3-D Steady Temperature Field in a Laminated Rectangular Plate

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Abstract—According to the exact three-dimensional (3D) thermal theory, the steady temperature distribution in a laminated rectangular plate with zero temperature conditions on four lateral surfaces was studied. An analytical method was developed to solve the temperature field in the plate. Firstly, the general solution of the temperature field in a single-layer rectangular plate, which exactly satisfies the governing thermal differential equation, was derived out. Then, the temperature and heat flux relationships between the upper surface and the lower surface of the single-layer plate were obtained. Based on the continuity of the temperature and the heat flux on the interface of two adjacent layers, the temperature and the heat flux between the lowest layer and the top layer of the laminated plate were recursively obtained by using the transfer matrix method. The unknown coefficients in the solutions for every layer were uniquely determined by the use of the temperature conditions at the upper and lower surfaces of the plate. The temperature distribution in the laminated plate was given by substituting the unknown coefficients obtained back to the recurrent formulae and the solutions. The convergence of the solutions has been checked based on the number of series term. Comparing the results with those obtained from the finite element method, the accuracy and correctness of the present method were demonstrated. Finally, the effects of surface temperatures, thickness, layer number and material properties of the plate on the temperature distribution were discussed in detail.

Keywords-Three-dimensional temperature field;Laminated plate; Exact solution; Displacement and stress; Transfer matrix method

I. INTRODUCTION

In recent years, laminated plates, especially subjected to thermal loads, have received a considerable attention. The stresses and non-uniform deformations appear in the laminated plate even for a complete free laminated plate because of the inhomogeneous thermal expansion coefficients in the plate. Therefore, the study on temperature distribution in laminated plates has particular importance for the structural safety analysis.

Some analytical solutions were reported. Delouei et al. [1] presented an exact analytical solution for transient heat conduction in cylindrical multilayered composites. Kayhani et al. [2] presented a steady analytical solution for heat conduction in a cylindrical multilayered laminate with different fiber directions among layers. The Sturm-Liouville theorem was used to derive the appropriate Fourier transformation. Beck et al. [3] proved that in some cases, by means of the Kayhani' approach the poorly-convergent or non-convergent series can be replaced by the closed-form algebraic solutions. Ma and Chang [4] analyzed the steady-state temperature field and heat flux field in a multi-layered media with anisotropic properties subjected to surface temperature. Savoia and Reddy [5] considered the polynomial and exponential temperature distributions through the thickness and presented the temperature analysis for multilayered plates subjected to thermal loads. Hsieh and Ma [6] provided the analytical solution for heat conduction in an anisotropic thin-layer media with embedded heat sources. Haji-Sheikh et al. [7] presented the mathematical formulation of the steady-state temperature field in multi-dimensional and multi-layer bodies.Norouzi et al. [8] presented an exact analytical solution for steady conductive heat transfer in multilayer spherical fiber reinforced composite laminates. Kayhani et al. [9] gave a steady analytical solution for heat conduction in a cylindrical multilayer composite laminate where the fiber direction may vary between layers.

In this paper, we use the exact three dimensional (3D) thermal theory to study the steady temperature distribution in a laminated rectangular plate with zero temperature conditions on four lateral surfaces. Firstly, the general solutions of the temperature distribution in a single-layer rectangular plate are derived out. Then, the formulae of temperature and heat flux between two adjacent layers are obtained based on the continuity of the temperature and the heat flux on the interface of two adjacent layers, which can be provided recursively from the lower surface to the upper surface of the laminated plate by using the transfer matrix method. Finally, the unknown coefficients are determined by using the upper surface and lower surface conditions of the laminated plate. Excellent convergence has been achieved and the numerical results agree well with the finite element solutions.

II. BASIC EQUATIONS AND SOLUTION OF SINGLE LAYER

Consider a laminated rectangular plate with length *a*, width *b* and thickness *h* ($h=\sum_{i=1}^{p} h_i$), as shown in Fig .1.

