

A Linear time-Varying Approximation Method to Nonlinear Structural Dynamic Systems

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Abstract—In order to obtain the response of nonlinear structural dynamic system effectively, a linear, time-varying approximation method is proposed in this paper. The technique is based on the replacement of the original nonlinear system by a sequence of linear time-varying systems, whose solutions will converge to the solution of the nonlinear problem in the global time domain. In this paper, the solutions of a nonlinear time-unvarying system and a nonlinear time-varying system are separately obtained by this iteration approach. This method is powerful for obtaining the accurate response of nonlinear structural system, and the only condition required for its application is a mild Lipschitz condition which must be satisfied by the nonlinear state matrix. Moreover, the choose of initial function is important for changing the convergence rate of this iteration technique. Results indicate that the closer the initial function is to the true nonlinear solution, the sooner this convergence will be achieved.

Keywords—nonlinear; structural dynamics; linear time-varying; global iteration; identification

I. INTRODUCTION

Nonlinear structural dynamics phenomena are widely encountered in real-world complex structures [1-3]. For example, the mass distribution and stiffness of a large flexible space sail-mast would change as it deploys, which introduce nonlinearities to its dynamic model; truss structures with large scale motions could introduce clearance nonlinearities arising from joints connecting individual components and geometric nonlinearities arising from large deflections of oscillating structural members; and structures could be subjected to nonlinear excitations. Research of the nonlinear structural dynamic systems is becoming more and more important in many engineering fields, especially in the applied fields of mechanical engineering and aerospace engineering.

The research of nonlinear structural dynamic systems can be divided into two broad categories: research of forward problems and research of inverse problems. In particular, research of forward problems require a priori knowledge of the characteristics of dynamic system and the system input. It's mainly about the research of solution, stability, bifurcation and chaos of the systems, establishing the equation of motion with related principles of physics and mechanics [4]. Explicit or implicit time integration algorithms are usually adopted to solve the nonlinear structural dynamic response, but strict condition of

stability and strengthen is proposed as the solving accuracy becomes high. On the contrary, research of inverse problems allow both system input and characteristic parameters to be unknown. It is mainly about identification of nonlinear system, including identification of nonlinear time-unvarying and nonlinear time-varying dynamic system. Identification results reflect system input-output relationship or physical parameters such as nonlinear time-varying coefficients, mass, stiffness and damping of the system. Much effort has been devoted in nonlinear time-varying system identification [1], and a great amount of achievement has been witnessed. Nowadays, identification of nonlinear time-varying system is mainly based on time series analysis method and neural network method [5-7], but these methods are not applicable for getting high identification accuracy of complex structural systems such as continua. As we know, identification technique of linear time-varying systems is mature [8-9]. Therefore, it is of great significance to identify a nonlinear time-varying structural dynamic system by translating it into a linear time-varying one.

In this paper, a linear time-varying approximation method is proposed, which is based on the replacement of the original nonlinear structural dynamic system by a sequence of linear time-varying systems, for obtaining global solutions to nonlinear problems. This “global iteration technique” is verified effectively by two examples in this paper.

II. SYSTEM DESCRIPTION AND LINEAR TIME-VARYING APPROXIMATION METHOD

The general motion equation of a nonlinear structural dynamic system can be described as

$$\mathbf{M}(t)\ddot{\mathbf{x}} = \mathbf{f}_{ext}(t) - \mathbf{f}_{int}(\mathbf{x}, \dot{\mathbf{x}}; t) \quad (1)$$

with initial condition: $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(0) = \dot{\mathbf{x}}_0$, where \mathbf{x} is the displacement response of n dimensions; $\mathbf{M}(t)$ is mass matrix of $n \times n$ dimensions; $\mathbf{f}_{ext}(t)$ is the external force to the system; $\mathbf{f}_{int}(\mathbf{x}, \dot{\mathbf{x}}; t)$ is the internal force of the system, which is a linear or nonlinear function of \mathbf{x} and its first derivate $\dot{\mathbf{x}}$.

