Journal of Nonlinear Mathematical Physics Volume 14, Number 3 (2007), 462–473 Article

Partial integrability of the anharmonic oscillator

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Received January 29, 2007; Accepted in Revised Form April 9, 2007

Abstract

We consider the anharmonic oscillator with an arbitrary-degree anharmonicity, a damping term and a forcing term, all coefficients being time-dependent:

$$u'' + g_1(x)u' + g_2(x)u + g_3(x)u^n + g_4(x) = 0, \qquad n \text{ real}.$$

Its physical applications range from the atomic Thomas-Fermi model to Emden gas dynamics equilibria, the Duffing oscillator and numerous dynamical systems. The present work is an overview which includes and generalizes all previously known results of partial integrability of this oscillator. We give the most general two conditions on the coefficients under which a first integral of a particular type exists. A natural interpretation is given for the two conditions. We compare these two conditions with those provided by the Painlevé analysis.

1 Introduction

The harmonic oscillator is the simplest approximation to a physical oscillator and, when perturbation terms are taken into account, the resulting *anharmonic oscillator* is governed by the nonlinear differential equation

$$E \equiv u'' + g_1 u' + g_2 u + g_3 u^n + g_4 = 0, \; n(n-1)g_3 \neq 0, \tag{1.1}$$

where $'$ denotes the derivative with respect to the independent time or space variable x, $g_1(x)$ a damping factor, $g_2(x)$ a time-dependent frequency coefficient, $g_3(x)$ the simplest possible anharmonic term, $g_4(x)$ a forcing term.

As to the anharmonicity exponent n, it can be either real if $u(x)$ is real positive, which is the case for Lane-Emden [24] gas dynamics equilibria, rational, like $n = 3/2$ in the Thomas and Fermi [31, 17] atomic model, or more usually integer: -3 for the Ermakov [12] or Pinney [28] equation, 3 for the Duffing oscillator [10].

For generic values of the coefficients, this equation is equivalent to a third order autonomous dynamical system, which generically admits no closed form general solution. The purpose of this article is to review all the nongeneric situations for which there exist exact analytic results, such as a first integral or a closed form solution, either particular or general. This can only happen when the coefficients satisfy some constraints.

The paper is organized as follows. In Section 2, we give a Lagrangian and a Hamiltonian formulation for any value of the coefficients (n, g_i). This generalizes all the previous particular results, obtained for values of (n, g_1, g_2, g_3, g_4) equal to:

$(5; 0, \text{const}, \text{const}, 0)$ [3],

$(5; 2/x, 0, 1, 0)$ [26, Eq. (3.7)],

$(n; g_1, 0, g_3, 0)$ [30, 29, 16],

$(n; 0, \text{const}, ax^\alpha, 0)$ [1],

$(n; g_1, 0, 1, 0)$ [25],

$(n; g_1, g_2, g_3, 0)$ [15], [23, Section 6.74, vol. 1].

In Section 3, we provide two conditions on (n, g_i) which are sufficient to ensure the existence of a first integral.

In Section 4, we give a natural interpretation of these two conditions.

Finally, in section 5, we perform the Painlevé analysis of (1.1). Most of this work has already been done by Painlevé and Gambier [19]. Indeed, the ordinary differential equation (ODE) (1.1) belongs, at least for specific values of n and maybe after a change $u \mapsto u^N$ of the dependent variable u in case n is not an integer, to the class of second order ODEs which they studied and classified. However, as opposed to these classical authors, we do not request the full Painlevé integrability of the ODE, only some partial integrability, and this requires some additional work. In particular, we compute the condition for the absence of any infinite movable branching, i.e. a multivaluedness which occurs at a location depending on the initial conditions. Such a condition, like for linear ODEs, arises from any integer value of the difference of the two Fuchs indices, whether positive or negative, and we check that this condition is a differential consequence of the two conditions for the existence of a particular first integral. This detailed Painlevé analysis of equation (1.1) happens to be an excellent example for several features of Painlevé analysis which are most of the time overlooked.

