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On a new technique to manufacture isochronous Hamiltonian systems: classical and quantal treatments

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Abstract

We discuss a new technique to ω-modify *real* Hamiltonians so that they become *isochronous* while remaining *real*. Although the ω-modified Hamiltonians thereby obtained often yield, in the *classical* context, *singular* motions, we exhibit and investigate simple examples when this does *not* (quite) happen. We also identify *quantized* versions of some of these *isochronous* models featuring *equispaced* spectra, and observe that there are cases when in the *classical* context the motions run into singularities while in the *quantal* context it is nevertheless justified to arrive at nonsingular models with *equispaced* spectra.

1 Introduction

Recently a technique was introduced associating to a Hamiltonian another, ω-modified, Hamiltonian, which reduces to the original one when the parameter ω vanishes, and for $\omega > 0$ features an *open*, hence *fully dimensional*, region in its phase space (possibly encompassing the entire phase space) where *all* its solutions are *isochronous*, i. e. *completely periodic* with the *same* fixed period $T = 2\pi/\omega$. [1] [2] This technique is applicable to a large class of Hamiltonians, justifying the assertion that *isochronous Hamiltonian systems are not rare*. But the ω-modified Hamiltonians it yields are *complex*; and although a *real* Hamiltonian producing the *same* dynamics can be manufactured from any (analytic) *complex* Hamiltonian via a standard technique (see for instance Ref. [3]), it is generally more complicated and it involves twice as many (*real*) canonical variables than those of the original model. In the present paper we discuss an alternative technique to ω-modify *real* Hamiltonians so that they become *isochronous* while remaining *real* [4]. Unfortunately the *real* ω-modified Hamiltonians thereby obtained often yield, in the *classical* context, *singular* motions; but *not always*, indeed we exhibit and investigate below simple examples when this need not happen. In most of these cases we moreover investigate *quantized* versions of these models, finding that, for appropriate quantization prescriptions, these quantum models exhibit *equispaced* energy spectra, with the spacing ΔE generally corresponding to the period $\tilde{T}$ of the corresponding classical motion via the standard formula,

$$\Delta E = \frac{h}{\tilde{T}} \ , \tag{1.1}$$

where $h \equiv 2\pi\hbar$ is Planck's constant. Remarkably, this happens even in some cases when in the *classical* context the motions run generally into *singularities*.

The new technique is described in Section 2, and the examples are discussed in Sections 3 respectively 4 in the *classical* respectively *quantal* contexts. For simplicity the treatment is restricted to systems with only one (canonical) coordinate q and correspondingly one (canonical) momentum p – even though one of the most appealing features of these techniques to identify *isochronous* systems is their applicability to problems involving an *arbitrary* number of canonical variables.

Added in proofs: another, more effective and more generally applicable, technique to modify Hamiltonians so that they become *isochronous* has now been found: F. Calogero and F. Leyvraz, "General technique to produce isochronous Hamiltonians", J. Phys. A: Math. Theor. **40** (2007), 12931-12944.

2 The new technique

Consider – in the classical context – a Hamiltonian $H\left(p,q\right)$ and assume that there exists a function $\tau\left(p,q\right)$ of the canonical variables whose Poisson bracket with this Hamiltonian $H\left(p,q\right)$ is unity,

$$\left[H\left(p,q\right), \tau\left(p,q\right)\right] = 1 \ . \tag{2.1}$$

Here and hereafter the Poisson bracket of two functions $F\left(p,q\right)$ and $G\left(p,q\right)$ of the canonical variables is defined as follows:

$$\left[F\left(p,q\right), G\left(p,q\right)\right] = \frac{\partial F\left(p,q\right)}{\partial p}\frac{\partial G\left(p,q\right)}{\partial q} - \frac{\partial G\left(p,q\right)}{\partial p}\frac{\partial F\left(p,q\right)}{\partial q} \ . \tag{2.2}$$

Since in the context of Hamiltonian dynamics the time evolution of any function $F\left(p,q\right)$ of the canonical variables is given by the formula

$$\dot{F}\left(p,q\right) = \left[H\left(p,q\right), F\left(p,q\right)\right] \ , \tag{2.3}$$

the property (2.1) entails that the function $\tau\left(p,q\right)$ evolves under the flow due to the Hamiltonian $H\left(p,q\right)$ as time itself:

$$\dot{\tau}\left(p,q\right) = 1 \ , \quad \tau\left(p,q\right) = \tau_0 + t \ . \tag{2.4}$$

Here and throughout a superimposed dot denotes differentiation with respect to the variable t ("time"), while clearly $\tau_0 = \tau\left(p_0, q_0\right)$ where (here and below) $p_0 \equiv p\left(0\right), q_0 \equiv q\left(0\right)$ are the initial values of the canonical variables.

The "old" technique [1] [2] to generate an *isochronous* Hamiltonian is based on the introduction of the ω-modified Hamiltonian

$$\tilde{H}\left(p,q;\omega\right)=\left[1+i\omega\tau\left(p,q\right)\right]H\left(p,q\right)\ . \tag{2.5}$$

Here and hereafter i is the *imaginary* unit, $i^2=-1$, and ω is a *positive* constant, $\omega>0$, to which we associate the period

$$T=\frac{2\pi}{\omega}\ . \tag{2.6}$$

The presence of the *imaginary* unit in the definition (2.5) makes the *complex* character of this ω-modified Hamiltonian $\tilde{H}\left(p,q;\omega\right)$ evident. Hence the dynamics yielded by $\tilde{H}\left(p,q;\omega\right)$ entails that the canonical variables p and q become *complex* numbers. Of course by introducing their *real* and *imaginary* parts, say

$$q=x+iy\ ,\quad p=p_x+ip_y, \tag{2.7}$$

one can again deal only with *real* variables; and it is well known (see for instance Ref. [3]) that exactly the *same* dynamics yielded by the *complex* Hamiltonian $\tilde{H}\left(p,q;\omega\right)$ for the two *complex* canonical variables p,q (hence, via (2.7), for the 4 *real* variables p_x,p_y,x,y) is generally produced by the *real* Hamiltonian (note the minus sign in the right-hand side!)

$$H_R\left(p_x,p_y,x,y;\omega\right)=\mathrm{Re}\left[\tilde{H}\left(p_x-ip_y,x+iy;\omega\right)\right] \tag{2.8}$$

featuring the 4 *canonical* variables p_x,p_y,x,y. Note that dealing with *real* Hamiltonians is generally *necessary* (if not *sufficient*) in order to get, after the transition to quantum mechanics, *Hermitian* Hamiltonian operators, as required for a proper physical interpretation.