Suppose that $\mathbf{f}_{int}(\mathbf{x}, \dot{\mathbf{x}}; t)$ is differentiable at $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$, for all t , and $\mathbf{f}_{int}(\mathbf{x}, \dot{\mathbf{x}}; t) \equiv \mathbf{0}$, for all t . This indicates that

the internal force is equal to zero when the system is static. Then $\mathbf{f}_{\text{int}}(\mathbf{x}, \mathbf{x}\dot{t})$ can be written in the form

$$\mathbf{f}_{\text{int}}(\mathbf{x}, \mathbf{x}\dot{t}) = -(\mathbf{A}_1(\mathbf{x}, \mathbf{x}\dot{t})\mathbf{g}\mathbf{x} + \mathbf{A}_2(\mathbf{x}, \mathbf{x}\dot{t})\mathbf{g}\mathbf{x}\dot{t}) \quad (2)$$

for some differentiable, matrix-valued function $\mathbf{A}_1(\mathbf{x}, \mathbf{x}\dot{t})$ and $\mathbf{A}_2(\mathbf{x}, \mathbf{x}\dot{t})$, which are also functions of state variants.

By substitute (2) into (1), the motion equation of the system can be expressed as

$$\mathbf{M}(t)\mathbf{x}\ddot{t} = \mathbf{A}_1(\mathbf{x}, \mathbf{x}\dot{t})\mathbf{g}\mathbf{x} + \mathbf{A}_2(\mathbf{x}, \mathbf{x}\dot{t})\mathbf{g}\mathbf{x}\dot{t} + \mathbf{f}_{\text{ext}}(t) \quad (3)$$

Reduce the equation order of (3), and the state space expression of the dynamic system can be obtained as follows

$$\begin{bmatrix} \mathbf{x}\dot{t} \\ \mathbf{x}\ddot{t} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{n \times n} \\ -\mathbf{M}^{-1}(t)\mathbf{A}_1(\mathbf{x}, \mathbf{x}\dot{t}) & -\mathbf{M}^{-1}(t)\mathbf{A}_2(\mathbf{x}, \mathbf{x}\dot{t}) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}\dot{t} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}(t)\mathbf{f}_{\text{ext}}(t) \end{bmatrix} \quad (4)$$

Suppose that $\mathbf{y} = [\mathbf{x}; \mathbf{x}\dot{t}]$, $\mathbf{u}(t) = [\mathbf{0}; \mathbf{M}^{-1}(t)\mathbf{f}_{\text{ext}}(t)]$, and $\mathbf{A}(\mathbf{y}, t) = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{n \times n} \\ -\mathbf{M}^{-1}(t)\mathbf{A}_1(\mathbf{x}, \mathbf{x}\dot{t}) & -\mathbf{M}^{-1}(t)\mathbf{A}_2(\mathbf{x}, \mathbf{x}\dot{t}) \end{bmatrix}$, then (4) can be described as

$$\mathbf{y}\dot{t} = \mathbf{A}(\mathbf{y}, t)\mathbf{y} + \mathbf{u}(t) \quad (5)$$

with the initial condition: $\mathbf{y}(0) = \mathbf{y}_0 = [\mathbf{x}_0; \mathbf{x}_0\dot{t}]$.