For convenience, we use the notation

$$\operatorname{Log} G_1(x) = \int^x g_1(t)\mathrm{d}t, \qquad \gamma_3 = \operatorname{Log} g_3, \qquad \gamma_4 = \operatorname{Log} g_4, \tag{1.2}$$

and the convention that function G_1 implicitly contains an arbitrary multiplicative constant; letter K, with or without subscript, denotes an arbitrary constant. Function G_1 frequently occurs, for the way to suppress term $g_1 u'$ in (1.1) is to perform the change of function $u \to G_1^{-1/2} u$.

2 Lagrangian and Hamiltonian formulations

For every value of (n, g_i), including the logarithmic case $n = -1$, the anharmonic oscillator can be put in Lagrangian form

$$\left(\frac{\partial L}{\partial u'}\right)' - \frac{\partial L}{\partial u} = 0, \tag{2.1}$$

or in Hamiltonian form

$$q' = \frac{\partial H}{\partial p},\ p' = -\frac{\partial H}{\partial q}, \tag{2.2}$$

as shown by the explicit expressions for L, H, q, p

$$L(u,u',x) = G_1\left[u'^2 - 2g_3\int_0^u u^n \mathrm{d}u - g_2u^2 - 2g_4u\right] + \frac{1}{2}\left(hu^2\right)', \tag{2.3}$$

$$H(q,p,x) = G_1\left[u'^2 + 2g_3\int_0^u u^n \mathrm{d}u + g_2u^2 + 2g_4u\right] - \frac{1}{2}h'u^2, \tag{2.4}$$

$$q = u,\ p = 2G_1u' + hu, \tag{2.5}$$

in which h is an arbitrary gauge function of x.

3 Particular first integral

According to Noether theorem, one can find first integrals by looking at the infinitesimal symmetries of the Lagrangian. For a detailed review of this Lie symmetries approach to the anharmonic oscillator, the interested reader can refer to [14]. Since the dependence of ODE (1.1) in u is rather simple, let us determine under which conditions on parameters (n, g_i) there exists a particular first integral containing the same kind of terms than the Hamiltonian

$$I = f_1u'^2 + f_2\int_0^u u^n \mathrm{d}u + f_3uu' + f_4u^2 + f_5u + f_6, \tag{3.1}$$

in which the six functions f_i of x are to be determined.

Eliminating u'' between I' and E, we obtain

$$\begin{aligned} I' - (2f_1u' + f_3u)E \equiv\, & f_2'\int_0^u u^n \mathrm{d}u - g_3f_3u^{n+1} + (f_2 - 2g_3f_1)u^nu' + f_6' \\ & +(f_1' + f_3 - 2g_1f_1)u'^2 + (f_3' + 2f_4 - g_1f_3 - 2g_2f_1)uu' \\ & +(f_4' - g_2f_3)u^2 + (f_5 - 2g_4f_1)u' + (f_5' - g_4f_3)u. \end{aligned} \tag{3.2}$$

Out of the nine monomials $\int_0^u u^n \mathrm{d}u, u^{n+1}, u^nu', u'^2, u^2, uu', u', u, 1$, only eight are linearly independent since $n(n-1) \neq 0$, thus generating eight linear homogeneous differential equations in six unknowns, hence generically two conditions on (n, g_i). Note that, even in the logarithmic case $n = -1$, the first generated equation is $f_2' - (n+1)g_3f_3 = 0$. Functions f_2 to f_6 are given by

$$f_2 = 2g_3f_1, \tag{3.3}$$

$$f_3 = 2g_1f_1 - f_1', \tag{3.4}$$

$$f_4 = (g_1^2 + g_2 - g_1')f_1 - \frac{3}{2}g_1f_1' + \frac{1}{2}f_1'', \tag{3.5}$$

$$f_5 = 2g_4f_1, \tag{3.6}$$

$$f_6 = \delta_{n,-1}\int^x g_3f_3\mathrm{d}x, \tag{3.7}$$

and function f_1 must be a nonzero solution common to the three linear equations

$$\left[-2(n+1)g_1g_3 + 2g_3'\right]f_1 + (n+3)g_3f_1' = 0, \tag{3.8}$$

$$(-2g_1g_2 + 2g_1g_1' - g_1'' + g_2')f_1 + (g_1^2 + 2g_2 - \frac{5}{2}g_1')f_1' - \frac{3}{2}g_1f_1'' + \frac{1}{2}f_1''' = 0, \tag{3.9}$$