The "new" technique [4] to generate an *isochronous* Hamiltonian is based on the introduction of the ω-modified Hamiltonian

$$\check{H}\left(p,q;\omega\right)=\left[1+\omega^2\tau^2\left(p,q\right)\right]H\left(p,q\right)\ . \tag{2.9}$$

This Hamiltonian is now generally *real*. In the following part of this section we show why it is justified to expect that the dynamics yielded by this Hamiltonian is *isochronous* (with period $T/2$, see (2.6), or possibly a small *integer* multiple of this basic period). Accordingly, hereafter the time evolution under consideration is that yielded by the Hamiltonian $\check{H}\left(p,q;\omega\right)$, namely the time evolution of any function $F\left(t\right)\equiv F\left(p,q\right)$ of the canonical variables $p\equiv p\left(t\right)$ and $q\equiv q\left(t\right)$ is now characterized by the rule

$$\dot{F}=\left[\check{H},F\right]\ . \tag{2.10}$$

It is then easily seen that (2.9) together with (2.1) yield

$$\dot{\tau}=1+\omega^2\tau^2\ , \tag{2.11a}$$

entailing

$$\tau\left[p\left(t\right),q\left(t\right)\right]\equiv\tau\left(t\right)=\frac{\tan\left[\omega\left(t-t_0\right)\right]}{\omega}\ , \tag{2.11b}$$

with t_0 a *real* constant identified mod $(T/2)$, in the context of the initial-value problem, by the assignment

$$t_0 = -\frac{\arctan\left[\omega\tau\left(0\right)\right]}{\omega} \equiv -\frac{\arctan\left\{\omega\tau\left[p\left(0\right), q\left(0\right)\right]\right\}}{\omega} . \tag{2.11c}$$

Likewise (2.9) entails

$$\dot{H} = \omega^2 H\left[\tau^2, H\right] = -2\omega^2 H\tau\left[H, \tau\right] \ = -2\omega^2 H\tau, \tag{2.12a}$$

hence via (2.1) and (2.11b)

$$\dot{H} = -2\omega H \tan\left[\omega\left(t - t_0\right)\right] \ , \tag{2.12b}$$

hence

$$H\left(t\right) = H\left(t_0\right)\cos^2\left[\omega\left(t - t_0\right)\right] \ . \tag{2.12c}$$

Note the consistency of these formulas, via the trigonometric identity

$$\cos^2\left(z\right)\left[1 + \tan^2\left(z\right)\right] = 1 \ , \tag{2.13}$$

with the time-independence of the Hamiltonian $\check{H}\left(p, q; \omega\right)$: see (2.9), (2.11b) and (2.12c). Indeed the time evolution (2.12c) of the original, unmodified Hamiltonian $H\left(p, q\right)$ could have been directly obtained from the constancy of the Hamiltonian $\check{H}\left(p, q; \omega\right)$ via (2.9), (2.11b) and (2.13).

The solution of the initial-value problem characterizing the time evolution of the canonical variables p and q can then be obtained directly (i. e., bypassing the need to integrate the Hamiltonian equations of motion) by solving for these variables the two relations

$$H\left(p, q\right) = H\left(p_0, q_0\right)\cos^2\left[\omega\left(t - t_0\right)\right] \ , \tag{2.14a}$$

$$\tau\left(p, q\right) = \frac{\tan\left[\omega\left(t - t_0\right)\right]}{\omega} \ , \tag{2.14b}$$

with

$$t_0 = -\frac{\arctan\left[\omega\tau\left(p_0, q_0\right)\right]}{\omega} \ . \tag{2.14c}$$

The fact that in these relations the time dependence is *periodic* with period $T/2$ indicates that the resulting time dependence of the canonical variables p and q shall generally also be *periodic* with this period (or some multiple of it, due to the inversion) confirming the *isochronous* character of the dynamics induced by the ω-modified Hamiltonian $\check{H}\left(p, q; \omega\right)$; but the *singular* character of the time dependence of τ, see (2.14b), is likely to affect the time dependence of the canonical variables, causing them to become *singular*. This last phenomenon, however, does not *always* happen, as the specific examples discussed in the following Section 3 demonstrate—and as was already shown by the example treated in Ref. [4].

3 Classical examples

In this section we treat tersely *eight* examples, displaying the classical dynamics yielded by *isochronous* Hamiltonians $\check{H}(p,q;\omega)$ obtained by applying the trick (2.9).

3.1 Four examples

Consider the following class of Hamiltonians:

$$H(p,q) = \frac{1}{2}\left[\frac{p}{f'(q)}\right]^{2/k} , \tag{3.1a}$$

with k an *a priori arbitrary* number (of course *nonvanishing*, $k \neq 0$) and $f(q)$ an *a priori arbitrary* function. Here and below primes denote, as usual, differentiations with respect to the argument of the function they are appended to. The justification for the somewhat peculiar notation used in this definition (3.1a) is given by the neat look of the solutions of the equations of motion yielded by this Hamiltonian, see below (but in fact we will treat in detail only the simple case with $k = 1$). This notation is also suggestive of the existence of a canonical transformation – whose explicit determination can be left to the diligent reader – relating this Hamiltonian to that characterizing free motion. Of course – unless the exponent $2/k$ in (3.1a) is a *positive integer* – this time evolution can be expected to make sense only if the canonical momentum p remains *nonnegative* throughout the time evolution, $p(t) \geq 0$, and likewise the canonical coordinate q guarantees that the quantity $f'[q(t)]$ remains positive throughout the time evolution, $f'[q(t)] > 0$.