$\mathbf{A}(\mathbf{y}, t)$ is called nonlinear state matrix. If $\mathbf{A}(\mathbf{y}, t)$ satisfies local Lipschitz continuity (the usual minimum assumption for the existence and uniqueness of solutions), the nonlinear system (5) can be approximated by the following sequence of linear, time-varying approximations:

$$\begin{cases} \mathbf{y}\dot{t}^{[1]}(t) = \mathbf{A}(\mathbf{y}_0, t)\mathbf{y}^{[1]}(t) + \mathbf{u}^{[1]}(t), & \mathbf{y}^{[1]}(0) = \mathbf{y}_0 \\ \mathbf{y}\dot{t}^{[i-1]}(t) = \mathbf{A}(\mathbf{y}^{[i-2]}(t), t)\mathbf{y}^{[i-1]}(t) + \mathbf{u}^{[i-1]}(t), & \mathbf{y}^{[i-1]}(0) = \mathbf{y}_0 \\ \mathbf{y}\dot{t}^{[i]}(t) = \mathbf{A}(\mathbf{y}^{[i-1]}(t), t)\mathbf{y}^{[i]}(t) + \mathbf{u}^{[i]}(t), & \mathbf{y}^{[i]}(0) = \mathbf{y}_0 \end{cases} \quad (6)$$

for $i \geq 1$, where the initial function $\mathbf{y}^{[0]}(t)$ is usually taken to be the initial conditions \mathbf{y}_0 , being possible to generalize and use other functions.

The solutions of this sequence, $\{\mathbf{y}^{[i]}(t)\}_{i \geq 1}$, each of which satisfies a linear time-varying system, can be found numerically and converge to the solution of the nonlinear system given in (5), in the sense that

$$\mathbf{y}(t) = \lim_{i \rightarrow \infty} \{\mathbf{y}^{[i]}(t)\} \quad (7)$$

uniformly for t in any compact interval, $[0, \tau]$.

Consider a linear structural dynamic system with constant coefficients, the state space equation of motion can be expressed as

$$\begin{bmatrix} \mathbf{x}\dot{t} \\ \mathbf{x}\ddot{t} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f}(t) \end{bmatrix} \quad (8)$$

where \mathbf{u} and \mathbf{v} are separately displacement vector and velocity vector; \mathbf{M} , \mathbf{C} and \mathbf{K} are separately mass matrix, damping matrix and stiffness matrix; $\mathbf{f}(t)$ is the external force to the system.

According to (4) and (8), equivalent stiffness matrix $\mathbf{K}(t)$ and equivalent damping matrix $\mathbf{C}(t)$ can be obtained as follows

$$\mathbf{K}(t) = \lim_{i \rightarrow \infty} \mathbf{A}_1(\mathbf{y}^{[i]}(t), t) \quad (9)$$

$$\mathbf{C}(t) = \lim_{i \rightarrow \infty} \mathbf{A}_2(\mathbf{y}^{[i]}(t), t) \quad (10)$$

III. NUMERICAL SIMULATIONS

In this section the structural dynamic equations of one nonlinear time-unvarying system and one nonlinear time-varying system are considered and it will be shown how its nonlinear solution vector $\mathbf{x}(t)$ can be approximated by using the "global iteration technique" previously proposed. The displacement responses obtained through this method match nicely the real response.

A. Nonlinear Time-unvarying System

Consider a Van der Pol oscillator with the following governing equation of motion

$$\mathbf{x}\ddot{t} + 0.6(1 - \mathbf{x}(t))\mathbf{x}\dot{t} + 5.96\mathbf{x}(t) = 0$$

with the initial conditions: $\mathbf{x}(0) = 1$ mm and $\mathbf{x}\dot{t}(0) = 1$ mm/s.

Suppose that $\mathbf{y}_1(t) = \mathbf{x}(t)$ and $\mathbf{y}_2(t) = \mathbf{x}\dot{t}(t)$, and the state space expression of the system can be described as

$$\begin{bmatrix} \mathbf{x}\dot{t} \\ \mathbf{x}\ddot{t} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5.96 & -0.6(1 - \mathbf{y}_1^2) \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \quad (11)$$

In order to verify the accuracy of this method proposed previously, the system responses calculated by the Runge Kutta numerical algorithm are regarded as referenced real solution.