$$(-2g_1g_4 + 2g_4')f_1 + 3g_4f_1' = 0. \tag{3.10}$$

Each above equation can be integrated once,

$$K_1 = f_1^{n+3} G_1^{-2n-2} g_3^2, \tag{3.11}$$

$$K_2 = f_1^2 G_1^{-2} g_2 + \int \left[((g_1^2 - g_1') f_1 - \frac{3}{2} g_1 f_1' + \frac{1}{2} f_1'')' f_1 G_1^{-2} \right] \mathrm{d}x, \tag{3.12}$$

$$K_3 = f_1^3 G_1^{-2} g_4^2. \tag{3.13}$$

Whatever be (n, g_i), the function f_1 can always be computed from (3.9); depending on (n, g_4), it is also given by

$$n \ne -3: \quad f_1 = G_1^{2(n+1)/(n+3)} g_3^{-2/(n+3)} \tag{3.14}$$

$$g_4 \ne 0: \quad f_1 = G_1^{2/3} g_4^{-2/3} \tag{3.15}$$

$$n = -3: \quad f_1 = -g_3^{-1} y^2, \; y = \text{ general solution of}$$

$$\left[g_3^{-1} y^3 (y'' - \frac{1}{2} \frac{g_3'}{g_3} y' - g_2 y) \right]' = 0 \tag{3.16}$$

where the constants K_1 and K_3 have been absorbed in the definition of G_1. The only case in which equation (3.16) needs to be considered is $n = -3, g_4 = 0$, and its solution can be found in [12, 19] [22, ¶14.33] [28] [2, Eq. E12] [5].

Once f_1 is determined, f_2 and f_5 are given by (3.3), (3.7), and f_3, f_4 by the three following expressions, corresponding to cases (3.14), (3.15), (3.16) respectively,

$$\frac{f_3}{f_1} = \begin{cases} \dfrac{2}{n+3} \left(2g_1 + \gamma_3' \right) \\ \dfrac{2}{3} (2g_1 + \gamma_4') \\ -\gamma_3' - \dfrac{f_1'}{f_1} \end{cases} \tag{3.17}$$

$$\frac{f_4}{f_1} = \begin{cases} g_2 + \dfrac{-2(n-1) g_1^2 - (n+3)(2g_1' + \gamma_3'') - (n-5) g_1 \gamma_3' + 2\gamma_3'^2}{(n+3)^2} \\ g_2 + \dfrac{2}{9} g_1^2 - \dfrac{2}{3} g_1' - g_1 \gamma_4' - 2\gamma_4'^2 - \gamma_4'' \\ g_2 + \dfrac{1}{4} \gamma_3'^2 + \dfrac{1}{2} \gamma_3'' + \dfrac{3}{4} \gamma_3' \dfrac{f_1'}{f_1} + \dfrac{1}{2} \dfrac{f_1''}{f_1}. \end{cases} \tag{3.18}$$

Parameters (n, g_i) must satisfy the conditions, polynomial in $n, g_1, g_2, \gamma_3', \gamma_4'$, resulting from the elimination of f_1 between the three linear equations (3.8)-(3.10). There are two such conditions when $g_3 g_4$ is nonzero, and only one when it is zero. The simplest choice of these two conditions is (the labelling refers to the contributing g_i's):

$$g_4 \ne 0: C_{134} \equiv \quad 2n g_1 - 3\gamma_3' + (n+3)\gamma_4' = 0, \tag{3.19}$$

$$\begin{aligned} g_4 \ne 0: C_{124} \equiv \quad & 4g_1^3 - 18 g_1 g_2 - 18 g_1'' + 27 g_2' + (6g_1^2 - 36 g_2 + 27 g_1')\gamma_4' \\ & -6 g_1 \gamma_4'^2 - 4\gamma_4'^3 + 9 g_1 \gamma_4'' + 18 \gamma_4' \gamma_4'' - 9\gamma_4''' = 0 \end{aligned} \tag{3.20}$$

