Journal of Nonlinear Mathematical Physics Volume 14, Number 3 (2007), 389–397 Article

# A two-point boundary value problem on a Lorentz manifold arising in A. Poltorak's concept of reference frame

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*Received October 17, 2006; Accepted in Revised Form February 9, 2007*

**Abstract**

In A. Poltorak's concept, the reference frame in General Relativity is a certain manifold equipped with a connection. The question under consideration here is whether it is possible to join two events in the space-time by a time-like geodesic if they are joined by a geodesic of the reference frame connection that has a time-like initial vector. This question is interpreted as whether an event belongs to the proper future of another event in the space-time in case it is so in the reference frame. For reference frames of two special types some geometric conditions are found under which the answer is positive.

## 1 Introduction

Let $\mathsf{M}$ be a 4-dimensional Lorentz manifold (space-time of General Relativity, for principal notions of GR, see [4, 8]). For definiteness sake, hereafter we use the Lorentz metric with signature $(-,+,+,+)$.

In [5, 6] A. Poltorak suggested a concept in which a reference frame in GR is defined as a certain smooth manifold with a connection. In the most simple cases this is the Minkowski space with its natural flat connection but in more complicated cases some more general manifolds and connections may also appear.

In the reference frame, the Gravitation field is described as a $(1,2)$-tensor $G$ that on any couple of vector fields $X$ and $Y$ takes the value

$$G(X,Y) = \nabla_X Y - \bar{\nabla}_X Y,$$

where $\bar{\nabla}$ is the covariant derivative of the Levi-Civita connection of the Lorentz metric while $\nabla$ is the covariant derivative of the connection in the reference frame. Denote by

$\frac{D}{d\tau}$ the covariant derivative of the connection in the reference frame along a given world line with respect to a certain parameter $\tau$. Then the geodesic $m(\tau)$ of the Levi-Civita connection in $\mathsf{M}$ (a world line in the absence of force fields except the gravitation) is described in the reference frame by the equation (here $G_m(X,Y)$ is the value of $G(X,Y)$ at point $m$)

$$\frac{D}{d\tau}m'(\tau) = G_{m(\tau)}(m'(\tau), m'(\tau)). \tag{1.1}$$

Notice that the right-hand side of (1.1) is quadratic in velocities $m'(\tau)$.

We refer the reader to [5, 6] for more details about the Poltorak's concept and for physical interpretation of the covariant derivative of connection in the reference frame, of the tensor $G$ and several other objects associated with it. The subsequent development of this idea can be found in [7].

We suggest a version of the concept where the manifold of reference frame is the tangent space $\mathsf{T}_m\mathsf{M}$ at an event $m \in \mathsf{M}$ and a Lorentz-orthonormal basis $e_\alpha$, where $\alpha = 0,1,2,3$, is specified (the time-like vector $e_0$ is the observer's 4-velocity). We suppose that this reference frame is valid in a neighborhood $\mathcal{O}$ of the origin in $\mathsf{T}_m\mathsf{M}$, which is identified with a neighborhood $\mathcal{U}$ of the event $m$ by the exponential map of the Levi-Civita connection of the Lorentz metric (the normal chart).

We deal with two choices of connection on the manifold $\mathsf{T}_m\mathsf{M}$. In the first one, we consider on $\mathsf{T}_m\mathsf{M}$ its natural flat connection of Minkowski space (the main case taken into account by A. Poltorak). In the second case, we involve a Riemannian connection of a certain (positive definite) Riemannian metric on $\mathsf{T}_m\mathsf{M}$. This case is motivated by a natural development of the idea yielding Euclidean models in the Quantum Field Theory. Observe that the above-mentioned Riemannian connection may not be the Levi-Civita one, and a non-zero torsion connection compatible with the metric is also allowed. In principle, this allows us to consider electromagnetic interactions as well.

For two reference frames mentioned above, we investigate the question of whether it is possible to connect two events $m_0$ and $m_1$ in $\mathsf{M}$ by a time-like geodesic if they are connected in the reference frame by a geodesic of the corresponding connection whose initial vector is time-like, i.e., lies inside the light cone in the space $\mathsf{T}_{m_0}\mathsf{M}$. This question can be interpreted as follows: does the event $m_1$ belong to the proper future of the event $m_0$ if this is true in the reference frame? The fact that it is not always so is illustrated by the following

**Example.** For the sake of simplicity we are dealing with a 2-dimensional space-time. Consider the manifold $R^2_+ = \{(q^0, q^1) \in R^2 | q^0 > 0\}$ with the Lorentz metric $g = -dq^0 \otimes dq^0 + (q^0)^{4/3} dq^1 \otimes dq^1$. This is a 2-dimensional Lorentz submanifold in the Einstein-de Sitter space-time (see [8]). The flat Minkowski connection of the metric $h = -dq^0 \otimes dq^0 + dq^1 \otimes dq^1$ plays the role of connection in reference frame while $g$ is the metric transferred into the reference frame from the space-time by exponential map as it is described above.