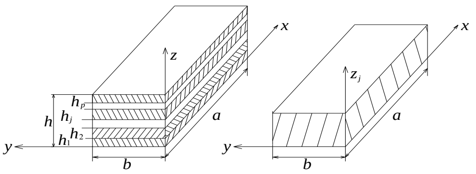

Figure 1. Cartesian coordinates of the laminated plate

The plate is composed of *p* layers. For the *i*th layer, its thickness is h_i and thermal conductivity is K_i. We consider the plate having a constant temperature value on four lateral surfaces and take this value as the datum mark of temperature field in the plate. Without losing the generality, we can further think the temperatures on four lateral surfaces to be zero. The upper surface and lower surface of the plate are subjected to steady temperatures load $t_2(x)$ and $t_1(x)$, respectively.

In the local Cartesian coordinate system xyz_i, the heat conduction equation [10] in the layer *i* is:

$$\frac{\partial^2 T_i(x,y,z_i)}{\partial x^2}+\frac{\partial^2 T_i(x,y,z_i)}{\partial y^2}+\frac{\partial^2 T_i(x,y,z_i)}{\partial {z_i}^2}=0 \tag{1}$$

where $T_i(x,y,z_i)$ is the temperature distribution in the ith layer under the local coordinates x-y-z_i with origin at the lower right corner of the layer. The temperatures on four lateral surfaces are:

$$T_i(0,y,z_i)=T(a,y,z_i)=0,\quad T_i(x,0,z_i)=T(x,b,z_i)=0; \tag{2}$$

The relationships of the temperature and the heat flux on the interface of two adjacent layers are (i=1, 2, … , p):

$$\begin{cases} T_{i+1}(x,y,z_{i+1})\Big|_{z_{i+1}=0}=T_i(x,y,z_i)\Big|_{z_i=h_i} \\ k_{i+1}\dfrac{\partial T(x,y,z_{i+1})}{\partial z_{i+1}}\Bigg|_{z_{i+1}=0}=k_i\dfrac{\partial T(x,y,z_i)}{\partial z_i}\Bigg|_{z_i=h_i} \end{cases} \tag{3}$$

The solution of Equation (1) can be given in the double sinusoidal series form of

$$T_i(x,y,z_i)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} t_{mn}^i(z_i)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{4}$$

It is obvious that Equation (4) satisfies Equation (2) exactly.

Substituting Equation (4) into Equation (1), we obtain a series of ordinary differential equations of second order with constant coefficients about the coordinate z_i (i=1, 2,…,p). Finally, $T_i(x,y,z_i)$ can be worked out:

$$T_i(x,y,z_i)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(e^{\alpha_{mn}z_i}E_{mn}^i+e^{-\alpha_{mn}z_i}F_{mn}^i)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{5}$$

$$\alpha_{mn}=\sqrt{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}}$$

where, $E_{mn}{}^i$ and $F_{mn}{}^i$ are the unknown coefficients which can be determined by the temperature conditions on the upper and lower surfaces of the ith layer plate.

III. RECURSIVE FORMULAE FOR TEMPERATURE AND HEAT FLUX

From the above analysis, the temperature and heat flux in the plate can be described in the compact form, i.e.

$$\begin{bmatrix} T(x,y,z_i) \\ k_i\dfrac{\partial T(x,y,z_i)}{\partial z_i} \end{bmatrix}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\phi_{mn}^i(z_i)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{6}$$

where $\phi_{mn}^i(z_i)$ is the unknown functions expressed in the column form about the local coordinate z_i . According to Equation (4), $\phi_{mn}^i(z_i)$ can be described as

$$\phi_{mn}^i(z_i)=\varphi_{mn}^i(z_i)\Lambda_{mn}^i \tag{7}$$

in which, $\varphi_{mn}^i(z_i)$ and Λ_{mn}^i are

$$\varphi_{mn}^i(z_i)=\begin{bmatrix} e^{\alpha_{mn}z_i} & e^{-\alpha_{mn}z_i} \\ k_i\alpha_{mn}e^{\alpha_{mn}z_i} & -k_i\alpha_{mn}e^{-\alpha_{mn}z_i} \end{bmatrix},\ \Lambda_{mn}^i=\begin{bmatrix} E_{mn}^i \\ F_{mn}^i \end{bmatrix}$$