It is easily verified that a quantity $\tau(p,q)$ whose Poisson bracket with this Hamiltonian equals unity, see (2.1), reads

$$\tau(p,q) = kf(q)\left[\frac{p}{f'(q)}\right]^{(k-2)/k} . \tag{3.1b}$$

Clearly the corresponding ω-modified Hamiltonian (2.9) reads then

$$\check{H}(p,q;\omega) = \frac{1}{2}\left\{\left[\frac{p}{f'(q)}\right]^{2/k} + k^2\omega^2 f(q)^2 \left[\frac{p}{f'(q)}\right]^{2(k-1)/k}\right\} . \tag{3.2a}$$

The alert reader will easily recognize that, by appropriate canonical transformations, this Hamiltonian can be transformed into that of the standard *linear* harmonic oscillator or into that characterizing free motion. Using this finding, or exploiting the treatment of the preceding section, the time evolutions of the canonical variables p and q induced by this ω-modified Hamiltonian, (3.2a), are then easily obtained. The canonical coordinate turns out to be given by the solution of the following (non-differential) equation (entailing the inversion of the function $f(q)$):

$$f[q(t)] = A\frac{\sin(\omega t + \theta)}{\omega}[\cos(\omega t + \theta)]^{1-k} , \tag{3.2b}$$

and the corresponding canonical momentum reads

$$p(t) = (kA)^{k/(2-k)} f'[q(t)][\cos(\omega t + \theta)]^k . \tag{3.2c}$$

Here (and always below) A and θ are two arbitrary constants, to be fixed by the initial data $q(0) \equiv q_0, \; p(0) \equiv p_0$ in the context of the initial-value problem.

Let us now illustrate these findings by making some specific assignments of the number k and of the function $f(q)$. The alert reader is welcome to try other choices.

Example 3.1.

$$k = 1 \;, \quad f(q) = q; \tag{3.3a}$$

$$H(p,q) = \frac{p^2}{2} \;, \quad \check{H}(p,q;\omega) = \frac{1}{2}\left(p^2 + \omega^2 q^2\right) \;; \tag{3.3b}$$

$$q(t) = A\frac{\sin(\omega t + \theta)}{\omega} \;, \quad p(t) = A\cos(\omega t + \theta) \;. \tag{3.3c}$$

No comment.

Example 3.2.

$$k = 1 \;, \quad f(q) = a\log\left(\frac{q}{b}\right) \;; \tag{3.4a}$$

$$\check{H}(p,q;\omega) = \frac{1}{2}\left\{\left(\frac{qp}{a}\right)^2 + \left[\omega a\log\left(\frac{q}{b}\right)\right]^2\right\} \;; \tag{3.4b}$$

$$q(t) = b\exp\left[\frac{A\sin(\omega t + \theta)}{a\omega}\right] \;, \tag{3.4c}$$

$$p(t) = A\frac{a}{b}\exp\left[-\frac{A\sin(\omega t + \theta)}{a\omega}\right]\cos(\omega t + \theta) \;. \tag{3.4d}$$

Here a, b are two *arbitrary* (*real*) constants ($b > 0$). Clearly throughout the time evolution $q(t)$ is *positive*, $q(t) > 0$, while $p(t)$ generally changes sign: and the solution is generally *nonsingular* and *periodic* with period T, see (2.6).

Example 3.3.

$$k = 1 \;, \quad f(q) = a\log\left[\beta\log\left(\frac{q}{b}\right)\right] \;; \tag{3.5a}$$

$$\check{H}(p,q;\omega) = \frac{1}{2}\left\{\left[\frac{pq}{a}\log\left(\frac{q}{b}\right)\right]^2 + \left\{\omega a\log\left[\beta\log\left(\frac{q}{b}\right)\right]\right\}^2\right\} \;; \tag{3.5b}$$

$$q(t) = b\exp\left\{\frac{1}{\beta}\exp\left[\frac{A\sin(\omega t + \theta)}{a\omega}\right]\right\} \;, \tag{3.5c}$$

$$p(t) = A\frac{a\beta}{b}\exp\left\{-\frac{1}{\beta}\exp\left[\frac{A\sin(\omega t + \theta)}{a\omega}\right]\right\}\exp\left[-\frac{A\sin(\omega t + \theta)}{a\omega}\right]\cos(\omega t + \theta) \;. \tag{3.5d}$$

Here a, b are two *arbitrary* constants ($b > 0$), and β is an *arbitrary positive* number, $\beta > 0$. Again, throughout the time evolution $q(t)$ is *positive*, $q(t) > 0$, while $p(t)$ generally changes sign: and the solution is generally *nonsingular* and *periodic* with period T, see (2.6).

Example 3.4.

$$k=1, \quad f\left(q\right)=a\left(\frac{q}{a}\right)^{1/j}-c\ ; \tag{3.6a}$$

$$\check{H}\left(p,q;\omega\right)=\frac{1}{2}\left\{\left[jp\left(\frac{q}{a}\right)^{(j-1)/j}\right]^2+\omega^2\left[a\left(\frac{q}{a}\right)^{1/j}-c\right]^2\right\}\ ; \tag{3.6b}$$

$$q\left(t\right)=a\left[\frac{c\omega+A\sin\left(\omega t+\theta\right)}{a\omega}\right]^j\ , \tag{3.6c}$$

$$p\left(t\right)=\frac{A}{j}\left[\frac{c\omega+A\sin\left(\omega t+\theta\right)}{a\omega}\right]^{1-j}\cos\left(\omega t+\theta\right)\ . \tag{3.6d}$$

Here a and c are two *arbitrary*, but *positive*, constants, $a>0$, $c>0$, while j is an *a priori arbitrary* (*real* and *nonvanishing*) number, and $A^2=2\check{H}$. The solution is clearly *real*, *nonsingular* and *periodic* with period T for *all* values of the number j provided the initial data entail the restriction $A^2<(c\omega)^2$, implying $|q(t)|\le|a|\,|2c/a|^j$, $|p(t)|<|c\omega/j|\,|2c/a|^{1-j}$; otherwise both $q(t)$ and $p(t)$ become *singular* at the finite time $t=\left[-\theta-\arcsin\left(c\omega/A\right)\right]/\omega$ (defined $\bmod\,(T)$, see (2.6)), although the singularity is only of polar type if j is *integer* affecting only $p(t)$ if j is *positive,* $j=2,3,...$ and only $q(t)$ if j is *negative,* $j=-1,-2,...$ (and obviously there is no singularity at all in the *trivial* case with $j=1$). Hereafter (in particular, in the discussion of the corresponding quantal models), we limit our consideration to *integer* values of the parameter j. Moreover, even in this case some difficulties may arise when j is *even*: then whenever q reaches a vanishing value there is a square-root ambiguity in $q^{1/j}$ that makes the Hamiltonian $\check{H}\left(p,q;\omega\right)$ *multivalued* (the appropriate assignment might or might not involve a change of sign); hence in this case a possible multivaluedness of the Hamiltonian $\check{H}\left(p,q;\omega\right)$ as a function of q (see (3.6b)) should be reckoned with. This might be a source of difficulty, albeit only in the quantal context, see below: in the classical context the Hamiltonian $\check{H}\left(p,q;\omega\right)$ is of course constant throughout the time evolution, and $q(t)$ does not vanish – nor diverge – if $A^2<(c\omega)^2$, see (3.6c). Note moreover that no multivaluedness of the Hamiltonian may occur if j is an *odd* integer, since in the *real* domain the root of $q^{1/j}$ is then unambiguous, having the same sign as q itself.