uniquely defined as, respectively, the condition independent of g_2 and the one independent of (n, g_3). By elimination, one obtains the condition independent of g_4 and the one independent of g_1,

$$C_{123} \equiv \quad -4 \left(2g_1 + \gamma_3' \right)^3 + (n+3)[(n-3)(-4g_1^3 - 2g_1'' - 4g_2\gamma_3' - \gamma_3''')$$

$$+ 2(n+3)(n-1)g_1 g_2 + n(-8g_1 g_1' - 6g_1\gamma_3'^2) + (n+3)^2 g_2'$$
$$+ (n-9)(-2g_1^2\gamma_3' - g_1'\gamma_3') - 3(n-1)g_1\gamma_3'' + 6\gamma_3'\gamma_3''] = 0, \quad (3.21)$$

$$g_4 \neq 0 : C_{234} \equiv n^3 g_2' + n^2(-g_2\gamma_3' - \gamma_3''' + \frac{3}{2}\gamma_3''\gamma_4' + \frac{1}{2}\gamma_3'\gamma_4'' - 2\gamma_4'\gamma_4'' + \gamma_4''')$$
$$- \frac{3}{2}\gamma_3'^2\gamma_4' + \frac{1}{2}\gamma_3'^3 - n^2(n-1)g_2\gamma_4'$$
$$- \frac{1}{2}(n^2-3)\gamma_3'\gamma_4'^2 + \frac{1}{2}(n^2-1)\gamma_4'^3 = 0. \quad (3.22)$$

For $(n+3)g_4 \neq 0$, any two of the above four conditions are functionally independent. For $n = -3$, one has $27C_{123} - 4C_{134}^3 = 0$ and independent conditions are C_{134} and C_{234}. All above conditions admit an integrating factor, a natural consequence of the integrated forms (3.11)–(3.13). This is evident for C_{134}; for each of the three others, it is sufficient to integrate it as a first order linear inhomogeneous ODE in g_2,

$$g_4 \neq 0 : K_{134} \equiv G_1^{2n} g_3^{-3} g_4^{n+3}, \quad (3.23)$$

$$g_4 \neq 0 : K_{124} \equiv \left[g_2 - \frac{2}{9}g_1^2 - \frac{2}{3}g_1' + \frac{1}{9}g_1\gamma_4' + \frac{1}{9}\gamma_4'^2 - \frac{1}{3}\gamma_4''\right] G_1^{-8/3} g_4^{-4/3}, \quad (3.24)$$

$$K_{123} \equiv [(n+3)^2 g_2 - (n+3)(2g_1' + \gamma_3'') - 2(n+1)g_1^2$$
$$- (n-1)g_1\gamma_3' + \gamma_3'^2]^{n+3} G_1^{2(n-1)} g_3^{-4}, \quad (3.25)$$

$$g_4 \neq 0 : K_{234} \equiv \left[g_2 + \frac{(n+2)\gamma_3'\gamma_4' - \gamma_3'^2 - (n+1)\gamma_4'^2 + 2n(\gamma_3'' - \gamma_4'')}{2n^2}\right] \times$$
$$g_3^{-4/3} g_4^{4/n}. \quad (3.26)$$

In the Duffing case $n = 3$, condition C_{123} has already been given [15], together with its integrated form K_{123} [13].

4 Interpretation of the two conditions

A very simple interpretation can be given for the two conditions. Indeed, the form of equation (1.1) is invariant under the simultaneous change of dependent and independent variables

$$u(x) \to U(X) : \qquad u = \alpha(x)U, X = \xi(x), \quad (4.1)$$

where α and ξ are two arbitrary gauge functions. The transformed ODE reads

$$U'' + \frac{1}{\xi'}\left[g_1 + 2\frac{\alpha'}{\alpha} + \frac{\xi''}{\xi'}\right]U' + \frac{1}{\xi'^2}\left[g_2 + g_1\frac{\alpha'}{\alpha} + \frac{\alpha''}{\alpha}\right]U$$
$$+ \frac{\alpha^{n-1}}{\xi'^2}g_3 U^n + \frac{1}{\alpha\xi'^2}g_4 = 0. \quad (4.2)$$

