Journal of Nonlinear Mathematical Physics Volume 14, Number 3 (2007), 375–388 ARTICLE

A Whittaker-Shannon-Kotel'nikov sampling theorem related to the Askey-Wilson functions

Fethi BOUZEFFOUR

Institut Préparatoire aux Etudes d'Ingénieur de Bizerte Tunisie
E-mail: bouzeffourfethi@yahoo.fr

Received October 7, 2006; Accepted in Revised Form February 14, 2007

Abstract

A Whittaker-Shannon-Kotel'nikov sampling theorem related to the Askey-Wilson functions is proved. Applications to finite continuous Askey-Wilson transform are given.

1 Introduction

Recently there has been interest in the study of certain class of integral transforms which contains the continuous Jacobi, Gegenbauer, Legendre, Laguerre and Hermite transforms as special cases. The importance of these integral transforms lies not only in their intrinsic properties but also in their connection with sampling theory and signal analysis. They lead to various sampling expansions similar to the one given by the celebrated Whittaker -Shannon-Kotelnikov sampling theorem [17] which states that if $\hat{f}\,(t)$ is band-limited to $[-\pi, \pi]$, then $\hat{f}\,(t)$ can be expanded in the form

$$\hat{f}\,(t) = \sum_{n=-\infty}^{\infty} \hat{f}\,(n)\,\frac{\sin \pi\,(t-n)}{\pi\,(t-n)}.$$

More recently q-analogues of theses results have appeared. In [13] considering a q-exponential function, a q-version of the classical WSK was proved. The q-version of Higgins' result was established in [1].

The Askey-Wilson functions of first and second kind are introduced by M. E. H. Ismail and Mizan Rahman, see [9]. Erik Koelink and Jasper V. Stokman establish the L^2-theory for the Askey-Wilson transform and determine its inversion formula [14]. The Askey-Wilson functions of first kind is also subject of study in Suslov's papers ([15j],[16]) in which Fourier-Bessel type orthogonality relations are derived. Our first step in this paper is to find a raising operator for the Wilson function $u_\mu\,(x, a, b, c, d)$ are then used to investigate the properties of the functions as solutions of q-difference equation. In Section 2 we state the q-sampling theorem. Section 3 contains serval remarks, some of which we define the finite continuous Wilson transform and we study some of its properties.

We first remind the reader of the notations to be used. A q-shifted factorial is defined by [6]

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad n = 1, 2, ..., \infty \tag{1.1}$$

and more generally

$$(a_1, ..., a_s; q)_n = \prod_{k=0}^{s}(a_k; q)_n, \quad n = 0, 1, 2, ..., \infty$$

A basic hypergeometric series is

$${}_r\varphi_s\left(\begin{array}{c} a_1, ..., a_r \\ b_1, ..., b_s \end{array}\middle| q; z\right) = \sum_{k=0}^{\infty} \frac{(a_1, ..., a_r; q)_k}{(b_1, ..., b_s, q; q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} z^k. \tag{1.2}$$

We employ the compact notation

$${}_{r+1}W_r(a_1; a_4, ..., a_{r+1}; q, z),$$

for the very-well poised ${}_{r+1}\varphi_r$ series by

$${}_{r+1}\varphi_r\left(\begin{array}{c} a_1, q\sqrt{a_1}, -q\sqrt{a_1}, a_4, ..., a_{r+1} \\ \sqrt{a_1}, -\sqrt{a_1}, q\frac{a_1}{a_4}, ..., q\frac{a_1}{a_{r+1}} \end{array}\middle| q; z\right). \tag{1.3}$$

Given a function f defined on $(-1,1)$ we set $\breve{f}(e^{i\theta}) := f(x),\ x = \cos\theta$. In other words we think of $f(\cos\theta)$ as a function of $e^{i\theta}$. In this notation the Askey-Wilson finite difference operator $\mathcal{D}_q$ is defined by [7]

$$(\mathcal{D}_q f)(x) = \frac{\breve{f}(q^{\frac{1}{2}}e^{i\theta}) - \breve{f}(q^{-\frac{1}{2}}e^{i\theta})}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})i\sin\theta}. \tag{1.4}$$

In particular $\mathcal{D}_q$ is well-defined on $H_{\frac{1}{2}}$, where

$$H_\nu := \{f : f(\frac{1}{2}(z + \frac{1}{z})) \text{ is analytic for } q^\nu < |z| < q^{-\nu}\}.$$

In [7] M. E. H. Ismail developed a calculus for $\mathcal{D}_q$, and establish a Strum-Liouville theory of a second order Askey-Wilson operators. We shall use the inner product associated with the Chebyshev weight $(1-x^2)^{-1/2}$ on $(-1,1)$, namely

$$< f, g >= \int_{-1}^{1} f(x)\overline{g(x)}\frac{dx}{\sqrt{1-x^2}}. \tag{1.5}$$

The q-integration by parts is given by see [7]

$$\begin{aligned} < f, g >= \frac{\pi\sqrt{q}}{1-q}[f(\frac{1}{2}(q^{\frac{1}{2}} + q^{-\frac{1}{2}})\overline{g(1)} - f(-\frac{1}{2}(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \\ \times\overline{g(-1)}]- < \sqrt{1-x^2}f, \mathcal{D}_q\left(\frac{g(x)}{\sqrt{1-x^2}}\right) >, \end{aligned} \tag{1.6}$$

$$for \quad f,\ g \in H_{\frac{1}{2}}.$$

The Askey-Wilson polynomials $p_n(x;a,b,c,d)$ defined by

$$p_n(\cos\theta;a,b,c,d) = (ab,ac,ad;q)_n a^{-n}{}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}\middle| q;q\right). \quad (1.7)$$