Specify an event $m_0 = (a_0, b_0) \in R^2_+$. The proper future $I_h$ of $m_0$ with respect to the flat Minkowski connection is the interior of the light cone at $m_0$, i.e., the events from this future are located between the lines $(a_0 + t, b_0 + (a_0^{-2/3})t)$ and $(a_0 + t, b_0 - (a_0^{-2/3})t)$ where $t > 0$. On the other hand, the proper future $I_g$ of $m_0$ with respect to the Levi-Civita connection of $g$ consists of the events located between the curves $(a_0 + t, b_0 + 3\sqrt[3]{a_0 + t})$

and $(a_0+t, b_0-3\sqrt[3]{a_0+t})$, $t>0$. One can easily see that the closure $\bar{I}_g$ (it is simply called "future"), except the event $m_0$, completely belongs to the open set $I_h$. This means that if an event $m_1 \in I_h$ is "far enough" from $m_0$ and "close enough" to the boundary of the light cone $I_h$, it may not lie in $I_g$, hence it may not be connected with $m_0$ by a time-like geodesic of the Levi-Civita connection of $g$. But by construction $m_1$ is connected with $m_0$ with a time-like geodesic of flat Minkowski connection (its initial vector is time-like).

Notice also that if $m_1$ is conjugate with $m_0$ along all geodesics, joining them in reference frame, it can be impossible to connect $m_0$ and $m_1$ by a geodesic of another connection (in particular, of the connection in the space-time, see general examples of this sort in [1, 2, 3]).

For two above-mentioned kinds of reference frames we find geometric conditions that in any case guarantee that the answer to the above question is positive. The conditions take the form of a certain interrelation between the tensor $G$ and some geometric charateristics, measuring in particular "how far" $m_1$ is from $m_0$ and "how close" $m_1$ is to the boundary of "proper future" of $m_0$ and to conjugate points (if they exist) in the reference frame.

Below we use the following technical statement.

**Lemma 1.** *Specify arbitrary positive real numbers $\varepsilon$, $T$ and $C$. Let a number $b$ be such that $0 < b < \frac{\varepsilon}{(\varepsilon+C)^2}$. Then there exists a sufficiently small positive real number $\varphi$ such that*

$$b((\varepsilon T^{-1} - \varphi) + CT^{-1})^2 < \varepsilon T^{-2} - \varphi T^{-1}. \tag{1.2}$$

**Proof.** For $b$ satisfying the hypothesis of the Lemma, we have

$$b(\varepsilon T^{-1} + CT^{-1})^2 < \varepsilon T^{-2}.$$

The continuity obviously implies that there exists a sufficiently small real number $\varphi > 0$ such that $(\varepsilon T^{-1} - \varphi) > 0$ and (1.2) holds. ■

## 2 The reference frame with flat connection

In this section we investigate the reference frame of the first type, mentioned above, i.e., the manifold of reference frame is $\mathsf{T}_m\mathsf{M}$ with a certain basis $e_\alpha$, where $\alpha = 0, 1, 2, 3$, such that the time-like vector $e_0$ is the 4-velocity of a certain observer, and the connection in $\mathsf{T}_m\mathsf{M}$ is the flat connection of the Minkowski space.

In this case, it is convenient to consider $\mathcal{O}$ as a domain in a linear space, on which there are given the Lorentz metric, the tensor $G$ and other objects described in Introduction. It is also convenient to use the usual facts of Linear Algebra. In particular, the tangent space $\mathsf{T}_{\bar{m}}\mathsf{M}$ at $\bar{m} \in \mathcal{O}$ can be identified with $\mathsf{T}_m\mathsf{M}$ by a translation and for any $\bar{m} \in \mathcal{O}$, we may consider the light cone in $\mathsf{T}_{\bar{m}}\mathsf{M}$ (generated by Lorentz metric tensor at $\bar{m}$) as lying in $\mathsf{T}_m\mathsf{M}$ but depending on $\bar{m}$.

