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Second-order recursion operators of third-order evolution equations with fourth-order integrating factors

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Abstract We report the recursion operators for a class of symmetry integrable evolution equations of third order which admit a fourth-order integrating factor. Under some assumptions we obtain the complete list of equations, one of which is a special case of the Schwarzian Korteweg-de Vries equation.

We consider the third-order evolution equation

$$u_t = u_{3x} + F(u, u_x, u_{xx}) \tag{1}$$

and require that (1) admits a recursion operator of the form

$$R[u] = D_x^2 + G_1 D_x + G_0 + I_1 D_x^{-1} \circ J_1 + I_2 D_x^{-1} \circ J_2 \tag{2}$$

where $G_j = G_j(u, u_x, u_{xx}, \ldots)$, I_i are Lie point symmetries and J_i integrating factors for (1), with

$$\frac{\partial J_i}{\partial u_{4x}} \neq 0 \tag{3}$$

for J_1 and/or J_2 such that

$$J_i = \hat{E}_u \Phi_i^t.$$

Here $\hat{E}_u$ is the Euler operator

$$\hat{E}_u = \frac{\partial}{\partial u} - D_x \circ \frac{\partial}{\partial u_x} + D_x^2 \circ \frac{\partial}{\partial u_{2x}} - \cdots$$

and Φ^t a conserved density for the evolution equation. We recall that if (1) admits a recursion operator and an infinite hierarchy of higher order conservation laws in involution,

$$D_t\Phi_i^t + D_x\Phi_i^x = 0,$$

then (1) is said to be symmetry integrable [2].

As usual subscripts denote partial derivatives, so that

$$u_x = \frac{\partial u}{\partial x}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{3x} = \frac{\partial^3 u}{\partial x^3}, \quad \dots$$

Proposition: Equations of the form (1) which admit recursion operators of the form (2) under condition (3) are exhausted by the following three cases (all λ's are arbitrary constants and c is an arbitrary but nonzero constant):

1. The equation

$$u_t = u_{3x} - \frac{3}{2}u_x^{-1}u_{xx}^2 + \lambda_1 u_x^{-1} + \lambda_2 u_x^3 + \lambda_3 u_x + \lambda_0, \tag{4}$$

admits the recursion operator

$$\begin{aligned} R[u] = {} & D_x^2 - 2\left(\frac{u_{xx}}{u_x}\right)D_x + \frac{u_{3x}}{u_x} - \frac{u_{xx}^2}{u_x^2} - \frac{2\lambda_1}{3u_x^2} + 2\lambda_2 u_x^2 + k_0 \\ & - u_x D_x^{-1}\circ\left(\frac{u_{4x}}{u_x^2} - 4\frac{u_{xx}u_{3x}}{u_x^3} + 3\frac{u_{xx}^3}{u_x^4} + 2\lambda_2 u_{xx} - 2\lambda_1\frac{u_{xx}}{u_x^4}\right) \\ & - \frac{8}{3}\lambda_1 D_x^{-1}\circ\left(\frac{u_{xx}}{u_x^3}\right). \end{aligned} \tag{5}$$

2. The equation

$$u_t = u_{3x} - \frac{3}{2}\left(\frac{u_x}{u_x^2 - c}\right)u_{xx}^2 + \lambda_1(u_x^2 - c)^{3/2} + \lambda_2 u_x^3 + \lambda_3 u_x + \lambda_0 \tag{6}$$

with $c \neq 0$ admits the recursion operator

$$\begin{aligned} R[u] = {} & D_x^2 - 2\left(\frac{u_x u_{xx}}{u_x^2 - c}\right)D_x + \frac{u_x u_{3x}}{u_x^2 - c} - \frac{u_x^2 u_{xx}^2}{(u_x^2 - c)^2} + 2\lambda_1 u_x(u_x^2 - c)^{1/2} \\ & + 2\lambda_2 u_x^2 + \frac{2}{3}\lambda_3 + k_0 - \lambda_1 c^2 D_x^{-1}\circ\left(\frac{u_{xx}}{(u_x^2 - c)^{3/2}}\right) \\ & - u_x D_x^{-1}\circ\left(\frac{u_{4x}}{u_x^2 - c} - \frac{4u_x u_{xx} u_{3x}}{(u_x^2 - c)^2} + \frac{(3u_x^2 + c)u_{xx}^3}{(u_x^2 - c)^3}\right. \\ & \left. + \lambda_1\frac{(2u_x^2 - 3c)u_x u_{xx}}{(u_x^2 - c)^{3/2}} + 2\lambda_2 u_{xx}\right). \end{aligned} \tag{7}$$

3. The equation

$$u_t = u_{3x} - \frac{3}{4} u_x^{-1} u_{xx}^2 + \lambda_1 u_x^{3/2} + \lambda_2 u_x^2 + \lambda_3 u_x + \lambda_0, \tag{8}$$

admits the recursion operator

$$\begin{aligned} R[u] &= D_x^2 - \left(\frac{u_{xx}}{u_x}\right) D_x + \frac{1}{2}\frac{u_{3x}}{u_x} - \frac{1}{4}\frac{u_{xx}^2}{u_x^2} + \lambda_1 u_x^{1/2} + \frac{4\lambda_2}{3} u_x + k_0 \\ &-\frac{1}{2} D_x^{-1} \circ \left(\frac{u_{4x}}{u_x} - 2\frac{u_{xx}u_{3x}}{u_x^2} + \frac{u_{xx}^3}{u_x^3} + \frac{3\lambda_1}{4}\frac{u_{xx}}{u_x^{1/2}} + \frac{4\lambda_2}{3} u_{xx} \right) \\ &+\frac{\lambda_1}{8} u_x D_x^{-1} \circ \left(\frac{u_{xx}}{u_x^{3/2}} \right). \end{aligned} \tag{9}$$

Some remarks are in order:

Equation (4) is listed in [3] as eq. (4.1.16). The recursion operator for the case $\lambda_1 = \lambda_2 = \lambda_3 = 0$ is given in [1]. With all λ's zero (4) is known as the Schwazian KdV equation. Equation (6) with $c = -1$ is listed in [3] as eq. (4.1.14). For (6) with $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_0 = 0$ and $c = \pm 1$ the recursion operator is given in [4]. Equation (8) is listed in [3] as eq. (4.1.13).

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