When $\max(|a|,|b|,|c|,|d|) < 1$, the Askey-Wilson polynomials satisfy the orthogonality relation([7],15.2.4)

$$\int_{-1}^{1} w(x;a,b,c,d)p_n(x;a,b,c,d)p_m(x;a,b,c,d)dx = h_n\delta_{m,n}, \quad (1.8)$$

where

$$w(x,a,b,c,d) = \frac{(1-x^2)^{-\frac{1}{2}}(e^{2i\theta},e^{-2i\theta};q)_\infty}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},ce^{i\theta},ce^{-i\theta},de^{i\theta},de^{-i\theta};q)_\infty},\ x=\cos\theta \quad (1.9)$$

and

$$h_n = \frac{2\pi(abcdq^{2n};q)_\infty(abcdq^{n-1};q)_n}{(q^{n+1},abq^n,acq^n,adq^n,bcq^n,bdq^n,cdq^n;q)_\infty}.$$

The Rodrigues formula

$$\begin{aligned} & w(x,a,b,c,d)p_n(x;a,b,c,d) \quad (1.10) \\ &= \left(\frac{q-1}{2}\right)^n q^{\frac{n(n-1)}{4}}\mathcal{D}_q^n[w(x,aq^{\frac{n}{2}},bq^{\frac{n}{2}},cq^{\frac{n}{2}},dq^{\frac{n}{2}})]. \end{aligned}$$

In [7] gives a generalization of Nassrallah and Rahman's extension of the Askey-Willson integral

$$\begin{aligned} &\int_0^\pi {}_8W_7(dg/q;h,r,g/f,de^{i\theta},de^{-i\theta};q,gf/hr)w(x;a,b,c,d)\frac{(ge^{i\theta},ge^{-i\theta};q)_\infty}{(fe^{i\theta},fe^{-i\theta};q)_\infty}d\theta \\ &= \frac{2\pi(abcd,dg,g/d;q)_\infty}{(q,ab,ac,ad,bc,bd,cd,df,f/d;q)_\infty} \quad (1.11) \\ &\times{}_5\phi_4\left(\begin{matrix} ad,bd,cd,g/f,dg/hr \\ abcd,qd/f,dg/h,dg/r \end{matrix}\middle| q;q\right) \\ &+\frac{2\pi(fg/h,fg/r,dg/hr;q)_\infty}{(dg/h,dg/r,fg/hr;q)_\infty} \\ &\times{}_5\phi_4\left(\begin{matrix} af,bf,cf,g/d,gf/hr \\ abcf,qf/d,gf/h,gf/r \end{matrix}\middle| q;q\right). \end{aligned}$$

2 The Askey-Wilson function

The Askey-Wilson function is defined by [9]

$$\begin{aligned} u_\mu(\cos\theta,a,b,c,d) &= \frac{(qa/d,bcq^\mu,q^{1-\mu}e^{i\theta}/d,q^{1-\mu}e^{-i\theta}/d;q)_\infty}{(q^{1-\mu}a/d,bc,qe^{i\theta}/d,qe^{-i\theta}/d;q)_\infty} \\ &\times {}_8W_7(q^{-\mu}a/d\ ;q^{-\mu},q^{1-\mu}/bd,q^{1-\mu}/cd,ae^{i\theta},ae^{-i\theta};q,bcq^\mu). \end{aligned}$$

We first compute the action of the Askey-Wilson operator (1.4) in the Askey-Wilson functions.

Proposition 1. *The function $u_\mu(x;a,b,c,d)$ satisfy*

$$\mathcal{D}_q u_\mu(x;a,b,c,d) = -\frac{2q}{(1-q)d}\frac{(1-q^{-\mu})(1-abcdq^{\mu-1})}{(1-ab)(1-ac)(1-bc)} \times u_{\mu-1}(x;aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}). \tag{2.1}$$

Proof. From Bailey formula (III.36) of [6], we have

$$\begin{aligned} u_\mu(x;a,b,c,d) &= \frac{(bcq^\mu, q^{1-\mu}/ad;q)_\infty}{(bc,q/ad;q)_\infty}\ {}_4\varphi_3\left(\begin{matrix} q^{-\mu}, abcdq^{\mu-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}\ \middle|\ q;q\right) \\ &+ \frac{(q^{-\mu}, abcdq^{\mu-1}, qb/d, qc/d, ae^{i\theta}, ae^{-i\theta};q)_\infty}{(ab,ac,bc,ad/q,qe^{i\theta}/d,qe^{-i\theta}/d;q)_\infty} \\ &\times {}_4\varphi_3\left(\begin{matrix} q^{1-\mu}/ad, bcq^\mu, qe^{i\theta}/d, qe^{-i\theta}/d \\ qb/d, qc/d, q^2/ad \end{matrix}\ \middle|\ q;q\right), \end{aligned}$$

it suffice to compute the action of the Askey-Wilson operator on the ${}_4\varphi_3-$function. First observe that

$$\mathcal{D}_q(ae^{i\theta}, ae^{-i\theta};q)_n = -2a\frac{1-q^n}{1-q}(aq^{\frac{1}{2}}e^{i\theta}, aq^{\frac{1}{2}}e^{-i\theta};q)_{n-1}. \tag{2.2}$$