This ends our display of examples based on the *ansatz* (3.1a). Note that we did not manufacture any example yielding *nonsingular* solutions with $k\neq1$.

3.2 Three other examples

Other examples obtain from the Hamiltonian

$$H=\frac{1}{2}f\left(q\right)\exp\left(2ap\right) \tag{3.7a}$$

with a an *arbitrary positive* constant, $a>0$, and $f(q)$ an *a priori arbitrary* function.

It is easily verified that a quantity $\tau(p,q)$ whose Poisson bracket with this Hamiltonian equals unity, see (2.1), reads now as follows:

$$\tau(p,q) = \frac{q+c}{af(q)} \exp(-2ap) \ , \tag{3.7b}$$

with c an arbitrary constant. Hence the corresponding ω-modified Hamiltonian (2.9) now reads

$$\check{H}(p,q;\omega) = \frac{1}{2}\left[f(q)\exp(2ap) + \frac{\omega^2 (q+c)^2}{a^2 f(q)} \exp(-2ap)\right] \ . \tag{3.8a}$$

Again, the alert reader will easily recognize that, by appropriate canonical transformations, this Hamiltonian can be transformed into that of the standard *linear* harmonic oscillator, or into that yielding free motion. Using this finding, or exploiting the treatment of the preceding section, the time evolutions of the canonical variables p and q induced by this ω-modified Hamiltonian, (3.8a), are easily obtained:

$$q(t) = -c + A\frac{\sigma(\omega t+\theta)\sin[2(\omega t+\theta)]}{2\omega} \ , \tag{3.8b}$$

$$p(t) = \frac{1}{2a}\log\left\{\frac{A\cos^2(\omega t+\theta)}{af[q(t)]}\right\} \ . \tag{3.8c}$$

Here $A = 2a\check{H}$ and $\theta = -\omega t_0$ are fixed by the initial data, while the constant σ in the right-hand side of (3.8b) is a sign, $\sigma = \pm 1$, that we introduce at this stage to take account of the possibility – discussed below – that a change of sign occur where $p(t)$ is singular (see (3.8c) and below). These formulas indicate that, for a large class of assignments of the function $f(q)$ the two canonical variables $q(t)$, $p(t)$ are *periodic* with period $\tilde{T} = T/2$, see (2.6); but the canonical momentum $p(t)$ becomes generally *singular* at a finite time (i. e., whenever $t = t_0 + T/4 \mod (T/2)$), and there is no assignment (of course, it should be independent of the initial data) of the function $f(q)$ to cure this defect. Note that, on the other hand, the canonical coordinate $q(t)$ is well defined throughout its time evolution, see (3.8b).

The fact that, for *generic* initial data, the solution of this problem becomes *singular* at a *finite* time is a serious drawback to assign to it any *physical* relevance: there is indeed an intrinsic ambiguity on how to continue the solution for the canonical momentum $p(t)$ beyond the singularity – an ambiguity that is also reflected in the presence of the (possibly time dependent) sign σ in the nonsingular expression (3.8b) of the canonical coordinate $q(t)$. To clarify the matter we now analyze *two* simple cases: they are quite special, but also interesting for their quantal treatment provided in the next section.

Example 3.5.

$$f(q) = b(q+c) \ , \tag{3.9a}$$

entailing that the ω-modified Hamiltonian (3.8a) reads

$$\check{H}(p,q;\omega) = \frac{b(q+c)}{2}\left[\exp(2ap) + \left(\frac{\omega}{ab}\right)^2 \exp(-2ap)\right] \ . \tag{3.9b}$$

A naive treatment based on (3.8b) and (3.8c) would yield the solution

$$p\,(t) = \frac{1}{2a}\log\left\{\frac{\omega\cot\,(\omega t+\theta)}{ab}\right\} \tag{3.9c}$$

with (3.8b), which is however unacceptable since it entails not only that the canonical momentum $p\,(t)$ has a singularity (it blows up) whenever $t = -\theta/\omega \mod (T/4)$, but that it even loses the property to be *real* for $-\theta/\omega - T/4 < t < -\theta/\omega \mod (T/2)$, see (3.9c). One might try to cure this lack of reality by replacing (3.9c) with

$$p\,(t) = \frac{1}{4a}\log\left\{\frac{\omega^2\cot^2\,(\omega t+\theta)}{a^2b^2}\right\}\ , \tag{3.9d}$$

but this formula is actually inconsistent with the equation of motion entailed by the Hamiltonian (3.9b),

$$\dot{p}\,(t) = -\frac{b}{2}\left[\exp\,(2ap) + \left(\frac{\omega}{ab}\right)^2\exp\,(-2ap)\right] < 0\ , \tag{3.9e}$$

as it is plain from this formula that, as long as $p\,(t)$ is *real*, it should be a *decreasing* function of time; while according to (3.9d) $p\,(t)$ diverges to *negative* infinity whenever $-\,(\theta/\omega) + T/4 \mod (T/2)$ but to *positive* infinity whenever $t = -\theta/\omega \mod (T/2)$. This indicates that a more appropriate prescription must be introduced to continue the function $p\,(t)$ beyond the points $t = -\theta/\omega \mod (T/4)$ where it blows up; of course a prescription that maintains the reality of $p\,(t)$ as well as the validity of the equation of motion (3.9e) at all other times. Such a continuation is yielded by the requirement that the Hamiltonian $\check{H}\,(p, q; \omega)$ maintain its value across the singularity, reading as follows:

$$q(t) = -c + A\frac{|\sin 2(\omega t+\theta)|}{2\omega} \tag{3.10a}$$

where again $A = 2a\check{H}$ (and we did set $\sigma\,(x) = \mathrm{sign}(x)$, see (3.8b)), and

$$p\,(t) = \frac{1}{2a}\log\frac{\omega}{ab} + \frac{1}{2a}\mathrm{arcsinh}\,[\cot 2(\omega t+\theta)]\ . \tag{3.10b}$$

Let us re-emphasize that the canonical coordinate $q\,(t)$ (see (3.10a)) is a continuous function of time and does satisfy its Hamiltonian equation of motion at all times except for those special values at which $p\,(t)$ blows up. Likewise, apart from the jump from $-\infty$ to ∞ at the singular times $t = -\theta/\omega \mod (T/4)$, the canonical momentum $p(t)$ is always monotonically decreasing, as was to be expected from the equation of motion (3.9e), which is indeed satisfied at all times except those at which the jumps occur. Note moreover the neat form of the solution (3.10b), with the only singularities being now the jumps due to the singularities of the cotangent function.