Let us adjust the two functions α, ξ so as to make two of the four new coefficients as simple as possible. One of the three possible ways is to cancel the damping term by the choice $\xi' = \alpha^{-2}G_1^{-1}$, which reduces ODE (4.2) to

$$U'' + \alpha^4 G_1^2\left[g_2 + g_1\frac{\alpha'}{\alpha} + \frac{\alpha''}{\alpha}\right]U + \alpha^{n+3}G_1^2 g_3 U^n + \alpha^3 G_1^2 g_4 = 0. \quad (4.3)$$

Canceling the new g_2 coefficient amounts to solving the general linear second order ODE for α, which is possible (from the point of view of Painlevé, adopted here) but does not lead to an explicit value of α. This reduced form is then

$$U'' + h_3 U^n + h_4 = 0, \tag{4.4}$$

and this means that one can freely set $g_1 = g_2 = 0$ in (1.1) without altering its global properties (existence of first integrals, Painlevé property, etc). Instead of that, one can make constant either the reduced g_3 coefficient iff $(n+3)g_3 \neq 0$ by choosing $\alpha^{n+3} = G_1^{-2} g_3^{-1}$, or the reduced g_4 coefficient iff $g_4 \neq 0$ by the choice $\alpha = G_1^{-2/3} g_4^{-1/3}$ (let us recall that G_1 implicitly contains an arbitrary multiplicative constant).

We are thus led to the reduced forms

$$\begin{aligned}
g_4 \neq 0: \quad & g_1 \mapsto 0,\ g_4 \mapsto 1, \\
& g_2 \mapsto \left[g_2 - \frac{2}{9} g_1^2 - \frac{2}{3} g_1' + \frac{1}{9} g_1 \gamma_4' + \frac{1}{9} \gamma_4'^2 - \frac{1}{3} \gamma_4''\right] \times \\
& \quad G_1^{-8/3} g_4^{-4/3}, \\
& g_3 \mapsto g_3 G_1^{-2n/3} g_4^{-(n+3)/3},
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
n \neq -3: \quad & g_1 \mapsto 0,\ g_3 \mapsto 1, \\
& g_2 \mapsto [g_2 - \frac{1}{n+3}(2g_1' + \gamma_3'') + \frac{1}{(n+3)^2}(-2(n+1)g_1^2 \\
& \quad - (n-1) g_1 \gamma_3' + \gamma_3'^2)] G_1^{2(n-1)/(n+3)} g_3^{-4/(n+3)}, \\
& g_4 \mapsto g_4 G_1^{2n/(n+3)} g_3^{-3/(n+3)},
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
n = -3, g_4 = 0: \quad & g_1 \mapsto 0,\ g_3 \mapsto g_3 G_1^2, \\
& g_2 \mapsto \left[g_2 + g_1 \frac{\alpha'}{\alpha} + \frac{\alpha''}{\alpha}\right] \alpha^4 G_1^2 \mapsto 0.
\end{aligned} \tag{4.7}$$

Then the interpretation is obvious: any reduced coefficient distinct from 0 or 1 is the r.h.s. of one of the integrated conditions (3.23)–(3.26). Conversely, any integrated condition is one of the remaining coefficients when two coefficients have been made constant by a choice of gauge. For instance, K_{234} is the reduced g_2 coefficient associated to reduced coefficients g_3 and g_4 equal to unity.

This can also be seen in a more elementary way. In a gauge (α, ξ) such that $g_1 = 0, g_3' g_4' = 0$, an expression for the first integral is

$$g_1 = 0, g_3' g_4' = 0 : I_0 = u'^2 + 2g_3 \int_0^u u^n \mathrm{d}u + g_2 u^2 + 2g_4 u, \tag{4.8}$$

and, from the relation

$$I_0' - 2u'E \equiv 2g_3' \int_0^u u^n \mathrm{d}u + g_2' u^2 + 2g_4' u, \tag{4.9}$$

one deduces that the two other coefficients g_2 and g_3 or g_4 must be constant.