This leads to

$$\begin{aligned} &\mathcal{D}_q\ {}_4\varphi_3\left(\begin{matrix} q^{-\mu}, abcdq^{\mu-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q,q\right) \\ &= -\frac{2qa}{(1-q)}\frac{(1-q^{-\mu})(1-abcdq^{\mu-1})}{(1-ab)(1-ac)(1-ad)} \\ &\times\ {}_4\varphi_3\left(\begin{matrix} q^{1-\mu}, abcdq^{\mu}, aq^{\frac{1}{2}}e^{i\theta}, aq^{\frac{1}{2}}e^{-i\theta} \\ abq, acq, adq \end{matrix}; q,q\right). \end{aligned}$$

On the other hand

$$\begin{aligned} &\frac{(ae^{i\theta}, ae^{-i\theta};q)_\infty}{(qe^{i\theta}/d, qe^{-i\theta}/d;q)_\infty}\ {}_4\varphi_3\left(\begin{matrix} q^{1-\mu}/ad, bcq^\mu, qe^{i\theta}/d, qe^{-i\theta}/d \\ qb/d, qc/d, q^2/ad \end{matrix}; q,q\right) \\ &= \sum_{n=0}^{\infty}\frac{(q^{1-\mu}/ad, bcq^\mu;q)_n q^n}{(q,qb/d,qc/d,q^2/ad;q)_n}\frac{(ae^{i\theta}, ae^{-i\theta};q)_\infty}{(q^{1+n}e^{i\theta}/d, q^{1+n}e^{-i\theta}/d;q)_\infty} \end{aligned}$$

and

$$\mathcal{D}_q\frac{(ae^{i\theta}, ae^{-i\theta};q)_\infty}{(q^{1+n}e^{i\theta}/d, q^{1+n}e^{-i\theta}/d;q)_\infty} = \frac{1-q^{n+1}/ad}{2(q-1)a}\frac{(aq^{\frac{1}{2}}e^{i\theta}, aq^{\frac{1}{2}}e^{-i\theta};q)_\infty}{(q^{\frac{3}{2}+n}e^{i\theta}/d, q^{\frac{3}{2}+n}e^{-i\theta}/d;q)_\infty}$$

so that

$$\begin{aligned} &\mathcal{D}_q[\frac{(ae^{i\theta}, ae^{-i\theta};q)_\infty}{(qe^{i\theta}/d, qe^{-i\theta}/d;q)_\infty}\ {}_4\varphi_3\left(\begin{matrix} q^{1-\mu}/ad, bcq^\mu, qe^{i\theta}/d, qe^{-i\theta}/d \\ qb/d, qc/d, q^2/ad \end{matrix}\ \middle|\ q;q\right)] \\ &= \frac{2q(1-\frac{ad}{q})}{(1-q)d}(\frac{(aq^{\frac{1}{2}}e^{i\theta}, aq^{\frac{1}{2}}e^{-i\theta};q)_\infty}{(q^{\frac{1}{2}}e^{i\theta}/d, q^{\frac{1}{2}}e^{-i\theta}/d;q)_\infty} \\ &\times {}_4\varphi_3\left(\begin{matrix} q^{1-\mu}/ad, bcq^\mu, q^{\frac{1}{2}}e^{i\theta}/d, q^{\frac{1}{2}}e^{-i\theta}/d \\ qb/d, qc/d, q/ad \end{matrix}\ \middle|\ q;q\right). \end{aligned}$$

It can be shown that relation follows from this. ■

By iterating the formula (2.1) one can derive

Corollary 2. *The Askey-Wilson function has the Rodrigues type property*

$$\mathcal{D}_q^k u_\mu(x;a,b,c,d) = (-1)^k \frac{(2q)^k q^{-\frac{k(k-1)}{4}}}{(1-q)^k d^k} \frac{(q^{-\mu};q)_k (abcdq^{\mu-1};q)_k}{(ab,ac,bc;q)_k} \times u_{\mu-k}(x; aq^{\frac{k}{2}}, bq^{\frac{k}{2}}, cq^{\frac{k}{2}}, dq^{\frac{k}{2}}). \tag{2.3}$$

Proposition 3. *The Askey-Wilson function satisfy*

$$\mathcal{D}_q[w(x,aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}})u_{\mu-1}(x;aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}})] = -\frac{2d}{1-q}(1-ab)(1-ac)(1-bc)w(x,a,b,c,d)u_\mu(x;a,b,c,d).$$

Proof. We use the formula

$$\frac{\mathcal{D}_q w(x,aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}})}{w(x,a,b,c,d)} = \frac{2}{q-1}[2(1-abcd)\cos\theta - (a+b+c+d) + abc + abd + acd + bcd].$$

This leads to

$$\begin{aligned}
&\mathcal{D}_q[w(x,aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}})\,{}_4\varphi_3\left(\begin{matrix} q^{1-\mu}, abcdq^{\mu}, aq^{\frac{1}{2}}e^{i\theta}, aq^{\frac{1}{2}}e^{-i\theta} \\ abq, acq, adq \end{matrix} \middle| q;q\right)] \\
&= \sum_{n=0}^{\infty} \frac{(q^{1-\mu}, abcdq^{\mu-1};q)_n q^n}{(q,abq,acq,adq;q)_n} \mathcal{D}_q w(x,aq^{n+\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}}) \\
&= \frac{2(1-ab)(1-ac)(1-ad)}{a(q-1)} \sum_{n=0}^{\infty} \frac{(q^{1-\mu}, abcdq^{\mu};q)_n}{(q,ab,ac,ad;q)_{n+1}} \\
&\quad w(x,aq^n,b,c,d)[2aq^n\cos\theta(1-abcdq^n) - aq^n(b+c+d) \\
&\quad + a^2q^n(bc+bd+cd-1) + abcdq^n]
\end{aligned}$$