Of course the arguments that have led us to this solution have required some appeal to "physical commonsense", without which no unique prescription could be given to continue the solution of the Hamiltonian ODEs beyond the points at which their solution become singular.

Finally let us point out that both $q(t)$ and $p(t)$ are periodic with period $T/4$, see (3.10). In the following section we shall see that an appropriate quantal treatment of this problem is – somewhat remarkably – devoid of difficulties and that it yields an equispaced spectrum with spacing $4\hbar\omega$, consistently, via (2.6), with (1.1).

Example 3.6.

$$f(q) = \Omega^2 (q+c)^2 \ , \tag{3.11a}$$

entailing that the ω-modified Hamiltonian (3.8a) reads

$$\check{H}(p,q;\omega) = \frac{1}{2}\left[\Omega^2 (q+c)^2 \exp(2ap) + \left(\frac{\omega}{a\Omega}\right)^2 \exp(-2ap)\right] \ , \tag{3.11b}$$

and the corresponding *classical* solution reads

$$p(t) = \frac{1}{2a}\log\left\{\frac{\omega^2}{A^2\Omega^2 \sin^2(\omega t+\theta)}\right\} \ , \tag{3.11c}$$

again with (3.8b) where however now $\sigma = 1$ for all time. Clearly this solution (3.11c) is again *periodic* with period $T/2 = \pi/\omega$, but $p(t)$ again blows up (now always to *positive* infinity) whenever $t = -(\theta/\omega) \mod (T/2)$.

As in the previous case, this solution has in fact entailed the assignment of an appropriate prescription – consistent with the fact that both $q(t)$ and $p(t)$ remain *real* throughout the time evolution, that they satisfy the Hamiltonian equations of motion at all times (of course except those at which singularities occur), and that the Hamiltonian remains constant across these singularities. In the following section we find that – again somewhat remarkably – with an appropriate prescription the *quantized* version of this model encounters no problems and yields an equispaced spectrum with spacing $2\hbar\omega$, again consistent with (1.1) via (2.6).

Example 3.7. To discuss a third variant we now generalize the Hamiltonian (3.8a) by using the following expression of τ rather than (3.7b),

$$\tau(p,q) = \frac{q+c}{af(q)}\exp(-2ap) + K \ , \tag{3.12}$$

with K an *a priori arbitrary* constant. This clearly still satisfies (2.1) and yields a more general expression for $\check{H}(p,q;\omega)$ than (3.8a), namely

$$\check{H}(p,q;\omega) = \frac{(1+\omega^2K^2)f(q)}{2}e^{2ap} + \frac{\omega^2(q+c)^2}{2a^2f(q)}e^{-2ap} + \frac{\omega^2K}{a}(q+c) \ , \tag{3.13a}$$

which shall of course also be *isochronous.* If we now assume the following *imaginary* values for the parameters K and a,

$$K = i\frac{\cosh\chi}{\omega} \ , \quad a = i\alpha \ , \tag{3.13b}$$

where χ and α are two arbitrary *real* numbers, and make the following choice for $f(q)$,

$$f(q) = \frac{\omega}{\alpha\sinh\chi}(q+c) \ , \tag{3.13c}$$

then the Hamiltonian $\check{H}(p,q;\omega)$ is given by the following *real* expression:

$$\check{H}(p,q;\omega) = \frac{\omega}{\alpha}(q+c)(\cosh\chi - \sinh\chi\cos 2\alpha p) \ . \tag{3.13d}$$

This is indeed *isochronous*, but in a somewhat peculiar sense, since the equation of motion for p reads

$$\dot{p} = -\frac{\omega}{\alpha}\left[\cosh\left(\chi\right) - \sinh\left(\chi\right)\cos\left(2\alpha p\right)\right] < 0 \ , \tag{3.13e}$$

so that $p(t)$ is a monotonically decreasing function on the real line. The equations of motion are easily solved to yield

$$q(t) = \left[q(0) + c\right]\frac{\exp\left(\chi\right)\cos^2\left[\omega\left(t+t_0\right)\right] + \exp\left(-\chi\right)\sin^2\left[\omega\left(t+t_0\right)\right]}{\exp\left(\chi\right)\cos^2\left(\omega t_0\right) + \exp\left(-\chi\right)\sin^2\left(\omega t_0\right)} - c \ , \tag{3.13f}$$

$$p(t) = -\frac{1}{\alpha}\arctan\left\{\exp\left(-\chi\right)\tan\left[\omega\left(t+t_0\right)\right]\right\} \ , \tag{3.13g}$$

$$t_0 = -\frac{1}{\omega}\arctan\left\{\exp\left(\chi\right)\tan\left[\alpha p(0)\right]\right\} \ . \tag{3.13h}$$

It is thus seen that

$$q(t+\frac{T}{2}) = q(t) \ , \quad p(t+\frac{T}{2}) = p(t) - \frac{\pi}{\alpha} \ . \tag{3.13i}$$

If we therefore take p as an *angular* variable of period π/α, then the *isochrony* (with period $T/2$, see (2.6)) is restored.

Note that in this case no singularity difficulty arose. And another aspect highlighting the interest of this Hamiltonian model is its *quantization*, see the following section.