Geodesic lines in $\mathsf{T}_m\mathsf{M}$ with respect to the flat connection are straight lines. Thus, in this reference frame, the question under consideration takes the following form: *is it possible to connect the events $m_0$ and $m_1$ on $\mathsf{M}$ by a time-like geodesic if they are connected by the straight line $a(\tau)$ in $\mathcal{O}$ so that $a(0) = m_0$, $a(T) = m_1$ and that lies inside the light* *cone in* $\mathsf{T}_{m_0}\mathsf{M}$*?* Here $\tau$ is a certain parameter that can be, say, the proper time on $\mathsf{M}$ or the natural parameter in the reference frame, etc. Notice that in this case the fact that the straight line $a(\tau)$ belongs to the light cone in $\mathsf{T}_{m_0}\mathsf{M}$, is equivalent to the fact that the vector of derivative $a'(0) = \frac{d}{d\tau}a(\tau)_{|\tau=0}$ lies inside that cone, as it is postulated in the problem under consideration.

Since the covariant derivative with respect to the flat connection coincides in this case with the ordinary derivative in $\mathsf{T}_m\mathsf{M}$, equation (1.1) takes the form

$$\frac{d}{d\tau}m'(\tau) = G_m(m', m'). \tag{2.1}$$

Thus the main problem is reduced to the two-point boundary value problem for (2.1). Since the right-hand side of (2.1) has quadratic growth in velocities, for some couples of points the two-point problem may have no solutions.

Recall that the tangent space $\mathsf{T}_m\mathsf{M}$ to the Lorentz manifold $\mathsf{M}$ has the natural structure of a Minkowski space whose scalar product is the metric tensor of $\mathsf{M}$ at the event $m$. Since the specified basis $e_\alpha$, where $\alpha = 0, 1, 2, 3$, is Lorentz-orthonormal, their Minkowski scalar product of $X = X^\alpha e_\alpha$ and $Y = Y^\alpha e_\alpha$ has the form

$$X \cdot Y = -X^0Y^0 + X_iY^i,$$

where $X_i = X^i$ for $i = 1, 2, 3$ (we use the Einstein's summation convention). Introduce the Euclidean scalar product in $\mathsf{T}_m\mathsf{M}$ by changing the sign of the time-like summand, i.e., by setting

$$(X, Y) = X^0Y^0 + X_iY^i.$$

Hereafter in this section all norms and distances are determined with respect to the latter scalar product.

By a linear change of time introduce a parameter $s$ along $a(\cdot)$ such that for the line $\tilde{a}(s)$ obtained from $a(\tau)$, we get $\tilde{a}(0) = m_0$ and $\tilde{a}(1) = m_1$. Consider the Banach space $C^0([0, 1], \mathsf{T}_m\mathsf{M})$ of continuous curves in $\mathsf{T}_m\mathsf{M}$ with the usual supremum norm.

**Lemma 2.** *There exists a sufficiently small real number $\varepsilon > 0$ such that, for any curve $\tilde{v}(s)$ from the ball $U_\varepsilon \subset C^0([0, 1], \mathsf{T}_m\mathsf{M})$ of radius $\varepsilon$ centered at the origin, there exists a vector $\tilde{C}_{\tilde{v}} \in \mathsf{T}_m\mathsf{M}$ belonging to a certain bounded neighborhood of the vector $\tilde{a}'(0) = \frac{d}{ds}\tilde{a}(s)_{|s=0}$ and such that $\tilde{C}_{\tilde{v}}$ lies inside the light cone of the space $\mathsf{T}_{m_0}\mathsf{M}$ and the curve $m_0 + \int_0^s(\tilde{v}(t) + \tilde{C}_{\tilde{v}})dt$ takes the value $m_1$ at $s = 1$. The vector $\tilde{C}_{\tilde{v}}$ continuously depends on $\tilde{v}(\cdot)$ and $\|\tilde{C}_{\tilde{v}}\| < C$ for any curve $\tilde{v} \in U_\varepsilon$ for some $C > 0$.*

**Proof.** By explicit integration one can easily prove that $C_{\tilde{v}}$ such that $m_0 + \int_0^1(\tilde{v}(t) + \tilde{C}_{\tilde{v}})dt = m_1$ does exist and that it is continuous in $\tilde{v}$. Then by continuity, from the fact that the vector $\tilde{a}'(0)$ lies inside the light cone of the space $\mathsf{T}_{m_0}\mathsf{M}$, it follows that for a perturbation $\tilde{v}(\cdot)$ sufficiently small with respect to the norm, the vector $\tilde{C}_{\tilde{v}}$ also lies inside the same light cone. Take for $C$ the upper bound of the set of norms of vectors $\tilde{C}_{\tilde{v}}$ from the above-mentioned bounded neighborhood of $\frac{d}{ds}\tilde{a}(s)_{|s=0}$. ■

Notice that $C$ is an estimate of Euclidean distance between $m_0$ and $m_1$.