The term in square brackets on the right-hand side can be written as

$$[(1-abq^n)(1-acq^n)(1-adq^n) - (1-aq^ne^{i\theta})(1-aq^ne^{-i\theta})(1-abcdq^n)].$$

Therefore

$$\mathcal{D}_q[w(x, aq^{\frac{1}{2}}, bq^{\frac{1}{2}}, cq^{\frac{1}{2}}, dq^{\frac{1}{2}})_4\varphi_3\left(\begin{matrix} q^{1-\mu}, abcdq^{\mu}, aq^{\frac{1}{2}}e^{i\theta}, aq^{\frac{1}{2}}e^{-i\theta} \\ abq, acq, adq \end{matrix}; q, q\right)]$$

$$= \frac{2(1-ab)(1-ac)(1-ad)}{a(q-1)} w(x, a, b, c, d)$$

$$\times \sum_{n=1}^{\infty} \frac{(q^{1-\mu}, abcdq^{\mu}; q)_{n-1}(ae^{i\theta}, ae^{-i\theta}; q)_n}{(q, ab, ac, ad; q)_n}$$

$$[(1-q^{n-\nu})(1-abcdq^{\nu+n-1}) - (1-q^n)(1-abcdq^{n-1})]$$

$$= \frac{2(1-ab)(1-ac)(1-ad)}{a(q-1)} w(x, a, b, c, d)$$

$$\times {}_4\varphi_3\left(\begin{matrix} q^{-\mu}, abcdq^{\mu-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right).$$

On the other hand

$$w(x, aq^{\frac{1}{2}}, bq^{\frac{1}{2}}, cq^{\frac{1}{2}}, dq^{\frac{1}{2}}) \frac{(aq^{\frac{1}{2}}e^{i\theta}, aq^{\frac{1}{2}}e^{-i\theta}; q)_\infty}{(q^{\frac{1}{2}}e^{i\theta}/d, q^{\frac{1}{2}}e^{-i\theta}/d; q)_\infty}$$

$${}_4\varphi_3\left(\begin{matrix} q^{1-\nu}/ad, bcq^{\nu}, q^{\frac{1}{2}}e^{i\theta}/d, q^{\frac{1}{2}}e^{-i\theta}/d \\ qb/d, qc/d, q/ad \end{matrix}\middle| q; q\right)$$

$$= w(x, q^{\frac{1}{2}}/d, bq^{\frac{1}{2}}, cq^{\frac{1}{2}}, dq^{\frac{1}{2}})_4\varphi_3\left(\begin{matrix} q^{1-\nu}/ad, bcq^{\nu}, q^{\frac{1}{2}}e^{i\theta}/d, q^{\frac{1}{2}}e^{-i\theta}/d \\ qb/d, qc/d, q/ad \end{matrix}\middle| q; q\right),$$

so that

$$\mathcal{D}_q[w(x, aq^{\frac{1}{2}}, bq^{\frac{1}{2}}, cq^{\frac{1}{2}}, dq^{\frac{1}{2}}) \frac{(aq^{\frac{1}{2}}e^{i\theta}, aq^{\frac{1}{2}}e^{-i\theta}; q)_\infty}{(q^{\frac{1}{2}}e^{i\theta}/d, q^{\frac{1}{2}}e^{-i\theta}/d; q)_\infty}$$

$${}_4\varphi_3\left(\begin{matrix} q^{1-\nu}/ad, bcq^{\nu}, q^{\frac{1}{2}}e^{i\theta}/d, q^{\frac{1}{2}}e^{-i\theta}/d \\ qb/d, qc/d, q/ad \end{matrix}\middle| q; q\right)]$$

$$= \frac{2d}{q-1} w(x, 1/d, b, c, d) \sum_{n=0}^{\infty} \frac{(q^{1-\mu}/ad, bcq^{\mu}; q)_n (e^{i\theta}/d, e^{-i\theta}/d; q)_n}{(q, qb/d, qc/d, q/ad; q)_n}$$

$$\times[(1-\frac{b}{d}q^n)(1-\frac{c}{d}q^n)(1-q^n) - (1-\frac{q^n}{d}e^{i\theta})(1-\frac{q^n}{d}e^{-i\theta})(1-bcq^n)]$$

$$= \frac{2d}{(q-1)(1-q/ad)} w(x, 1/d, b, c, d)\{\sum_{n=0}^{\infty} \frac{(q^{1-\mu}/ad, bcq^{\mu}; q)_{n+1}(e^{i\theta}/d, e^{-i\theta}/d; q)_{n+1}}{(q, qb/d, qc/d, q^2/ad; q)_n}$$

$$- \sum_{n=0}^{\infty} \frac{(q^{1-\mu}/ad, bcq^{\mu}; q)_n (e^{i\theta}/d, e^{-i\theta}/d; q)_{n+1}}{(q, qb/d, qc/d, q^2/ad; q)_n}(1-\frac{q^{n+1}}{ad})(1-bcq^n)\}$$

$$= 2\frac{q(1-q^{-\mu})(1-abcdq^{\mu-1})}{(q-1)a(1-\frac{q}{ad})} w(x, a, b, c, d) \frac{(ae^{i\theta}, ae^{-i\theta}; q)_\infty}{(qe^{i\theta}/d, qe^{-i\theta}/d; q)_\infty}$$

$$\times {}_4\varphi_3\left(\begin{matrix} q^{1-\mu}/ad, bcq^{\mu}, qe^{i\theta}/d, qe^{-i\theta}/d \\ qb/d, qc/d, q^2/ad \end{matrix}; q, q\right).$$

■

We can now formulate the following theorem.