The Hamiltonian (2.4) is a first integral if and only if $g_1 = 0$ and all other g_i's are constant.

5 Painlevé analysis

Painlevé set up the problem of finding nonlinear differential equations able to define functions, just like the first order elliptic equation

$$u'^2 = 4u^3 - g_2 u - g_3,\ (g_2, g_3) \text{ complex constants}, \tag{5.1}$$

defines the elliptic function of Weierstrass $\wp(x, g_2, g_3)$, a doubly periodic function which includes as particular cases the well known trigonometric and hyperbolic functions. For a tutorial introduction, see the books [21, 7].

A by-product of this quest for new functions has been the construction of exhaustive lists of nonlinear differential equations, the general solution of which can be made single-valued (in more technical terms, without movable critical singularities, this is the so-called *Painlevé property* (PP)), which implies that their general solution is known in closed form. In particular, the list of second order first degree algebraic equations, i.e.

$$u'' = F(u', u, x), \tag{5.2}$$

with F rational in u', algebraic in u, analytic in x, which possess the PP has been established by Painlevé and Gambier [19].

These classical results apply to our problem only for those values of n for which Eq. (1.1), maybe after a monomial change of the dependent variable $u = U^k, k \in \mathcal{R}$, belongs to the class (5.2). These values, which include at least all the integers, are determined below. Then, the way those classical results can be applied is twofold.

1. Require the PP for our equation or its transform under $u = U^k$.
2. Restricting to the values of (n, g_i) for which the first integral (3.1) exists, check that the two conditions for the existence of this first integral imply the identical satisfaction of the necessary condition that Eq. (5.2) have no movable logarithmic branch points. Indeed, this is a classical result of Poincaré that the movable singularities (i.e. those which depend on the initial conditions) of first order algebraic ODEs can only be algebraic, i.e. $u \sim u_0(x - x_0)^p$, and never logarithmic, i.e. with some $\mathrm{Log}(x - x_0)$ term. Let us do that without too many technical considerations.

The above mentioned necessary condition that Eq. (5.2) have no movable logarithmic branch points can only be computed after performing the following steps (for the unabribged procedure, see [8, section 6.6]).

Step 1. For each *family of movable singularities*

$$u = \chi^p \sum_{j=0}^{+\infty} u_j \chi^j,\ u_0 \neq 0,\ \chi' = 1, \tag{5.3}$$

determine the *leading behaviour* (p, u_0). This is achieved by balancing the highest derivative u'' with a nonlinear term. Therefore, there exist two leading behaviours, denoted "family g_3" (balancing of u'' and $g_3 u^n$) and "family g_4" (balancing of u'' and g_4)

$$(g_3) : \qquad p = -\frac{2}{n-1},\ u_0 = \left[-2\frac{n+1}{(n-1)^2} g_3\right]^{\frac{1}{n-1}},\ n \neq -1, \tag{5.4}$$

$$(g_4) : \qquad p = 2,\ u_0 = -\frac{1}{2} g_4,\ g_4 \neq 0. \tag{5.5}$$

Step 2. For each family, compute the Fuchs indices, i.e. the roots i of the indicial equation of the linear equation obtained by linearizing (1.1) near its leading behaviour $u \sim u_0\chi^p$, and require every Fuchs index to be integer. This linearized equation is

$$(g_3) \ : \qquad \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + n g_3 (u_0\chi^2)^{n-1}\right) v = 0, \tag{5.6}$$

$$(g_4) \ : \qquad \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 0\right) v = 0, \tag{5.7}$$

and the Fuchs indices are obtained by requiring the solution $v = v_0\chi^{p+i}(1+O(\chi))$,

$$(g_3) \ : \qquad (i+1)\left(i - 2 - \frac{4}{n-1}\right) = 0, \tag{5.8}$$

$$(g_4) \ : \qquad (i+1)(i+2) = 0. \tag{5.9}$$

The diophantine condition that $i = 2 + 4/(n-1)$ be integer has a countable number of solutions since we have not yet put restrictions on n.