Displacement responses and velocity responses are obtained by the "global iteration technique" proposed in this paper after 12 iterations, as shown in Fig .1 and Fig .2 It is easy to see that in this case, the convergence of these linear solutions towards the nonlinear solution is clear, since after 12 iterations they match perfectly the true solutions of the nonlinear time-unvarying system. Therefore, the accuracy of this technique to solve nonlinear time-varying system is guaranteed successfully.

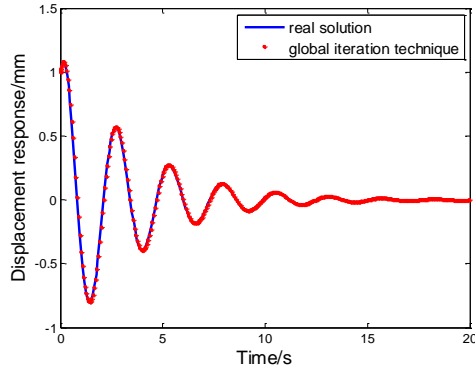


Figure 1. Displacement responses by this "global iteration technique"

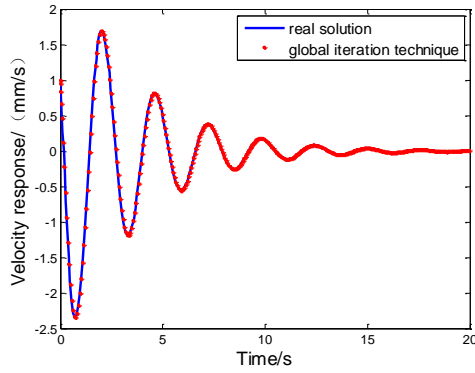


Figure 2. Velocity responses by this "global iteration technique"

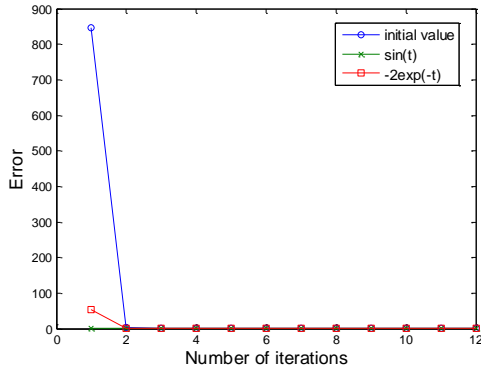


Figure 3. Comparison of the norm of the error for different initial conditions

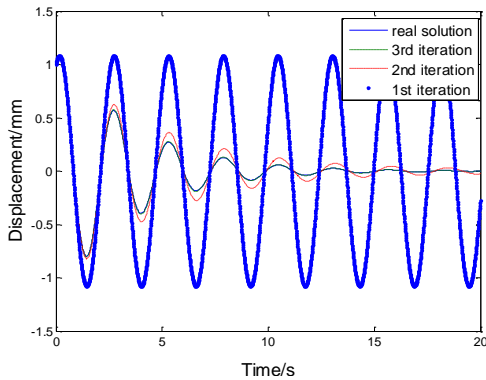


Figure 4. Solutions of different iteration steps by global iteration technique

Fig .3 shows that in the Van der Pol oscillator example, depending on the choice of the initial chosen solution, the error between the approximated solutions $\mathbf{x}^{[i]}(t)$ and the real solution $\mathbf{x}(t)$, converges to zero at a difference rate. For the nonlinear system in this example, the approximated solution matches nicely the real solution after just 2 global iterations, presenting high convergence rate.

For this linear time-varying method, after i iterations, the iterated solutions are close to each other and the $\mathbf{y}^{[i]}(t)$ solution from $\mathbf{y}^{[i]}(t) = \mathbf{A}(\mathbf{y}^{[i-1]}(t), t)\mathbf{y}^{[i]}(t) + \mathbf{u}^{[i]}(t)$ tends to the true nonlinear solution $\mathbf{y}(t)$ so this means that the closer the initial chosen solution is to the true nonlinear solution, the sooner this convergence will be achieved, as shown in Fig .4 These experimental results showed in Fig .3 confirm the statement above: there exists a difference in the convergence rate of the global iteration technique depending on the choice of the initial conditions substituted in the original matrix for the first iteration.