From Equation (7), the relationship between the upper surface and the lower surface in the *i*th layer can be obtained as follows:

$$\phi_{mn}^i(h_i)=\varphi_{mn}^i(h_i)\varphi_{mn}^i(0)^{-1}\phi_{mn}^i(0) \tag{8}$$

Based on the continuity of temperature and heat flux at the interface of two adjacent layers from Equation (3), one has

$$\phi_{mn}^i(h_i)=\phi_{mn}^{i+1}(0) \tag{9}$$

Thus, the temperature and the heat flux relationships between the lowest layer and the layer q (q=2, 3…p) are recursively obtained:

$$\begin{bmatrix} E_{mn}^q \\ F_{mn}^q \end{bmatrix}=[\varphi_{mn}^q(h_q)]^{-1}\left\{\prod_{j=q}^{1}\left[\varphi_{mn}^j(h_j)\varphi_{mn}^j(0)^{-1}\right]\right\}\varphi_{mn}^1(0)\begin{bmatrix} E_{mn}^1 \\ F_{mn}^1 \end{bmatrix} \tag{10}$$

IV. UNKNOWN COEFFICIENTS FOR TEMPERATURE FIELD

Consider that the upper and lower surfaces of the laminated plate is subjected to the steady state temperature loads $t_2(x, y)$ and $t_1(x, y)$ respectively, i.e.

$$T_1(x,y,0)=t_1(x,y),\quad T_p(x,y,h_p)=t_2(x,y) \tag{11}$$

We multiply Equation (11) by $\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}$ and integrate over x and y respectively:

$$\int_0^a\int_0^b T_1(x,y,0)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dydx=\int_0^a\int_0^b t_1(x,y)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dydx,$$

$$\int_0^a\int_0^b T_p(x,y,h_p)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dydx=\int_0^a\int_0^b t_2(x,y)\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}dydx \tag{12}$$

Substituting Equation (5) into Equation (12) gives

$$E_{mn}^1+F_{mn}^1=\omega_{mn}^1,\quad e^{\alpha_{mn}h_p}E_{mn}^p+e^{-\alpha_{mn}h_p}F_{mn}^p=\omega_{mn}^2 \tag{13}$$

where ω_{mn}^1 and ω_{mn}^2 can be expressed by:

$$\omega_{mn}^1=\frac{4}{ab}\int_0^a\int_0^b t_1(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy,$$

$$\omega_{mn}^2=\frac{4}{ab}\int_0^a\int_0^b t_2(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy \tag{14}$$

Simultaneously solving Equation (10) (taking q=p) and Equation (13), E_{mn}^1, F_{mn}^1, E_{mn}^p and F_{mn}^p can be uniquely determined. Taking E_{mn}^1 and F_{mn}^1 back to Equation (10), E_{mn}^q and F_{mn}^q (q=2, 3…p-1) for each layer can be solved. Finally, substituting the coefficients back into Equation (5) yields the temperature field within the laminated plate.

V. CONVERGENCE AND COMPARISON STUDIES

In order to verify the accuracy and correctness of the present method, some numerical calculations for temperature field are carried out. In the following numerical studies, all the computations were performed in double precision. We take the three-layered plates as example, which are widely used in various engineering. The top and lowest layers of the plate are made up of steel while the core layer is made up of concrete. The length is a=10m, the width is b=10m and the layer thicknesses are h_1=0.1m, h_2=0.8m, h_3=0.1m, respectively. The thermal conductivities for every layer are k_1=75.4 W/(m • °C), k_2=2.33 W/(m • °C), k_3=75.4 W/(m • °C). The upper and lower surfaces of the plate are subjected to different uniform steady-state temperature loads: $t_2(x, y)$ = 100°C and $t_1(x, y)$ = 20°C.