Theorem 4. *The function $u_\mu(x;a,b,c,d)$ is a solution of the equation*

$$L^{(a,b,c\ ,d)}y(x) = \frac{1}{w(x,a,b,c,d)}\mathcal{D}_q(w(x,aq^{\frac{1}{2}},bq^{\frac{1}{2}},cq^{\frac{1}{2}},dq^{\frac{1}{2}})\mathcal{D}_q y(x)) \\ = \lambda y(x), \tag{2.4}$$

witho λ given by

$$\lambda = \frac{4q}{(1-q)^2}(1-q^{-\mu})(1-abcdq^{\mu-1}).$$

Proof. Replace $\mathcal{D}_q u_\mu(x;a,b,c,d)$ in (2.4) by $u_{\mu-1}(x;a,b,c,d)$ then apply Proposition 3. Simple manipulations will establish (2.4). ■

3 q-Sampling theorem

We put

$$s_k(\mu;a,b,c,d) = \frac{1}{h_k}\int_{-1}^{1} p_k(x;a,b,c,d)u_\mu(x;a,b,c,d)w(x,a,b,c,d)dx.$$

Proposition 5. *We have*

$$s_k(n;a,b,c,d) = \frac{(-1)^n q^{-n(n-1)/2}}{(ab,ac,bc;q)_n}d^{-n}\delta_{n,k}.$$

Proof. For $\mu = n = 0,1,2,...$ the Askey-Wilson function $u_\mu(x,a,b,c,d)$ is just a multiple of the Askey-Wilson polynomial

$$u_n(x;a,b,c,d) = \frac{(-1)^n q^{-n(n-1)/2}}{(ab,ac,bc;q)_n}d^{-n}p_n(x;a,b,c,d). \tag{3.1}$$

The result follows from the orthogonality relation (1.8). ■

Proposition 6. *We have*

$$s_k(\mu;a,b,c,d) = \frac{q^k(1-abcdq^{2k-1})(q^{-\mu},abcdq^{\mu};q)_\infty}{d^k(q,ab,ac,bc;q)_k(q,abcdq^{k-1};q)_\infty(1-abcdq^{\mu+k-1})(1-q^{-\mu+k})}.$$

Proof. The coefficient $s_k(\mu;a,b,c,d)$ is given by

$$h_k s_k(\mu;a,b,c,d) =< \sqrt{1-x^2}u_\mu(x;a,b,c,d), w(x,a,b,c,d)p_k(x;a,b,c,d) > .$$

We use the q-integration by parts formula (1.6), the Rodrigues formula (1.10)

$$h_k s_k(\mu;a,b,c,d) \\ = [\frac{q-1}{2}]^k q^{\frac{k(k-1)}{4}} < \sqrt{1-x^2}u_\mu(x;a,b,c,d), \mathcal{D}_q^k w(x,aq^{\frac{k}{2}},bq^{\frac{k}{2}},cq^{\frac{k}{2}},dq^{\frac{k}{2}}) > \\ = [\frac{1-q}{2}]^k q^{\frac{k(k-1)}{4}} < \mathcal{D}_q^k u_\mu(x;a,b,c,d), \sqrt{1-x^2}w(x,aq^{\frac{k}{2}},bq^{\frac{k}{2}},cq^{\frac{k}{2}},dq^{\frac{k}{2}}) >$$

and by (2.1)

$$h_k s_k(\mu;a,b,c,d)=\frac{q^k(q^{-\mu};q)_k(abcdq^{\mu-1};q)_k}{d^k(ab,ac,bc;q)_k}$$
$$\times < u_{\mu-k}(x;aq^{\frac{k}{2}},bq^{\frac{k}{2}},cq^{\frac{k}{2}},dq^{\frac{k}{2}}),\sqrt{1-x^2}w(x,aq^{\frac{k}{2}},bq^{\frac{k}{2}},cq^{\frac{k}{2}},dq^{\frac{k}{2}}) > .$$

Apply the formula (2.3) with its parameters specialized as follows:

$$d\to aq^{\frac{k}{2}},\ g\to q^{1-\mu+\frac{k}{2}}/d,\ f\to q^{1-\frac{k}{2}}/d,\ h\to q^{1-\mu}/bd,\ r\to q^{1-\mu}/cd,\ a\to dq^{\frac{k}{2}}.$$

This give

$$\int_0^\pi {}_8W_7(aq^{-\mu+k}/d\ ;q^{-\mu+k},q^{1-\mu}/bd,q^{1-\mu}/cd,aq^{\frac{k}{2}}e^{i\theta},aq^{\frac{k}{2}}e^{-i\theta};q,bcq^{\mu})$$
$$\times w(x;aq^{\frac{k}{2}},bq^{\frac{k}{2}},cq^{\frac{k}{2}},dq^{\frac{k}{2}})\frac{(q^{1-\mu+\frac{k}{2}}e^{i\theta}/d,q^{1-\mu+\frac{k}{2}}e^{-i\theta}/d;q)_\infty}{(q^{1-\frac{k}{2}}e^{i\theta}/d,q^{1-\frac{k}{2}}e^{-i\theta}/d;q)_\infty}d\theta$$
$$=\frac{2\pi(abcdq^{2k},aq^{1-\mu+k}/d,q^{1-\mu}/ad;q)_\infty}{(q,abq^k,acq^k,adq^k,bcq^k,bdq^k,cdq^k,aq/d,q^{1-k}/ad;q)_\infty}$$
$$\times{}_5\phi_4\left(\begin{matrix}adq^k,abq^k,acq^k,q^{-\mu+k},abcdq^{-1+\mu+k}\\ abcdq^{2k},adq^k,abq^k,acq^k\end{matrix}\ \middle|\ q;q\right)$$
$$+\frac{2\pi(bcq^{1+k},aq^{1-\mu+k}/d,q^{-\mu+k},abcdq^{\mu+k-1};q)_\infty}{(q,q,abq^k,acq^k,bcq^k,bdq^k,cdq^k,qa/d,adq^{k-1},bcq^{\mu};q)_\infty}$$
$$\times{}_5\phi_4\left(\begin{matrix}q,qb/d,qc/d,q^{1-\mu}/ad,bcq^{\mu}\\ bcq^{1+k},q^{2-k}/ad,qb/d,qc/d\end{matrix}\ \middle|\ q;q\right).$$