Turn back to the parametrization of the line $a(\cdot)$ by the parameter $\tau$. Consider the Banach space $C^0([0, T], \mathsf{T}_m\mathsf{M})$.

**Lemma 3.** *Let a real number $k > 0$ be such that $T^{-1}\varepsilon > k$, where $\varepsilon$ is from lemma 2. Then for any curve $v(t)$ from the ball $U_k \subset C^0([0,T], \mathsf{T}_m\mathsf{M})$ of radius $k$ centered at the origin, there exists a vector $C_v \in \mathsf{T}_m\mathsf{M}$ from a bounded neighborhood of the vector $a'(0) = \frac{d}{d\tau}a(\tau)_{|\tau=0}$ such that the vector $C_v$ lies inside the light cone of the space $\mathsf{T}_{m_0}\mathsf{M}$ and the curve $m_0 + \int_0^\tau (v(t) + C_v)dt$ takes the value $m_1$ at $\tau = T$. The vector $C_v$ is continuous in $v(\cdot)$.*

**Proof.** Changing the time along $a(\tau)$ construct the straight line $\tilde{a}(s) = a(Ts)$ that meets the conditions $\tilde{a}(0) = m_0$ and $\tilde{a}(1) = m_1$ as in Lemma 2. For any curve $v(\cdot) \in U_k \subset C^0([0,T], \mathsf{T}_m\mathsf{M})$, construct the curve $\tilde{v}(s) = Tv(Ts)$ that lies in $U_\varepsilon \subset C^0([0,1], \mathsf{T}_m\mathsf{M})$, i.e., such that satisfies Lemma 2. In particular, for this curve, there exists a vector $\tilde{C}_{\tilde{v}}$ such that $\|\tilde{C}_{\tilde{v}}\| < C$ from Lemma 2. By explicit calculations one can easily derive that

$$m_0 + \in \mathsf{T}_0^1(\tilde{v}(s) + \tilde{C}_{\tilde{v}})ds = m_0 \in \mathsf{T}_0^T(v(t) + C_v)dt = m_1,$$

where $C_v = T^{-1}\tilde{C}_{\tilde{v}}$. ■

Notice that by construction $\|C_v\| < T^{-1}C$ for any $v \in U_k$.

For the tensor $G$, introduced above, define its norm $\|G_m\|$ by the standard formula $\|G_m\| = \sup\limits_{X \in \mathsf{T}_m\mathsf{M}, \|X\| \le 1} \|G_m(X,X)\|$. The definition immediately implies the estimate

$$\|G_m(X,X)\| \le \|G_m\|\|X\|^2 \ \text{ for any } X \in \mathsf{T}_m\mathsf{M}. \tag{2.2}$$

**Theorem 2.1.** *Let $m_0$ and $m_1$ be connected in $\mathcal{O}$ by a straight line $a(\tau)$ that lies inside the light cone of the space $\mathsf{T}_{m_0}\mathsf{M}$ and satisfies the conditions $a(0) = m_0$ and $a(T) = m_1$. Let $m_0$ and $m_1$ belong to a ball $V \subset \mathsf{T}_m\mathsf{M}$ such that for any $\hat{m} \in V$ the inequality $\|G_{\hat{m}}\| < \frac{\varepsilon}{(\varepsilon + C)^2}$ holds, where $\varepsilon$ and $C$ are from Lemma 2. Then on $\mathsf{M}$ there exists a time-like geodesic $m_0(\tau)$ of the Levi-Civita connection of the Lorentz metric such that $m_0(0) = m_0$ and $m_0(T) = m_1$.*

