cong T_6^C$ and $A_6 \leq A/N \leq S_6$, or $X_N \cong L_2(19)^6_{57}$ and $A/N \cong \text{PSL}(2, 19)$. For the latter case, $|A/N| = 2^2 \cdot 3^2 \cdot 5 \cdot 19$, which is impossible because A is a $\{2, 3, p\}$ -group. Thus, $A/N \cong A_6$ or S_6 , implying that A/N is arc-transitive on X_N , a contradiction.

Suppose that $N \cong Z_3$. Then $A/C \leq Z_2$. It follows that $p \nmid |C|$ and so $C > N$. Let M/N be a minimal normal subgroup of A/N contained in C/N . Suppose that M/N is solvable. By Claim II, N is a maximal normal 3-subgroup of A . Then M/N is an elementary abelian r -group for $r = 2$ or p . Similarly, we have $M \cong Z_6$ or Z_{3p} , contrary to Claim I. Thus, M/N is unsolvable, it follows that A/N is unsolvable. By Claim II again, we have X_N has valency 3 or 6, $K_u = 1$, $K = N \cong Z_3$ and $A/N \leq \text{Aut}(X_N)$. Suppose that X_N has valency 6. Then X_N is A/N -half-arc-transitive. By Proposition 2.4, $X_N \cong B^*(H(11))$ and $\text{PSL}(2, 11) \leq A/N \leq \text{PSL}(2, 11) \times Z_2$, or $X_N \cong B(\text{PG}(2, 5))$ and $\text{PSL}(3, 5) \leq A/N \leq \text{P}\Gamma\text{L}(3, 5) \times Z_2$, $X_N \cong G(2 \cdot 7, 6)$ and $L_2(7) \times Z_2 \leq A/N \leq S_7 \times Z_2$, or $X_N \cong O_3^C$ and $A_5 \leq A/N \leq S_5$. For the first two cases, $|\text{PSL}(2, 11)| = 2^2 \cdot 3 \cdot 5 \cdot 11 \nmid |A/N|$ or $|\text{PSL}(3, 5)| = 2^5 \cdot 3 \cdot 5^3 \cdot 31 \nmid |A/N|$, which is impossible because A is a $\{2, 3, p\}$ -group. For the last two cases, X_N is A/N -arc-transitive graph, a contradiction. Thus, X_N has valency 3, then $X_N \cong O_3$ and $A_5 \leq A/N \leq S_5$, or X_N is isomorphic to the Heawood graph and $A/N \cong \text{PGL}(2, 7)$. Then $M \cong A_5 \times Z_3$ or $\text{PSL}(2, 7) \times Z_3$. Set $L = A_5$ or

$\text{PSL}(2,7)$. Then L is a normal subgroup of A . Consider the quotient graph X_L . Then the length of the orbits of L is p or $2p$ where $p=5$ or 7 . Furthermore, $|L_v|>1$, it follows that X_L has valency 2. Then X_L is a 3- or 6-cycle. Assume that the induced subgraph $T=\langle B_i, B_{i+1} \rangle$ of $B_i \cup B_{i+1}$ where B_i and B_{i+1} are adjacent. Then T is a cubic edge-transitive graph of order $2p$ or $4p$. Furthermore, T is bipartite. By Proposition 2.4-2.5, we have T is isomorphic to Heawood graph and $p=7$, that is X has order 42.

IV. CONCLUSION

In the paper, we give the classification of hexavalent half-arc-transitive graphs of order $6p$. It is proved that the graph must have order 42 if hexavalent half-arc transitive graph of order $6p$ is available. In addition, from the proof we know that the quotient graph is a well-known graph-Heawood graph. However, we should further verify whether the graph belongs to half-arc transitive graph or not. In addition, we [6] proved that if tetravalent half-arc-transitive graphs of order $2pq$ exist, then $p-1$ is divisible by $2q$. In another paper[13], we showed that hexavalent half-arc-transitive graphs of order $4p$ exist if and only if $p-1$ is divisible by 12. Therefore, we guess that hexavalent half-arc-transitive graphs of order $2pq$ exist if and only if $p-1$ is divisible by $3q$, where q is a prime number no less than 5. In next paper, we wish to determine whether the 42-point half-arc-transitive graph exists or not. In addition, we hope to classify the hexavalent half-arc-transitive graphs of order $2pq$.

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REFERENCES

- [1] N. Biggs, Algebraic Graph theory (Second ed), Cambridge university Press, Cambridge, 1993.
- [2] W.T. Tutte, Connectivity in Graphs, University of Toronto Press, Toronto, 1966.
- [3] I.Z. Bouwer, Vertex and edge-transitive but not 1-transitive graphs, Canad.Math.Bull. 13(1970), 231-237.
- [4] Y.Q. Feng, J.H. Kwak, M.Y. Xu, Tetravalent half-arc-transitive graphs of order p^4 , European J. Combin. 29(2008), 555-567.
- [5] K. Kutnar, D. Marušič, P. Šparl, Classification of half-arc-transitive graphs of order $4p$, Euro J. Combinatorics 34(2013), 1158-1176.
- [6] Y.Q. Feng, J.H. Kwak, X. Wang, Tetravalent half-arc-transitive graphs of order $2pq$, J. Algebra Comb. 33(2011), 543-553.
- [7] B. Alspach, D. Marušič and L. Nowitz, Constructing graphs which are 1/2-transitive, J. Austral. Math.Soc. A 56(1994), 391-402.
- [8] Y.Q. Feng, K.S. Wang and C.X. Zhou, Tetravalent half-transitive graphs of order $4p$, European J. Combin. 28(2007), 726-733.
- [9] C.H. Li, Z.P. Lu and D. Marušič, On primitive permutation groups with small suborbits and their orbital graphs, J. Algebra. 279(2004), 749-770.
- [10] Y. Cheng and J. Oxley, On weakly symmetric graphs of order twice a prime, J. Combin. Theory B 42(1987), 196-211.
- [11] B. Alspach and M.Y. Xu, 1/2-transitive graphs of order $3p$, J. Algebraic Combin. 3(1994), 347-355.
- [12] R.J. Wang, M. Y. Xu, A classification of symmetric graphs of order $3p$, J. Combin. Theory B 58(1993) 197-216.
- [13] X.Y. Wang, Y.Q. Feng, Hexavalent half-arc-transitive graphs of order $4p$, European J. Combin. 30 (2009) 1263-1270.