B. Nonlinear Time-varying System

A planar two-link manipulator subjected to a varying end force is used as the example of nonlinear time-varying system, as shown in Fig .5. The links are assumed to be rigid ignoring deformation of links. The elasticity of joint 1 and 2 is represented by two rotational springs k_1 and k_2 , which is nonlinear. The system damping is assumed to be viscous, constant, and concentrated at the joints. The damping coefficients at joints 1 and 2 are c_1 and c_2 , respectively. The angles φ_1 and φ_2 denote the angular positions of the links relative to the x-axis. A time-varying force $f(t)$ acts at the free end of the second link, making the angle φ_3 with the x-axis. φ_3 is assumed to be constant during vibration for simplicity in this paper. A detailed dynamic model and the stability of such system can be found in reference [10]. For the sake of simplicity, it is assumed that the links of the manipulator are uniform cylinders of equal length l and mass m . When an initial disturbance is applied, the links vibrate around their equilibrium positions φ_{10} and φ_{20} . The actual angular position of the links become $\varphi_1 = \varphi_{10} + \varphi_{11}$ and $\varphi_2 = \varphi_{20} + \varphi_{21}$.

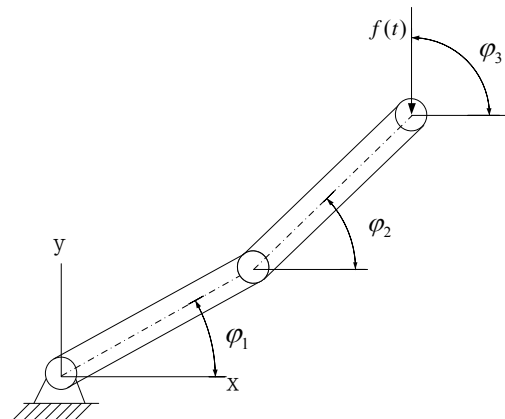


Figure 5. Sketch of a planar two-link manipulator

The governing equation of motion of this system can be expressed as

$$\mathbf{M}\ddot{\boldsymbol{\varphi}} + \mathbf{C}\dot{\boldsymbol{\varphi}} + \mathbf{K}(\boldsymbol{\varphi}, t) = \mathbf{0} \quad (12)$$

in which

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 \cos(\varphi_{10} - \varphi_{20}) \\ a_2 \cos(\varphi_{10} - \varphi_{20}) & a_3 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}$$

$$K_1 = k_1 + k_2 - a_4 \sin \varphi_{10} - f(t)l \cos(\varphi_{10} - \varphi_3),$$

$$K_2 = K_3 = -k_2,$$

$$K_4 = k_2 - a_5 \sin \varphi_{20} - f(t)l \cos(\varphi_{20} - \varphi_3),$$

$$\boldsymbol{\varphi} = [\varphi_{11} \quad \varphi_{21}]^T,$$

$$k_1 = 80(1 + (\cos \varphi_{11})^2) \text{ Nm/rad},$$

$$k_2 = 80(1 + (\cos \varphi_{21})^2) \text{ Nm/rad},$$

$$a_1 = 4ml^2/3, a_2 = ml^2/2, a_3 = ml^2/3,$$

$$a_4 = 3ml^2 g/2, a_5 = ml^2 g/2,$$

$$f(t) = 20 - 10 \sin(\pi(t - 0.5)/3) \text{ N}$$

In the simulation, the following numerical quantities are used: the length $l = 1 \text{ m}$, the mass $m = 1 \text{ kg}$, the damping $c_1 = c_2 = 0.8 \text{ Nm/rad}$, the force application angle $\varphi_3 = 90^\circ$. The testing configuration of the manipulator is set at $\varphi_{10} = 0^\circ$ and $\varphi_{20} = 90^\circ$. The initial conditions of this system are $\boldsymbol{\varphi} = [0 \quad 0]^T$ and $\dot{\boldsymbol{\varphi}} = [1 \quad 1]^T$.