3.3 One more example

Example 3.8. Another possibility is to take as point of departure the Hamiltonian of the standard Harmonic oscillator (with circular frequency Ω),

$$H\left(p,q\right) = \frac{1}{2}\left(p^2 + \Omega^2 q^2\right) \ . \tag{3.14}$$

It is then easily seen that a quantity $\tau\left(p,q\right)$ consistent with (2.1) reads

$$\tau\left(p,q\right) = \frac{1}{\Omega}\arctan\left(\frac{\Omega q}{p}\right) \ , \tag{3.15}$$

yielding the ω-modified Hamiltonian

$$\check{H}\left(p,q;\omega\right) = \frac{1}{2}\left(p^2+\Omega^2 q^2\right)\left\{1 + \left[\frac{\omega}{\Omega}\arctan\left(\frac{\Omega q}{p}\right)\right]^2\right\} \ . \tag{3.16a}$$

Via the treatment of the preceding section the time evolutions of the canonical variables p and q induced by this ω-modified Hamiltonian are then easily obtained:

$$q\left(t\right) = \frac{A}{\Omega}\sin\left[\frac{\Omega\tan\left(\omega t+\theta\right)}{\omega}\right]\cos\left(\omega t+\theta\right) \ , \tag{3.16b}$$

$$p(t) = A\cos\left[\frac{\Omega\tan(\omega t+\theta)}{\omega}\right]\cos(\omega t+\theta) \ . \tag{3.16c}$$

Clearly this remarkable time evolution is *periodic* with period T, see (2.6), and it is *nonsingular* in the sense that $q(t)$ and $p(t)$ have *always* well defined, finite values; however, while they both vanish for $t = -\theta/\omega + \pi/(2\omega) \mod (\pi/\omega)$, their time derivatives diverge in an oscillatory manner at these times.

This solution suggests – as the diligent reader will easily verify – that the following transformation from the coordinates q, p to the coordinates x, y,

$$q = \frac{y}{\Omega}\sin\left(\frac{\Omega x}{y}\right) \ , \quad p = y\cos\left(\frac{\Omega x}{y}\right) \ , \tag{3.17a}$$

is *canonical* and reduces the ω-modified Hamiltonian (3.16a) to the standard Hamiltonian characterizing the *linear* harmonic oscillator,

$$\hat{H}(y,x;\omega) = \frac{1}{2}\left(y^2+\omega^2x^2\right) \ ; \tag{3.17b}$$

and it is of course well known that, by the additional, quite analogous, *canonical* transformation

$$x = \frac{\eta}{\omega}\sin\left(\frac{\omega\xi}{\eta}\right) \ , \quad y = \eta\cos\left(\frac{\omega\xi}{\eta}\right) \ , \tag{3.17c}$$

this Hamiltonian becomes that describing free motion,

$$h(\eta,\xi) = \frac{1}{2}\eta^2 \ . \tag{3.17d}$$

Of course each of these Hamiltonians must be complemented by an appropriate specification of the phase space topology, for instance whether the canonical variable must be considered a *Cartesian* coordinate ranging from $-\infty$ to $+\infty$, a *radial* coordinate ranging from 0 to ∞ or an *angle* coordinate ranging from 0 to 2π and requiring that all physically significant quantities (and, in the quantal context, that the eigenfunctions or at least their moduli) be *periodic* in that variable with period 2π.

4 Quantal examples

As we have seen, each of the Hamiltonians of the examples displayed in the preceding Section 3 can be transformed via appropriate *canonical* transformations into that characterizing the standard *linear* harmonic oscillator with circular frequency ω (see (3.17b)) respectively into that characterizing free motion (see (3.17d)). (Indeed, this possibility is a standard result in classical mechanics, associated with the existence of a *canonical* transformation from the original canonical variables to action-angle variables). Thus if the quantization of any one of these models were effected *after* performing such *canonical* transformations its energy spectrum would obviously turn out to be *equispaced* with spacing $\hbar\omega = h/T$ respectively *continuous* – provided in both cases the quantization problem were treated in the "natural" phase space associated to the transformed Hamiltonians, namely the entire real axis for the coordinate (of course the outcome is different for different phase space assignments, see for instance [10]). But the quantization should actually be performed on the original Hamiltonian, without firstly subjecting it to any canonical transformation: and there is then no guarantee that by quantizing a classically *isochronous* system one obtains a quantal Hamiltonian yielding an *equispaced* spectrum, see indeed counterexamples in [5] [6] [7] [8] and below. However a natural hunch is that there generally exist an appropriate application of the quantization procedure – in particular, taking care appropriately of the ordering issue intrinsic in it – such that any (or at least "almost" any) classically *isochronous* Hamiltonian – which should of course be *real* to begin with – yield a quantal Hamiltonian operator that is *Hermitian* (or, more precisely, *self-adjoint*: see below) and features an *equispaced* spectrum; at least whenever the *isochronous* behavior holds, in the classical context, in the *entire* "natural" phase space (possibly up to a *lower-dimensional* phase space sector characterized by *singular* solutions). It is therefore of interest—as we do in this section—to explore such examples; indeed by studying analogous cases interesting features of the quantization process have already been brought to light in the past, see for instance Ref. [5] [6] [7].

Hence in this section we examine in a *quantal* context the examples treated in the *classical* context in the preceding section, although we do not treat each of them with the same thoroughness.

4.1 Four examples

Let us begin by providing a quantal treatment of the Hamiltonian (3.2a) with $k = 1$, without committing ourselves for the moment to any special assignment of the function $f(q)$.

To this end we set

$$q \Longrightarrow z\ , \quad p \Longrightarrow -i\hbar\frac{d}{dz}\ , \tag{4.1}$$

and associate to the classical Hamiltonian (3.2a) with $k = 1$ the following Hermitian operator:

$$\begin{aligned}\check{H}_Q \;=\; & -\frac{\hbar^2}{2}\left[\rho\frac{d}{dz}\frac{1}{f'(z)^2}\frac{d}{dz} + (\sigma-\rho)\frac{1}{f'(z)}\frac{d^2}{dz^2}\frac{1}{f'(z)}\right.\\ & \left.+\frac{1-\sigma}{2}\left(\frac{1}{f'(z)^2}\frac{d^2}{dz^2} + \frac{d^2}{dz^2}\frac{1}{f'(z)^2}\right)\right] + \frac{\omega^2 f(z)^2}{2}\ ,\end{aligned} \tag{4.2}$$

where ρ and σ are two *a priori* arbitrary (*real*) parameters, introduced to take into some account the ordering ambiguity; in particular the Weyl prescription (see for instance Section 5.2 of Ref. [9], or the Appendix of [5]) would assign the values $\rho = \sigma = 1/2$. The corresponding stationary Schrödinger equation then reads

$$\check{H}_Q\psi_n = E_n\psi_n\ , \tag{4.3}$$

with the energy eigenvalues E_n determined by the requirement that the corresponding eigenfunctions $\psi_n \equiv \psi_n(z)$ be *normalizable*.