Step 3. For each family, compute all the necessary conditions for the absence of movable logarithms (in short, no-log conditions), which might occur when one computes the successive coefficients u_j of (5.3). One can check that the family g_4 can never generate such no-log conditions. These conditions need not be computed on the original equation (1.1), they can be computed on any algebraic transform if this proves more convenient (indeed, movable logarithms are not affected by an algebraic transform on u), such as

$$\begin{aligned} u = U^k \ : \ & kUU'' + k(k-1)U'^2 + k g_1 UU' + g_2 U^2 \\ & + g_3 U^{2+(n-1)k} + g_4 U^{2-k} = 0. \end{aligned} \tag{5.10}$$

The transformed powers p are $p_3 = -2/((n-1)k), p_4 = 2/k$, and the Fuchs indices are unchanged.

The computation of the no-log conditions is impossible unless there exists a k making all the powers of U in (5.10) at least rational. In order to avoid the technical complications of dealing with rational values of the leading exponent p, we restrict to those values of n for which there exists a k making $2 + (n-1)k$ and, if g_4 is nonzero, $2 - k$ integer. The useful transforms are

$$\begin{aligned} u = v^{2/(n-1)} : \quad & \frac{2}{n-1} v v'' - 2\frac{n-3}{(n-1)^2} v'^2 + \frac{2}{n-1} g_1 v v' + g_2 v^2 \\ & + g_3 v^4 + g_4 v^{2-2/(n-1)} = 0, \end{aligned} \tag{5.11}$$

$$\begin{aligned} u = w^{1/(n-1)} : \quad & \frac{1}{n-1} w w'' - \frac{n-2}{(n-1)^2} w'^2 + \frac{1}{n-1} g_1 w w' + g_2 w^2 \\ & + g_3 w^3 + g_4 w^{2-1/(n-1)} = 0, \end{aligned} \tag{5.12}$$

$$\begin{aligned} u = V^{-2} : \quad & -2VV'' + 6V'^2 - 2g_1 VV' + g_2 V^2 \\ & + g_3 V^{4-2n} + g_4 V^4 = 0, \end{aligned} \tag{5.13}$$

$$\begin{aligned} u = W^{-1} : \quad & -WW'' + 2W'^2 - g_1 WW' + g_2 W^2 \\ & + g_3 W^{3-n} + g_4 W^3 = 0, \end{aligned} \tag{5.14}$$

which are polynomial if and only if

$$(5.11): \quad g_4 = 0 \text{ or } (g_4 \neq 0 \text{ and } \frac{2}{n-1} \in \mathcal{Z}), \tag{5.15}$$

$$(5.12): \quad g_4 = 0 \text{ or } (g_4 \neq 0 \text{ and } \frac{1}{n-1} \in \mathcal{Z}), \tag{5.16}$$

$$(5.13): \quad 2n \in \mathcal{Z}, \tag{5.17}$$

$$(5.14): \quad n \in \mathcal{Z}. \tag{5.18}$$

The original ODE (1.1) is identical to (5.11) for $n = 3$ and to (5.12) for $n = 2$.

To summarize, let us compute the no-log condition $Q_i = 0$ on the ODE for v (5.11). Unfortunately, one does not know how to obtain the dependence of Q_i on n, since n must first be given a numerical value before Q_i is computed; this makes uneasy the comparison with conditions (3.19)–(3.20), which depend on n.

To fix the ideas, a list of useful values of (n, i) is displayed in Table 1.

Table 1: Values of (i, n) for i integer $\in [-4, 10]$.

$2+\frac{4}{n-1}$	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
n	1/3	1/5	0	-1/3	-1	-3	∞	5	3	7/3	2	9/5	5/3	11/7	3/2