**Proof.** Consider the ball $U_K \subset C^0([0,T], \mathsf{T}_m\mathsf{M})$ of radius $K = T^{-1}\varepsilon - \varphi$ centered at the origin, where $\varphi$ is from Lemma 1. Since $K < T^{-1}\varepsilon$, the assertion of Lemma 3 is true for $U_K$ and the following completely continuous operator

$$Bv = \int_0^\tau G_{m_0 + \int_0^t (v(s)+C_v)ds}(v(t) + C_v, v(t) + C_v)dt$$

is well-posed on this ball. Let us show that this operator has a fixed point in $U_K$. Recall that, for any curve $v \in U_K$, its $C^0$-norm is not greater than $K = T^{-1}\varepsilon - \varphi$ and that by Lemma 3 $\|C_v\| < T^{-1}C$. Then the hypothesis of the theorem, eq. (2.2) and Lemmas 1, 2 and 3 imply that

$$\begin{aligned}
&\|G_{m_0 + \int_0^t (v(s)+C_v)ds}\left((v(t) + C_v), (v(t) + C_v)\right)\| \le \\
&\quad \le \|G_{m_0 + \int_0^t (v(s)+C_v)ds}\|((\varepsilon T^{-1} - \varphi) + CT^{-1})^2 < \\
&\quad < (T^{-2}\varepsilon - T^{-1}\varphi).
\end{aligned}$$

From the last inequality we obtain

$$\|\int_0^\tau G_{m_0 + \int_0^t (v(s)+C_v)ds}((v(t) + C_v), (v(t) + C_v))dt\| < (T^{-1}\varepsilon - \varphi) = K.$$

This means that the operator $\mathcal{B}$ sends the ball $U_K$ into itself and so, by Schauder's principle, it has a fixed point $v_0(t)$ in this ball. It is easy to see that $m_0(\tau) = m_0 + \int_0^\tau (v_0(t) + C_{v_0})dt$ is a solution of differential equation (2.1) such that $m_0(0) = m_0$ and $m_0(T) = m_1$. Notice that by the construction $m_0(\tau)$ is a geodesic of the Levi-Civita connection of the Lorentz metric on $\mathsf{M}$. The equality $\mathcal{B}v_0 = v_0$ and the definition of $\mathcal{B}$ implies that $v_0(0) = 0$, and hence $\frac{d}{d\tau}m_0(\tau)_{|\tau=0} = C_{v_0}$, where by Lemma 3 the vector $C_{v_0}$ lies in the light cone of the space $\mathsf{T}_{m_0}\mathsf{M}$, i.e., its Lorentz scalar square is negative. Since this scalar square of the vector of derivative is constant along the geodesic of Levi-Civita connection of the Lorentz metric, this geodesic is time-like. ■

**Remark 1.** Recall (see above) that $C$ estimates the Euclidean distance between $m_0$ and $m_1$. By construction, $\varepsilon$ in particular estimates in some sense "how close" $m_1$ is to the boundary of proper future of $m_0$ in the reference frame. This clarifies the meaning of condition $\|G_{\hat{m}}\| < \frac{\varepsilon}{(\varepsilon+C)^2}$.

# 3 The reference frame with Riemannian connection

In this section we investigate the reference frame at the event $m \in \mathsf{M}$ of the second type, mentioned in Introduction. Namely, the manifold here is absolutely the same as in the previous section: $\mathsf{T}_m\mathsf{M}$ with a specified orthonormal basis, while the connection may be not flat but it is supposed to be compatible with a certain (positive definite) Riemannian metric on $\mathsf{T}_m\mathsf{M}$ (see Introduction). Recall that we do not assume this connection to be torsionless.

Here, the question of existence of a geodesic of Levi-Civita connection on $\mathsf{M}$, that we are looking for, is reduced to the solvability of the two-point boundary value problem for equation (1.1) in the reference frame.

An important difference of this case from the case of flat connection is the fact that for non-flat connections the conjugate points may exist. This yields additional difficulties since there are examples (see [1, 2, 3]) showing that, for a couple of points, conjugate along all geodesics joining them, the boundary value problem for a second order differential equation may have no solutions at all (for the cases of smooth or continuous right-hand sides having quadratic or linear growth in velocities and even if its right-hand side is bounded). We suppose from the very beginning that $m_0$ and $m_1$ are connected by a geodesic in the reference frame, along which they are not conjugate. In this situation, we find conditions under which the problem for (1.1) is solvable.

The case of non-flat connection requires more complicated machinery than the previous one. In particular, we replace ordinary integral operators (used in the previous section) by *integral operators with parallel translation* introduced by Yu. Gliklikh (see, e.g., [1, 2]).