On the other hand

$$\begin{aligned}&{}_5\phi_4\left(\begin{matrix}adq^k,abq^k,acq^k,q^{-\mu+k},abcdq^{1-\mu+k}\\ abcdq^{2k},adq^k,abq^k,acq^k\end{matrix};q,q\right)\\ &={}_2\varphi_1\left(\begin{matrix}q^{-\mu+k},abcdq^{\mu+k-1}\\ abcdq^{2k}\end{matrix};q,q\right)\\ &=\frac{(abcdq^{\mu+k},q^{-\mu+k+1};q)_\infty}{(abcdq^{2k},q;q)_\infty},\end{aligned}\tag{3.2}$$

where we have used the q-Gauss sum ([6], $II.8$) to evaluate the sum (3.2). Apply the nonterminating form of the q-Saalschutz ([6], (2.10.11))with its parameters specialized as follows

$$a\to q,\ b\to bcq^{\mu},\ c\to q^{1-\mu}/ad,\ e\to q^{2-k}/ad,\ f\to bcq^{1+k},$$

this give

$$
\begin{aligned}
&{}_3\varphi_2\left(\begin{array}{c} q, bcq^{\mu}, q^{1-\mu}/ad \\ q^{2-k}/ad, bcq^{1+k} \end{array} \middle| q; q\right) \\
&= -\frac{(adq^{k-1}, q, bcq^{\mu}, q^{1-\mu}/ad, abcdq^{2k}; q)_\infty}{(q^{1-k}/ad, adq^k, abcdq^{\mu+k-1}, q^{k-\mu}, bcq^{1+k}; q)_\infty} \\
&\times {}_3\varphi_2\left(\begin{array}{c} adq^k, abcdq^{\mu+k-1}, q^{k-\mu} \\ adq^k, abcdq^{2k} \end{array} \middle| q; q\right) \\
&+\frac{(adq^{k-1}, bcq^k, q^{1+k-\mu}, abcdq^{k+\mu}; q)_\infty}{(adq^k, bcq^{1+k}, q^{k-\mu}, abcdq^{\mu+k-1}; q)_\infty}.
\end{aligned}
$$

By the q-Gauss sum ([6],II.8) we have

$$
\begin{aligned}
&{}_5\phi_4\left(\begin{array}{c} q, qb/d, qc/d, q^{1-\mu}/ad, bcq^{\mu} \\ bcq^{1+k}, q^{2-k}/ad, qb/d, qc/d \end{array}; q, q\right) \\
&= {}_3\varphi_2\left(\begin{array}{c} q, bcq^{\mu}, q^{1-\mu}/ad \\ bcq^{1+k}, q^{2-k}/ad \end{array}; q, q\right) \\
&= \frac{(1-bcq^k)(1-adq^{k-1})}{(1-abcdq^{\mu+k-1})(1-q^{k-\mu})} \\
&[1-\frac{(bcq^{\mu}, q^{1-\mu}/ad; q)_\infty}{(bcq^k, q^{1-k}/ad; q)_\infty}],
\end{aligned}
$$

so that

$$
\begin{aligned}
&s_k(\mu; a, b, c, d) \\
&= \frac{q^k (bcq^{\mu}, q^{1-\mu}/ad; q)_\infty}{d^k (ab, ac, bc, abcdq^{k-1}; q)_k (q, q, abcdq^{2k}, bcq^k, q^{1-k}/ad; q)_\infty} \\
&\times \frac{(q^{-\mu}, abcdq^{\mu}; q)_\infty}{(1-q^{-\mu+k})(1-abcdq^{\mu+k})} \\
&+ \frac{q^k}{d^k (ab, ac, bc, abcdq^{k-1}; q)_k (q, q, abcdq^{2k}; q)_\infty} \\
&\times \frac{(q^{-\mu}, abcdq^{\mu}; q)_\infty}{(1-abcdq^{\mu+k-1})(1-q^{-\mu+k})}[1-\frac{(bcq^{\mu}, q^{1-\mu}/ad; q)_\infty}{(bcq^k, q^{1-k}/ad; q)_\infty}] \\
&= \frac{q^k (1-abcdq^{2k-1})(q^{-\mu}, abcdq^{\mu}; q)_\infty}{d^k (q, ab, ac, bc; q)_k (q, abcdq^{k-1}; q)_\infty (1-abcdq^{\mu+k-1})(1-q^{-\mu+k})}.
\end{aligned}
$$

The following result is obvious. ■

Theorem 7. *We have*

$$
\begin{aligned}
u_\mu(x; a, b, c, d) = \sum_{k=0}^{\infty} &\frac{q^k (1-abcdq^{2k-1})}{d^k (q, ab, ac, bc; q)_k (q, abcdq^{k-1}; q)_\infty} \\
&\times \frac{(q^{-\mu}, abcdq^{\mu}; q)_\infty}{(1-abcdq^{\mu+k-1})(1-q^{-\mu+k})} p_k(x; a, b, c, d).
\end{aligned}
$$

To derive the sampling theorem associated to the Askey-Wilson function, we start by defining a notion of the q-band-limitedness.