We now define a new "space" variable x, and new eigenfunctions $\varphi_n \equiv \varphi_n(x)$, by setting

$$x = f(z)\ , \qquad \varphi_n(x) = \left[f'(z)\right]^{-1/2}\psi_n(z)\ . \tag{4.4}$$

Thereby the Schrödinger equation (4.3) takes the form

$$\bar{H}\varphi_n = E_n\varphi_n \ , \tag{4.5a}$$

with the new operator $\bar{H}$ defined as follows:

$$\bar{H} = \frac{1}{2}\left(-\hbar^2\frac{d^2}{dx^2} + \omega^2 x^2\right) + V_Q(x) \ , \tag{4.5b}$$

where the "quantum potential" $V_Q(x)$ is given by the formula

$$V_Q(x) = \frac{\hbar^2}{2[f'(z)]^4}\left\{\left(2\rho + \sigma - \frac{7}{4}\right)[f''(z)]^2 + \left(\frac{1}{2} - \rho\right) f'''(z)f'(z)\right\} . \tag{4.5c}$$

Here, of course, every occurrence of z should be replaced by its expression in terms of x (see (4.4)).

Note that the only place where the ordering prescription – namely, the values assigned to the two parameters ρ and σ – play any role is in this definition of the "quantum potential" $V_Q(x)$. Clearly a convenient assignment of these parameters is

$$\rho = \frac{1}{2} \ , \quad \sigma = \frac{3}{4} \ , \tag{4.6}$$

yielding an altogether vanishing $V_Q(x)$. Then the Schrödinger equation (4.5) can be solved in the standard manner, yielding for our Schrödinger equation (4.3) the following eigenvalues and eigenfunctions:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \ , \quad \psi_n(z) = \left[f'(z)\right]^{1/2} H_n\left[\frac{\omega f(z)}{\sqrt{\hbar}}\right] \exp\left\{-\frac{[\omega f\,(z)]^2}{2\hbar}\right\} \ . \tag{4.7}$$

Here and below $H_n(x)$ denotes the standard Hermite polynomial [13] of degree n. This assumes, of course, that these eigenfunctions $\psi_n(z)$ are normalizable. At the formal level, this does not depend on the choice of the function $f(q)$: indeed, as it is readily verified by change of variables in the relevant integrals, the scalar product of the eigenfunctions $\psi_n(z)$ is formally identical to those of the functions $\varphi_n(x)$ (see (4.4)). The only non-trivial issue concerns the range of variation of the two variables x and z: whenever it entails that x runs over the entire real line, all the above statements are correct.

There is moreover a (special) case in which the "quantum potential" $V_Q(x)$ plays only a trivial role (because it becomes *constant*) irrespective of the values of the parameters ρ and σ: this happens if

$$f\,(z) = a \log\left(\frac{z}{b}\right) \ , \tag{4.8a}$$

yielding

$$V_Q\,(x) = \frac{\hbar^2}{8a^2}\,(4\sigma - 3) \ . \tag{4.8b}$$

And finally another case yielding a Schrödinger equation (4.5) generally associated with an equispaced spectrum obtains for

$$f\,(z) = az^\gamma \ , \tag{4.9a}$$

with γ an arbitrary (*real*) number, yielding

$$V_Q(x) = \frac{\hbar^2(\gamma-1)}{2\gamma^2x^2}\left[\left(\rho+\sigma-\frac{5}{4}\right)\gamma-\sigma+\frac{3}{4}\right] = \frac{\hbar^2\lambda}{x^2} \ . \tag{4.9b}$$

This quantum potential vanishes not only for the assignment (4.6), but also if $\gamma = (3-4\sigma)/(5-4\rho-4\sigma)$ (and as well in the trivial $\gamma=1$ case), implying $\lambda=0$. Moreover, even when λ does not vanish (but provided it is not too negative, $\lambda > -1/4$), the resulting Hamiltonian has an *equispaced* energy spectrum (see (4.5)) with spacing $2\,\hbar\,\omega$, consistently with the fact that, in the classical context, the motions yielded by the Hamiltonian (4.5b) with the *positive* potential (4.9b) take place only on the – *positive* or *negative* – semiline and are *periodic* with period $T/2 = \pi/\omega$. On the other hand, if the Hamiltonian is defined on the whole real line $-\infty < x < \infty$, then in the quantal context each state is doubly degenerate, since the sign of the eigenfunction can be chosen at will on either half-line.

Let us now review how these findings apply, by discussing specifically the *examples 3.1-4* obtained from the Hamiltonian (3.2a) with $k=1$ via specific assignments of the functions $f(q)$.

Clearly no comment is needed on *example 3.1.*

To quantize the ω-modified Hamiltonian (3.4b) of *example 3.2* we replace via the standard quantization prescription (4.1) this Hamiltonian with the Hermitian operator

$$\check{H}_Q = \frac{1}{2}\left\{-\left(\frac{\hbar}{a}\right)^2\left[z^2\frac{d^2}{dz^2}+2z\frac{d}{dz}+1-\sigma\right]+\left[\omega a\log\left(\frac{z}{b}\right)\right]^2\right\} \ , \tag{4.10a}$$

where σ is as above (see (4.2)) and ρ has disappeared. Recall that these parameters (see (4.2)) are introduced to take some account of the ordering freedom in the transition from the classical to the quantal context. After changing from the z variable to the x variable via the transformation (4.4) with

$$x = a\sqrt{\frac{\omega}{\hbar}}\,\ln\left(\frac{z}{b}\right) \ , \tag{4.10b}$$

the corresponding stationary Schrödinger equation (4.5a) reads (see (4.5))

$$\hbar\left(-\frac{d^2}{dx^2}+x^2\right)\varphi_n(x)+\frac{\hbar^2}{8a^2}(4\sigma-3)\,\varphi_n(x) = E_n\varphi_n(x) \ , \tag{4.10c}$$

with the energy eigenvalues E_n determined by the requirement that the corresponding eigenfunctions $\psi_n(z)$ be normalizable in the interval $0 \le z < \infty$. Since x runs over the entire real axis as z goes from 0 to ∞ (we are of course assuming that b is *positive*), the corresponding functions $\varphi_n(x)$ (see (4.4)) are given by the usual eigenfunctions of the harmonic oscillator and the energy spectrum reads

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right)+\frac{\hbar^2}{8a^2}(4\sigma-3) \ , \tag{4.10d}$$

with the corresponding eigenfunctions reading

$$\psi_n(z) = z^{-1/2}H_n(x)\exp\left(-\frac{x^2}{2}\right) \ , \quad x = \left(\frac{\omega a^2}{\hbar}\right)^{1/2}\log\left(\frac{z}{b}\right) \ , \tag{4.10e}$$

up to a normalization factor. Note that these eigenfunctions vanish both at $z=0$ and, of course, as $z\to\infty$; and the corresponding energy spectrum is of course *equispaced*, indeed consistently with (1.1).