Six different terms N=5, 10, 15, 20, 25, 30 have been checked. Table 1 gives the solutions of temperature at x=2.7m, y=4.3m, z=0.05m; z=0.1m; z=0.3m; z=0.7m; z=0.9m; z=0.95m, respectively. It can be seen from Table 1 that the numerical results converge quickly with the increase of the series terms. The results for N=30 are the same as those for N=25 with three significance digits. This indicates an excellent convergence of the proposed method. Therefore, the number of terms of the Fourier series is fixed at N=25 in the following numerical computations.

Meanwhile, a finite element (FE) simulation using ANSYS has been carried out to verify the accuracy of the proposed method. Table 2 shows the comparison studies of the temperature at the points along the thickness: z=0.05m; z=0.1m; z=0.3m; z=0.7m; z=0.9m; z=0.95m with x=2.7, y= 4.3; x=1, y=2.5; x=1, y=0.7, respectively. It can be seen from Table 2 that the present solutions agree closely with the FE solutions. This validates the correctness of the proposed method.

Table 1 Convergence studies of temperature field

N	z=0.05	z=0.1	z=0.3	z=0.7	z=0.9	z=0.95
5	10.5	13.2	25.2	74.4	94.2	95.4
10	18.7	16.2	36.7	74.7	98.9	97.6
15	19.5	18.9	39.5	74.9	99.5	99.7
20	20.1	19.7	40.1	78.9	99.4	99.6
25	20.2	20.3	40.2	79.8	99.7	99.8
30	20.2	20.3	40.2	79.8	99.7	99.8

Table 2 Comparison studies of the temperature field from present solutions with FE solutions

Method	z=0.05	z=0.1	z=0.3	z=0.7	z=0.9	z=0.95
			x= 2.7, y= 4.3			
Present	20.2	20.3	40.2	79.8	99.7	99.8
ANSYS	20.1	20.5	39.9	80.1	99.7	99.8
			x= 1, y= 2.5			
Present	20.1	20.3	38.9	78.6	99.7	99.8
ANSYS	20.1	20.4	38.7	78.8	99.7	99.8
			x= 1, y= 0.7			
Present	20.1	20.2	35.6	74.9	99.7	99.8
ANSYS	20.1	20.3	35.4	75.1	99.7	99.8

VI. NUMERICAL EXAMPLES

In this section, three numerical examples are presented to show the applicability of the proposed method.

The first example is still the three-layered rectangular plate considered above. The upper surface of the plate is subjected to different uniform steady-state temperatures: $t_2(x, y)$ =30°C, 100°C, 200°C, respectively. The lower surface of the plate is subjected to the fixed temperature: $t_1(x, y)$ = 20°C. Fig .2 shows the temperature distribution along the y direction at x=2.3, z=0.3. It can be seen from Fig .2 that the temperature of the plate increases with the increase of the temperature on the upper surface except for the temperature at the edges. In Fig .3, the distribution of temperature along the thickness at x=2.7, y=4.3 is given. We can find that the slopes of the temperature change within the top and bottom layers of the plate are lower than that within the core layer. The reason is that the thermal conductivity in the top and bottom layers of the plate is larger than that in the core layer.