Definition 8. *We say that a function $f(\mu)$ is q-band-limited to $(-1,1)$ if it can be written in the form*

$$f(\mu)=\int_{-1}^{1} g\left(y\right) u_{\mu}(\cos\theta,a,b,c,d) w\left(x;a,b,c,d\right) dx,$$

where

$$g\in L^{2}\left(\left(-1,1\right),w\left(x;a,b,c,d\right)dx\right) \ .$$

Now we state the sampling theorem.

Theorem 9. *Let f be a q-bandlimited function. Then f has the sampling expansion*

$$f(\mu)=\sum_{n=0}^{\infty} S_{n}^{(\rho)}\left(\mu;q\right) f(n),$$

where

$$S_{n}^{(\rho)}\left(\mu\right)=\frac{(-1)^{n}q^{\binom{n+1}{2}}(1-\rho^{2}q^{2n})(q^{-\mu},\rho^{2}q^{\mu+1};q)_{\infty}}{(q,\rho^{2}q^{n};q)_{\infty}(1-\rho^{2}q^{\mu+n})(1-q^{-\mu+n})}, \tag{3.3}$$

and

$$\rho=\sqrt{q^{-1}abcd}. \tag{3.4}$$

Proof. The function f is q-bandlimited, then

$$f(\mu)=\int_{-1}^{1} g\left(x\right) u_{\mu}(\cos\theta,a,b,c,d) w\left(x;a,b,c,d\right) dx \tag{3.5}$$

where

$$g\in L^{2}\left(\left(-1,1\right),w\left(x;a,b,c,d\right)dx\right) \ .$$

Since $\{p_{n}\left(x,a,b,c,d\right)\}_{n=0}^{\infty}$ is a complete orthogonal set in a weighted

$$L^{2}\left(\left(-1,1\right),w\left(x;a,b,c,d\right)dx\right),$$

we have

$$g(x)=\sum_{n=0}^{\infty}\hat{g}_{n}p_{n}\left(x,a,b,c,d\right), \tag{3.6}$$

where

$$\hat{g}_{n}=h_{n}^{-1}\int_{-1}^{1} g\left(x\right)p_{n}\left(x,a,b,c,d\right)w\left(x;a,b,c,d\right)dx. \tag{3.7}$$

Subsisting (3.6) into (3.7)

$$f(\mu)=\sum_{n=0}^{\infty}\hat{g}_n\int_{-1}^{1}p_n\left(x,a,b,c,d\right)u_\mu(\cos\theta,a,b,c,d)w\left(x;a,b,c,d\right)dx. \tag{3.8}$$

But in view of (1.8) , equation (3.8) becomes

$$f(\mu)=\sum_{n=0}^{\infty}f(n)S_n^{(\lambda)}\left(\mu\right)).$$

■

4 Finite continuous Askey-Wilson transform

Let f be a function on $(-1,1)$ such that

$$\mathcal{F}_{a,b,c,d}(f)(\mu)=\int_{-1}^{1}f(x)u_\mu(\cos\theta;a,b,c,d)w(x,a,b,c,d)dx, \tag{4.1}$$

is well-defined for all $\mu\in\mathbb{C}$. Then $\mathcal{F}_{a,b,c,d}(f)$ is called the finite continuous Askey-Wilson transform.

Proposition 10. *For $f\in H_1$, we have*

$$\mathcal{F}_{a,b,c,d}(L^{(a,b,c,d)}f)(\mu)=\lambda\mathcal{F}_{a,b,c,d}(f)(\mu).$$

Consider the family of operator see ([7], 19.5.1)

$$S_r(f)(\cos\theta)=\frac{(q,r^2;q)_\infty}{2\pi}\int_0^\pi\frac{(e^{2i\phi},e^{-2i\phi};q)_\infty}{(re^{i(\phi+\theta)},re^{-i(\phi+\theta)},re^{i(\phi-\theta)},re^{-i(\phi+\theta)};q)_\infty}f(\cos\phi)d\phi,$$
$$r\in(0,1).$$

The operators S_r have the semigroup property

$$S_r\circ S_t=S_{rt}\text{ for }r,t,rt\in(0,1).$$

We define the operator

$$\chi_r=\frac{(are^{i\theta},are^{-i\theta},bre^{i\theta},bre^{-i\theta};q)_\infty}{(abr^2;q)_\infty}\circ S_t\circ\frac{(ab;q)_\infty}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta};q)_\infty}.$$

Proposition 11. *If $max(|a|,|b|,|r|)<1$, we have*

$$u_\mu(x,ar,br,cr^{-1},dr^{-1})=\chi_r(u_\mu(.,a,b,c,d))(x). \tag{4.2}$$

Proof. From Theorem 7, we have

$$\chi_r(u_\mu(.;a,b,c,d)) = \sum_{k=0}^{\infty} \frac{q^k(1-abcdq^{2k-1})}{d^k(q,ab,ac,bc;q)_k(q,abcdq^{k-1};q)_\infty} \times \frac{(q^{-\mu},abcdq^{\mu};q)_\infty}{(1-abcdq^{\mu+k-1})(1-q^{-\mu+k})} \chi_r(p_k(.;a,b,c,d)),$$

and by Theorem19.5.2, [7]

$$\chi_r(p_k(.;a,b,c,d))(x) = r^k \frac{(ab;q)_k}{(abr^2;q)_k} p_k(x;ar,br,c/r,d/r).$$