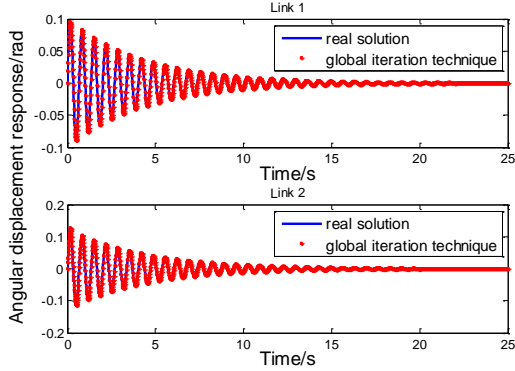


Figure 6. Angular displacement responses by this “global iteration technique”

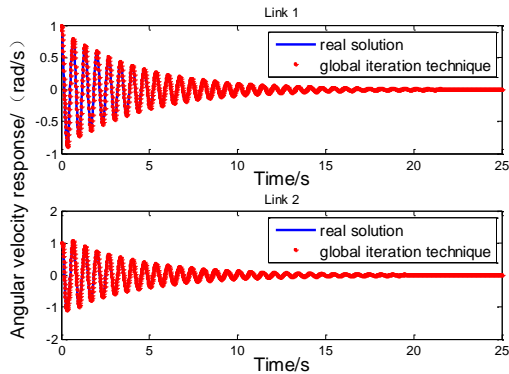


Figure 7. Angular velocity responses by this “global iteration technique”

Similarly to nonlinear time-unvarying systems, the equation order of nonlinear time-varying system needs to be reduced. Angular displacement responses and angular velocity responses of this nonlinear time-varying system are obtained by the “global iteration technique” proposed in this paper after 12 iterations, as shown in Fig .6 and Fig .8 The results demonstrate that the linear time-varying method proposed can accurately solve the nonlinear time-varying structural dynamic problems. In addition, the convergence rate of this system is rapid too, and the norm of the error of angular displacement responses is small enough after 2 iterations, as shown in Fig .8.

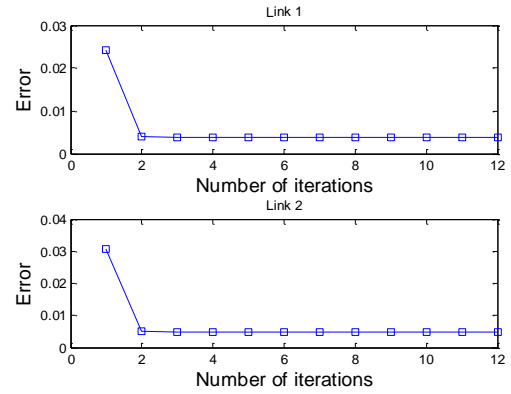


Figure 8. Norm of error of angular displacement for different iteration steps

IV. CONCLUSIONS

1) A linear time-varying method to nonlinear structural dynamic system was proposed in this paper, which is based on replacing the original nonlinear structural dynamic system with a sequence of linear time-varying systems, whose solutions will converge to the solution of the nonlinear problem.

2) The only condition required for this method is just a mild Lipschitz condition which must be satisfied by the nonlinear state matrix associated with the nonlinear system.

3) The closer the initial chosen function is to the true nonlinear solution, the sooner this convergence will be achieved.

4) The accuracy and rationality of this technique was verified by a nonlinear time-unvarying system and a nonlinear time-varying system.

5) How to identify nonlinear structural systems with this technique is a problem which needs further research.

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