Everywhere in this section the norms in the tangent spaces and the distances on manifolds are induced by the above positive definite Riemannian metric.

First, we describe some general constructions. Let $\mathcal{M}$ be a complete Riemannian manifold, on which a certain Riemannian connection is specified. Consider $p_0 \in \mathcal{M}$, $I = [0, l] \subset R$ and let $v : I \to \mathsf{T}_{p_0}\mathcal{M}$ be a continuous curve. Applying a construction of Cartan's development type, one can show (see, e.g., [1, 2]) that there exists a unique $C^1$-curve $p : I \to \mathcal{M}$ such that $p(0) = p_0$ and the vector $p'(t)$ is parallel along $p(\cdot)$ to the vector $v(t) \in \mathsf{T}_{p_0}\mathcal{M}$ for any $t \in I$.

Let $p(t) := Sv(t)$ be the curve constructed above from the curve $v(t)$. Thus the continuous operator $S : C^0(I, \mathsf{T}_{p_0}\mathcal{M}) \to C^1(I, \mathcal{M})$ that sends the Banach space $C^0(I, \mathsf{T}_{p_0}\mathcal{M})$ of continuous curves in $\mathsf{T}_{p_0}\mathcal{M}$ to the Banach manifold $C^1(I, \mathcal{M})$ of $C^1$-curves in $\mathcal{M}$ (the domain of all curves is $I$) is well-posed.

Notice that for a constant curve $v(t) \equiv X \in \mathsf{T}_{p_0}\mathcal{M}$ we get by construction that $Sv(t) = \exp X$, where exp is the exponential map of the given connection.

Instead of $\mathcal{M}$ we can consider the neighborhood $\mathcal{O}$ in $\mathsf{T}_m\mathsf{M}$ described in Introduction. Without loss of generality we may assume that the Riemannian metric on $\mathcal{O}$ is a restriction of a certain complete Riemannian metric on $\mathsf{T}_m\mathsf{M}$. Indeed, take a relatively compact domain $\mathcal{O}_1 \subset \mathcal{O}$ with smooth boundary such that $\mathcal{O}_1$ contains the points $0 \in \mathsf{T}_{m_0}\mathsf{M}$, $m_0$ and $m_1$ as well as the geodesic $\gamma(t)$, where $t \in [0, 1]$, joining $m_0$ and $m_1$ (if $\mathcal{O}$ is relatively compact one can take it for $\mathcal{O}_1$). Then it is possible to change the Riemannian metric outside $\mathcal{O}_1$ so that it becomes complete on $\mathsf{T}_m\mathsf{M}$, and to use $\mathcal{O}_1$ instead of $\mathcal{O}$. Thus, the operator $S$ is well-posed in this situation.

Let the points $m_0, m_1 \in \mathcal{O}$ be connected in $\mathcal{O}$ by a geodesic $\gamma(t)$ of the Riemanninan connection so that $\gamma(0) = m_0$ and $\gamma(1) = m_1$. In particular, we get $m_1 = \exp(\frac{d}{dt}\gamma(t)_{|t=0}) = S(\frac{d}{dt}\gamma(t)_{|t=0})$, where exp is the exponential map of the Riemannian connection. Let $m_0$ and $m_1$ be not conjugate along $\gamma(\cdot)$ and the vector $\frac{d}{dt}\gamma(t)_{|t=0}$ lie inside the light cone of $\mathsf{T}_{m_0}\mathsf{M}$.

Hereafter we denote by $U_k$ the ball of radius $k$ centered at the origin in a certain Banach space of continuous maps.

**Lemma 4.** *Let $\gamma(t)$ be a geodesic of the connection in the reference frame such that $\gamma(0) = m_0$ and $\gamma(1) = m_1$. Let also $m_0$ and $m_1$ be non-conjugate along $\gamma(\cdot)$. Then there exists a number $\varepsilon > 0$ and a bounded neighborhood $V$ of the vector $\frac{d}{dt}\gamma(t)_{|t=0}$ in $\mathsf{T}_{m_0}\mathcal{O}$ such that, for any curve $\tilde{u}(t) \in U_\varepsilon \subset C^0([0, 1], \mathsf{T}_{m_0}\mathcal{O})$, the neighborhood $V$ contains a unique vector $\tilde{C}_{\tilde{u}}$, depending continuously on $\tilde{u}$, such that $S(\tilde{u} + \tilde{C}_{\hat{u}})(1) = m_1$.*

Denote by $C$ an upper bound of the norms of vectors $\hat{C}_{\tilde{u}}$ in Lemma 4.