Simple manipulations will establish (4.2). ∎

Proposition 12. *The operator S_r is symmetric in $L^2((0,\pi),(e^{2i\theta},e^{-2i\theta};q)_\infty)$*

$$\int_0^\pi S_r(f)(\cos\theta)g(\cos\theta)(e^{2i\theta},e^{-2i\theta};q)_\infty d\theta = \int_0^\pi f(\cos\theta)S_r(g)(\cos\theta)(e^{2i\theta},e^{-2i\theta};q)_\infty d\theta.$$

Proof. For f,g in $L^2((0,\pi),(e^{2i\theta},e^{-2i\theta};q)_\infty)$

$$\int_0^\pi S_r(f)(\cos\theta)g(\cos\theta)(e^{2i\theta},e^{-2i\theta};q)_\infty d\theta = \int_0^\pi\int_0^\pi \frac{(e^{2i\phi},e^{-2i\phi};q)_\infty(e^{2i\theta},e^{-2i\theta};q)_\infty f(\cos\phi)g(\cos\theta)d\phi d\theta}{(re^{i(\theta+\phi)},re^{i(\theta+\phi)},re^{i(\theta+\phi)},re^{i(\theta+\phi)},re^{i(\theta+\phi)};q)_\infty}.$$

∎

Proposition 13. *If $max(|a|,|b|,|r|)<1$, we have*

$$\mathcal{F}_{ar,br,cr^{-1},dr^{-1}} = \mathcal{F}_{a,b,c,d} \circ {}^t\chi_r,$$

where

$${}^t\chi_r = (ce^{i\theta},ce^{-i\theta},de^{i\theta},de^{-i\theta};q)_\infty \circ S_r \circ \frac{1}{(c/re^{i\theta},c/re^{-i\theta},d/re^{i\theta},d/re^{-i\theta};q)_\infty}.$$

Put

$$\begin{aligned}
&\Phi(x;a,b,c,d,e,f,g)\\
&= \frac{(abcd,abcd/ef,abcd/eg,abcd/gf;q)_\infty}{(abcd/e,abcd/f,abcd/g,abcd/efg;q)_\infty}\, {}_5\varphi_4\left(\begin{matrix} de^{i\theta},de^{-i\theta},e,f,g\\ ad,bc,bd,qefg/abcd\end{matrix}\,\middle|\, q;q\right)\\
&+\frac{(e,f,g,abcd,a^2bcd^2/efg,ab^2cd^2/efg;q)_\infty}{(ad,bd,cd,abcd/e,abcd/f,abcd/g,efg/abcd;q)_\infty}\\
&\times\frac{(abc^2d^2/efg,de^{i\theta},de^{-i\theta};q)_\infty}{(abcd^2e^{i\theta}/efg,abcd^2e^{-i\theta}/efg;q)_\infty}\\
&\times{}_5\varphi_4\left(\begin{matrix} abcd/ef,abcd/eg,abcd/gf,abcd^2e^{i\theta}/efg,abcd^2e^{-i\theta}/efg\\ qabcd/efg,qa^2bcd^2/efg,qab^2cd^2/efg,ad/efg\end{matrix}\,\middle|\, q;q\right).
\end{aligned}$$

As application we give the image of $\Phi(x; a, b, c, d, e, f, g)$ under the finite continuous Askey-Wilson transform.

Proposition 14.

$$\begin{aligned}&\Phi(x; a, b, c, d, e, f, g) \\ &= \sum_{k=0}^{\infty} \frac{1 - abcdq^{2k-1}}{1 - abcdq^{-1}} \frac{(abcdq^{-1}, e, f, g; q)_k}{(q, ad, bd, cd, abcd/e, abcd/f, abcd/g; q)_k} \\ &\times (\frac{abcd^2}{efg})^k p_k(x; a, b, c, d).\end{aligned} \tag{4.3}$$

Proof. The result follow from the extension of the Baily formula ([10], (1.4)) with its parameters specialized as follows :

$$a \to abcdq^{-1}, \quad b \to ab, \ c \to ac, \ d \to g, \ e \to e, \ f \to f, \ g \to ae^{i\theta} \ , h \to ae^{-i\theta}.$$

■

Proposition 15. *We have*

$$\begin{aligned}&\mathcal{F}_{a,b,c,d}(\Phi(.; a, b, c, d, e, f, g)(\mu) \\ &= \frac{2\pi(q^{-\mu+1}, abcdq^{\mu}; q)_\infty}{(q, q, ab, ac, ad, bc, bd, cd; q)_\infty (1 - abcdq^{-1})} \\ &\times {}_8W_7(abcdq^{-1}; q^{-\mu}, abcdq^{\mu-1}, e, f, g; q, \frac{qabcd}{efg}).\end{aligned}$$

Proof. The formula (3.1) imply

$$\begin{aligned}&\mathcal{F}_{a,b,c,d}(\Phi(.; a, b, c, d, e, f, g))(n) \\ &= \frac{(-1)^n q^{-n(n-1)/2}}{(ab, ac, bc; q)_\infty} d^{-n} \int_{-1}^{1} \Phi(.; a, b, c, d, e, f, g) p_n(x; a, b, c, d) w(x, a, b, c, d) dx,\end{aligned}$$

and by Proposition 14, we have

$$\begin{aligned}&\mathcal{F}_{a,b,c,d}(\Phi(.; a, b, c, d, e, f, g))(n) \\ &= \frac{2\pi(abcd; q)_\infty}{(q, ab, ab, ac, ad, bc, bd, cd; q)_\infty} \\ &\times \frac{(e, f, g; q)_n}{(abcd/e, abcd/f, abcd/g; q)_n} (-1)^n q^{-n(n-1)/2} (\frac{abcd}{efg})^n.\end{aligned}$$

The result follows from Theorem 9. ■

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