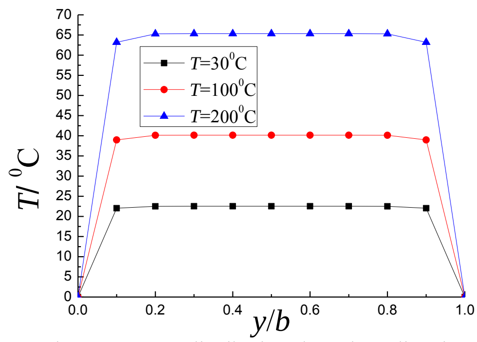

Figure 2. The temperature distribution along the y direction at x=2.3, z=0.3 for different boundary temperatures

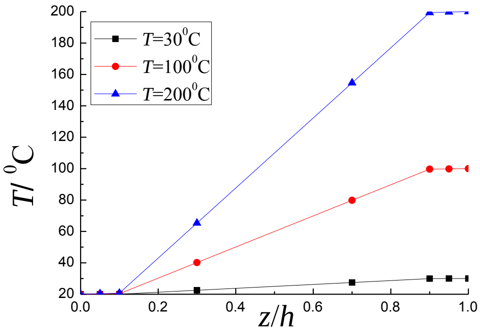

Figure 3. The temperature distribution along the thickness at x=2.7, y=4.3 for different boundary temperatures

The second example is the laminated rectangular plate of three layers with a=b=10m. The thickness is h and $h_1 = h_3 = 1/10h$, $h_2 = 4/5h$. Three different plate thicknesses are considered: h=1m, 2m, 4m, i.e. h/a=0.1, 0.2, 0.4, respectively. The upper surface of the plate is subjected to the uniform steady-state temperature: $t_2(x, y)$ = 100^{o}C. The lower surface of the plate is subjected to the temperature: $t_1(x, y)$ = 20^{o}C. Fig .4 shows the temperature distribution along the thickness at x=2.7, y=4.3. It can be seen from Fig .4 that for the thin plates with h/a=0.1 and h/a=0.2 the temperature variation along the plate thickness within the core are almost the same. However, for the thick plates such as h/a=0.4 the temperature variation within the core is obviously different from the thin plates. In Fig .5, the distribution of temperature along the y direction at x=2.3, z/h=0.3 is studied. It is seen that the temperature distribution is symmetrical in the y direction. We can find from Fig .5 that the temperature variation near to the edges is more remarkable that that within the interior of the plate especially for the thin plates.

The third example is for the comparison of the temperature distribution along the thickness and the length for an isotropic plate and two laminated rectangular plates. They are made up of three kinds of materials: the wood with k_1=0.1 W/(m • ^{o}C); the steel with k_2=50 W/(m • ^{o}C); the concrete with k_3=2 W/(m • ^{o}C). All the plates have the same length and width a=b=10m and the same thickness h=3m. The first plate is a single-layer rectangular plate made up of steel with the thickness h=3m. The second plate is a two-layer plate with h_1= h_2=1.5m. The upper layer of the plate is made up of steel and the lower layer of the plate is made up of wood. The third plate is a three-layer plate with h_1= h_2= h_3=1m. The top layer of the plate is made up of steel. The bottom layer of the plate is made up of wood. The core layer of the plate is made up of concrete. The upper and lower surfaces of the plate are subjected to different uniform steady-state temperatures: $t_2(x, y)$ = 100^{o}C and $t_1(x, y)$ = 20^{o}C. The temperature distribution at x=2.7, y=4.3 are given in Fig .6. It can be seen from Fig .6 that for different laminated materials, the temperature distributions are different even the plate has the same sizes. For the single-layer plate, the variation of the temperature along the plate thickness is almost linear. However for the laminated plates, the slopes of temperature variation in the different layers are different and the temperature variation in a layer can be nonlinear. In Fig .7, the temperature distribution along the y direction at x=2.3, z=2 is studied. It can be found from Fig .7 that the material properties of the plate have important effect on the temperature variation. The temperature in the plate made up of steel is the lowest while the temperature in the plate made up of three kinds of materials is the highest. The reason is that the thermal conductivity of the steel is larger than the wood and the concrete.