**Lemma 5.** *In conditions and notations of Lemma 4, let the numbers $k > 0$ and $T > 0$ satisfy the inequality $T^{-1}\varepsilon > k$. Then, for any curve $u(t) \in U_k \subset C^0([0, T], \mathsf{T}_{m_0}\mathcal{O})$ in a certain bounded neighborhood of the vector $T^{-1}\frac{d}{dt}\gamma(t)_{|t=0}$ in $\mathsf{T}_{m_0}\mathcal{O}$, there exists a unique vector $C_u$, depending continuously on $u$, such that $S(u + C_u)(T) = m_1$.*

The proofs of Lemmas 4 and 5 are quite analogous to that of Theorem 3.3 in [1] (see also proofs of Lemmas 1 and 2 in [9]). It should be pointed out that, as in Lemma 3, we have $C_v = T^{-1}C_{\tilde{v}}$, where $\tilde{v}(s) = Tv(Ts) \in U_\varepsilon \subset C^0([0, 1], \mathsf{T}_{m_0}\mathcal{O})$. Thus, for $u \in U_k$ from Lemma 5, the inequality $\|C_u\| < T^{-1}C$ holds, where $C$ is introduced after Lemma 4.

**Lemma 6.** *In the conditions and notations of Lemmas 4 and 5, the number $\varepsilon$ can be chosen so that for the curve $\tilde{u}(\cdot) \in U_\varepsilon \subset C^0([0, 1], \mathsf{T}_{m_0}\mathcal{O})$ the vector $\tilde{C}_{\tilde{u}}$ lies inside the light cone of the space $\mathsf{T}_{m_0}\mathsf{M}$ and, for the curve $u \in U_k \subset C^0([0, T], \mathsf{T}_{m_0}\mathcal{O})$, the vector $C_u$ also lies inside the light cone of the space $\mathsf{T}_{m_0}\mathsf{M}$.*

**Proof.** The fact that, for sufficiently small $\varepsilon > 0$, the vector $C_{\tilde{u}}$ belongs to the interior of the light cone, is derived from continuity consideration as in Lemma 2. For $C_u$, this statement follows from the fact that $C_v = T^{-1}C_{\tilde{v}}$, where $\tilde{v}(s) = Tv(Ts) \in U_\varepsilon \subset C^0([0, 1], \mathsf{T}_{m_0}\mathcal{O})$ (see above). ■

Hereafter we choose $\varepsilon$ satisfying the hypotheses of Lemmas 4 and 6.

Let $\gamma(t)$ be a $C^1$-curve given for $t \in [0, T]$, let $X(t, m)$ be a vector field on $\mathcal{O}$. Denote by $\Gamma X(t, \gamma(t))$ the curve in $\mathsf{T}_{\gamma(0)}\mathcal{O}$ obtained by parallel translation of vectors $X(t, \gamma(t))$ along $\gamma(\cdot)$ at the point $\gamma(0)$ with respect to the connection of the reference frame.

In complete analogy with the previous section, we introduce the norm $\|G_m\|$ by means of the norms of vectors with respect to the Riemannian metric on $\mathcal{O}$ as it was mentioned above. Obviously, eq. (2.1) is valid for $\|G_m\|$, as in the previous section.

With the help of $S$ and $\Gamma$ we construct the integral operator $B : U_k \to C^0([0, T], \mathsf{T}_{m_0}\mathsf{M})$ of the following form:

$$Bv = \int_0^\tau \Gamma G_{S(v(t)+C_v)} \left( \frac{d}{dt} S(v(t) + C_v), \frac{d}{dt} S(v(t) + C_v) \right) dt, \tag{3.1}$$

where $k$ and $T$ satisfy the conditions of Lemma 5. One can easily see that the operator $B$ is completely continuious.