The computation of Q_i for positive values of i is classical [19, 2]. Denoting for shortness $C_1 = C_{123}, C_2 = C_{134}$, one finds the following expressions Q_i for the indicated values of (n, g_4),

$$(-3, 0): Q_1 = C_1, \tag{5.19}$$

$$(5, 0): Q_3 = C_1, \tag{5.20}$$

$$(3, g_4): Q_4 = \pm \frac{1}{864}[-2g_3]^{-1/2}[(\gamma_3' - g_1)C_1 - C_1'] + \frac{1}{6}C_2, \tag{5.21}$$

$$(\frac{7}{3}, 0): Q_5 = (.g_1^2 + .g_2 + .g_1' + .g_1\gamma_3' + .\gamma_3'^2)C_1 + (.g_1 + .\gamma_3')C_1' + .C_1'', \tag{5.22}$$

$$(2, g_4): Q_6 = .C_1 + .C_1^2 + .C_1' + .C_1'' + .C_1''' + .C_2 + .C_2', \tag{5.23}$$

$$(\frac{9}{5}, 0): Q_7 = .C_1 + \ldots + .C_1^{(4)}, \tag{5.24}$$

$$(\frac{5}{3}, 0): Q_8 = .C_1 + \ldots + .C_1^{(5)}. \tag{5.25}$$

where dots stand for rational numbers when $i = 5$ and polynomials of g_1, g_2, γ_3, γ_4 when $i > 5$. Similar relations have been checked for $i = 9$ and $i = 10$ (Thomas-Fermi case) but are not reproduced here. Condition $Q_4 = 0$ contains a $\pm$ sign arising from the two possible choices for v_0 and is equivalent to the two conditions

$$(\gamma_3' - g_1)C_{123} - C_{123}' = 0, \qquad C_{134} = 0. \tag{5.26}$$

We therefore check the property that each Q_i is indeed a differential consequence of the two conditions $C_{123} = 0, C_{134} = 0$ for the existence of a first integral (3.1)

$$\forall i \in \mathcal{N}, \forall g_i : (C_1 = 0, C_2 = 0) \Rightarrow (Q_i = 0). \tag{5.27}$$

For negative [18, 6] values of the Fuchs index i, the results [6] are the following: the family g_4 never generates any no-log condition, and, for the family g_3, a no-log condition arises from the Fuchs index -1, and this condition is a differential consequence of conditions (3.19)–(3.20), at least for the examples handled $(n, r, g_4) = (1/5, -3, 0), (1/3, -4, 0)$. This is also an experimental verification of

$$\forall i \in \mathcal{Z}, \forall g_i : (C_1 = 0, C_2 = 0) \Rightarrow (Q_{-1} = 0) \tag{5.28}$$

and this relation cannot be reversed, as proven by Painlevé and Gambier. For instance, in the case of the Duffing oscillator $(n, i, g_4) = (3, 4, g_4)$, condition $Q_4 = 0$ implies the reducibility of v to the second Painlevé transcendent whereas the stronger conditions $C_1 = 0, C_2 = 0$ imply the reducibility of v to an elliptic function.

Remark. When one includes the contribution of the Schwarzian in the definition of the gradient of the expansion variable χ, as done in the invariant Painlevé analysis [4],

$$\chi' = 1 + \frac{S}{2}\chi^2, \tag{5.29}$$

all the computed no-log conditions $Q_i = 0$, equations (5.19)–(5.25), are independent of this Schwarzian S, as opposed e.g. to the Lorenz model [9]. This certainly indicates some hierarchy between the level of nonintegrability of these two dynamical systems.

Remark. For some small values of $|i|$, there is equivalence between the no-log condition and (3.19)–(3.20). This nongeneric situation occurs only for the following values of (n, i, g_4),

$(-3, 1, 0)$, i.e. the Ermakov-Pinney equation [12, 28],

$(5, 3, 0)$, i.e. an equation considered by Lane and Emden [24, 11], Chandrasekhar [3] and Logan [26, p. 52],

$(1/5, -3, 0)$, an equation which could deserve more study.

6 Conclusion

This work generalizes all previous results on the partial integrability of the anharmonic oscillator. It gives a natural interpretation of the two conditions for the existence of a particular first integral, in terms of reduced coefficients. Finally, this system is an excellent example to study several features of Painlevé analysis.

A good, recent bibliography can be found in Ref. [20].

Acknowledgments

The author wishes to thank M. Musette for fruitful discussions during the completion of this work and for her encouragement to publish these lecture notes, a first draft of which was delivered at a meeting in Dijon [27].

He thanks the IUAP Contract No. P4/08 funded by the Belgian government and acknowledges the financial support of the Tournesol grant T99/040.

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