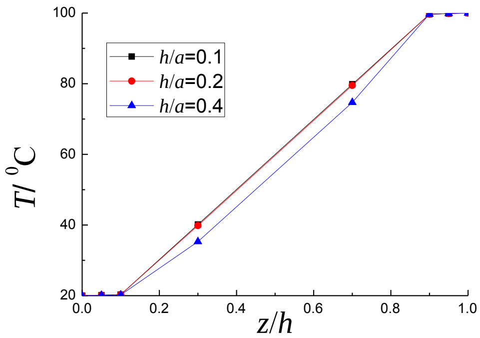

Figure 4. The temperature distribution along the thickness at x=2.7, y=4.3 for different thickness-width ratios h/a

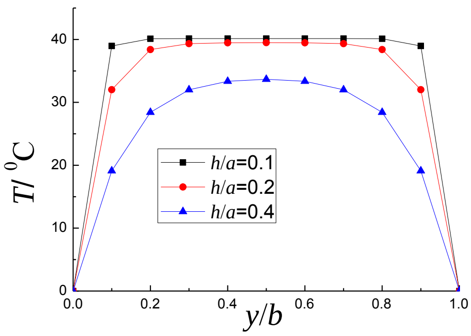

Figure 5. The temperature distribution along the y direction at x=2.3, z/h= 0.3 for different thickness-width ratios h/a

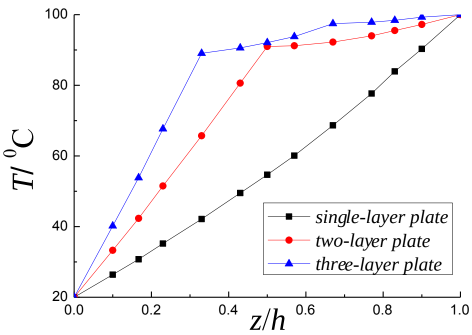

Figure 6. The temperature distribution along the thickness direction at x=2.7, y=4.3 for the plate made up of different materials

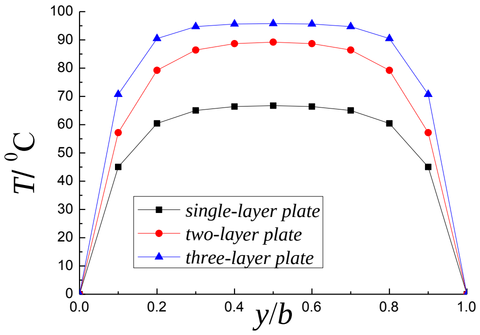

Figure 7. The temperature distribution along the y direction at x=2.3, z=2 for the plate made up of different materials

VII. CONCLUSIONS

The three-dimensional temperature field within a simply supported laminated plate has been investigated based on the exact three dimensional (3D) thermal theory. An analytical method is shown to get the temperature field in the plate. Firstly, the general solutions of a single-layer simply supported rectangular plate, which satisfies the governing differential equations and the thermal boundary conditions at the lateral edges of the plate, is derived. Then, the temperature and heat flux relationships between the upper surface and the lower surface in the single-layer plate are obtained. According to the continuity of the temperature and the heat flux on the interface of two adjacent layers, the recursive formulae of the temperature and the heat flux between the lowest layer and the top layer of laminated plate are derived out by using the transfer matrix method. Finally, the unknown coefficients in the solutions are determined by the use of the upper and lower surface conditions of the laminated plate. The distribution of temperatures in the plate is gained by substituting the unknown coefficients back to the recurrent formulae and the solutions. The solution obtained shows excellent convergence properties. The present method shows a good convergence. Comparing the numerical results with those gained from the finite element method, the accuracy and correctness of the present method are demonstrated. Finally, the effects of temperatures, thickness, layer number and material properties on the temperature distribution are discussed in detail. It is shown from the numerical results that the temperature solution increases with the increase of the temperature. The thickness, layer number and material properties have a significant effect on the temperature distribution in the laminated plates.

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