**Theorem 3.1.** *Let $m_0, m_1 \in \mathcal{O}$ and let there exist a geodesic $\gamma(\tau)$ of the connection of the reference frame such that $\gamma(0) = m_0$, $\gamma(T) = m_1$, $m_0$ and $m_1$ are not conjugate along $\gamma(\cdot)$ and the vector $\frac{d}{dt}\gamma(t)_{|t=0}$ lies inside the light cone of the space $\mathsf{T}_{m_0}\mathsf{M}$. If $m_0$ and $m_1$ belong to the ball $V \subset \mathcal{O}$, such that at any $m \in V$ the inequality $\|G_m\| < \frac{\varepsilon}{(\varepsilon+C)^2}$ holds, where $\varepsilon$ and $C$ are from Lemmas 4 and 6, then there exists a time-like geodesic $m_0(\tau)$ of the Levi-Civita connection of Lorentz metric on $\mathsf{M}$ such that $m_0(0) = m_0$ and $m_0(T) = m_1$.*

**Proof.** Let $k := T^{-1}\varepsilon - \varphi$, where $\varphi$ is from Lemma 1. For this $k$, the hypothesis of Lemma 5 is satisfied. Hence, on the ball $U_k \subset C^0([0, \mathsf{T}_1], \mathsf{T}_{m_0}\mathsf{M})$, the operator (3.1) is well-posed. Recall that, for the curve $v(\cdot) \in U_k$, its $C^0$-norm is not greater than $k$, that $\|C_v\| < T^{-1}C$ and that the parallel translation with respect to the Riemannian connection preserves the norms of vectors. Then taking into account the definition of operator $S$ we see that

$$\|\frac{d}{d\tau} S(v(\tau) + C_v)\| < (T^{-1}\varepsilon - \varphi) + T^{-1}C.$$

Hence, eq. (2.1), the hypothesis of the theorem and Lemma 1 imply that

$$\begin{aligned} &\|G_{S(v(\tau)+C_v)} \left( \frac{d}{d\tau} S(v(\tau) + C_v), \frac{d}{d\tau} S(v(\tau) + C_v) \right) \| \leq \\ &\quad \|G_{S(v(\tau)+C_v)}\|((T^{-1}\varepsilon - \varphi) + T^{-1}C)^2 < \\ &\quad (T^{-2}\varepsilon - T^{-1}\varphi) \end{aligned} \tag{3.2}$$

Since the operator $\Gamma$ of parallel translation preserves the norms of the vectors, from the last inequality we obtain:

$$\begin{aligned} &\| \int_0^\tau \Gamma G_{S(v(t)+C_v)} \left( \frac{d}{dt} S(v(t) + C_v), \frac{d}{dt} S(v(t) + C_v) \right) dt \| \leq \\ &\quad \int_0^\tau \|G_{S(v(t)+C_v)} \left( \frac{d}{dt} S(v(t) + C_v), \frac{d}{dt} S(v(t) + C_v) \right) \| dt < \\ &\quad (T^{-1}\varepsilon - \varphi) = k. \end{aligned} \tag{3.3}$$

This means that the completely continuous operator $B$ sends the ball $U_k$ into itself. Hence, by Schauder's principle, $B$ has a fixed point $v_0(\tau)$ in $U_k$. Then from the definition of operator $S$ and the usual properties of covariant derivative it follows that $m_0(\tau) = S(v(\tau)+C_v)$ is a solution of differential equation (1) (see [1, 2]). By its construction the curve $m_0(\tau)$ is a geodesic of the Levi-Civita connection of Lorentz metric on $\mathsf{M}$ and, for it, $m_0(0) = m_0$ and $m_0(T) = m_1$.

From the equality $Bv_0 = v_0$ and eq. (3.1) it follows that $v_0(0) = 0$. Then from the definition of operator $S$ it follows that $\frac{d}{d\tau}m_0(\tau)_{|\tau=0} = C_{v_0}$, where the vector $C_{v_0}$ belongs to the light cone of the space $\mathsf{T}_{m_0}\mathsf{M}$ by Lemma 6. This means that the Lorentz scalar square of the vector $C_{v_0}$ is negative. Since $C_{v_0}$ is the initial vector of derivative of the geodesic $m_0(\tau)$ of the Levi-Civita connection on $\mathsf{M}$ and since the Lorentz scalar square of the derivative vector along this geodesic is constant, the geodesic $m_0(\tau)$ is time-like. ■

The meaning of condition $\|G_m\| < \frac{\varepsilon}{(\varepsilon+C)^2}$ is analogous to that in the previous section (see Remark 1). But here $C$ estimates the Riemannian distance between $m_0$ and $m_1$ while $\varepsilon$ depends of the geometry in reference frame and in some sense estimates "how close" $m_1$ is to the boundary of "proper future" of $m_0$ and "how close" it is to conjugate points in the reference frame.

**Acknowledgments.** The research is supported in part by RFBR Grant No. 07-01-00137.

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