Likewise, to quantize the ω-modified Hamiltonian (3.5b) of *example 3.3* we replace via (4.1) this Hamiltonian with the Hermitian operator

$$\begin{aligned}\check{H}_Q \;=\;& -\frac{1}{2}\left(\frac{\hbar}{a}\right)^2\left\{z^2\log^2\left(\frac{z}{b}\right)\frac{d^2}{dz^2}+2z\log\left(\frac{z}{b}\right)\left[1+\log\left(\frac{z}{b}\right)\right]\frac{d}{dz}\right.\\&\left.+(\sigma-\rho)\log\left(\frac{z}{b}\right)+(1-\sigma)\left[\log^2\left(\frac{z}{b}\right)+3\log\left(\frac{z}{b}\right)+1\right]\right\}\\&+\frac{1}{2}\left\{\omega a\log\left[\beta\log\left(\frac{z}{b}\right)\right]\right\}^2\;, \qquad (4.11a)\end{aligned}$$

where ρ and σ are as above (see (4.2)). Recall that these parameters (see (4.2)) are introduced to take some account of the ordering freedom in the transition from the classical to the quantal context. Again it can be mapped by the transformation (4.4) with

$$x=a\sqrt{\frac{\omega}{\hbar}}\,\log\left[\beta\log\left(\frac{z}{b}\right)\right] \qquad (4.11b)$$

into the harmonic oscillator Hamiltonian. Here the range of values of z goes from b to ∞, and correspondingly the variable x ranges over the entire real axis, $-\infty<x<\infty$. The eigenfunctions $\varphi_n(x)$ of the transformed problem are therefore the standard eigenfunctions of the harmonic oscillator and the eigenfunctions $\psi_n(z)$ of the original Schrödinger equation read, up to a normalization factor,

$$\psi_n(z)=\left[z\log\left(\frac{z}{b}\right)\right]^{-1/2}H_n(x)\exp\left(-\frac{x^2}{2}\right)\;. \qquad (4.11c)$$

Again, these eigenfunctions vanish both at $z=b$ and as $z\to\infty$, and they are indeed *normalizable* (in the interval $b\le z<\infty$ – as the alert reader will easily verify). The spectrum is clearly *equispaced*:

$$E_n=\hbar\omega\left(n+\frac{1}{2}\right)\;,\quad n=0,1,2,\ldots\;, \qquad (4.11d)$$

indeed consistently with (1.1).

Next, to quantize the ω-modified Hamiltonian (3.6b) of *example 3.4* we replace via (4.1) this Hamiltonian with the Hermitian operator

$$\begin{aligned}\check{H}_Q \;=\;& -\frac{1}{2}(j\hbar)^2\left(\frac{z}{a}\right)^{2(j-1)/j}\left\{\frac{d^2}{dz^2}+2\left(\frac{j-1}{jz}\right)\frac{d}{dz}\right.\\&\left.+\frac{(j-1)\left[(1-\sigma)(j-2)-(\sigma-\rho)(j-1)\right]}{j^2z^2}\right\}\\&+\frac{1}{2}\omega^2a^2\left[\left(\frac{z}{a}\right)^{1/j}-\frac{c}{a}\right]^2\;, \qquad (4.12a)\end{aligned}$$

where ρ and σ are as above, see (4.2). The transformation (4.4) now reads

$$\psi_n(z)=z^{(1-j)/(2j)}\varphi_n(x)\;,\quad x=a\sqrt{\frac{\omega}{\hbar}}\left[\left(\frac{z}{a}\right)^{1/j}-\frac{c}{a}\right]\;. \qquad (4.12b)$$

There remains to discuss the range of z, and hence of x. If the integer j is *odd*, then $q(t)$ (see (3.6c)) ranges over the entire *real* axis, and thus so do z and x, irrespective of the sign of j. If, on the other hand, j is *even*, the Hamiltonian is *multivalued* and one needs to follow the correct branch, as discussed above in the classical context. While one could try and do something analogous in the quantum case, this would carry us too far from the main line of our work; for related work, however, see for example [11]. Hence hereafter j is an *odd integer*.

The eigenvalue equation then reads (see (4.5b) and (4.5c))

$$-\varphi_n''(x) + \frac{\lambda}{(x+\bar{x})^2}\varphi_n(x) + x^2\varphi_n(x) = 2\eta_n\varphi_n(x) \ , \tag{4.12c}$$

with

$$\bar{x} = \left(\frac{\omega c^2}{\hbar}\right)^{1/2} \ , \quad E_n = \hbar\omega\eta_n \ , \quad \lambda = 2(j-1)\left[\left(\sigma - \frac{3}{4}\right)j - \rho - \sigma + \frac{5}{4}\right]\hbar\omega \ . \tag{4.12d}$$

It is now plain that, for the assignment (4.6), λ vanishes, $\lambda = 0$ (see (4.12d)), and clearly this entails (see (4.12c)) that the formula

$$\psi_n(z) = z^{(1-j)/(2j)}\exp\left(-\frac{1}{2}x^2\right)H_n(x) \ , \quad x = a\sqrt{\frac{\omega}{\hbar}}\left[\left(\frac{z}{a}\right)^{1/j} - \frac{c}{a}\right] \ , \tag{4.12e}$$

provides a normalizable solution (in the interval $0 \le z < \infty$) of our Schrödinger equation (4.10c); and of course the corresponding energy eigenvalues yield the equispaced spectrum of the standard linear harmonic oscillator,

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \ , \quad n = 0, 1, 2... \ . \tag{4.12f}$$

This is a somewhat surprising result, since for arbitrary *odd integer* j the corresponding classical problem features a sector of its phase space where the time evolution runs into a singularity: when j is *positive* only the momentum $p(t)$ blows up, while the coordinate $q(t)$ always remains finite, whereas for $j < 0$ the opposite happens.

Note moreover that, if $\lambda \neq 0$, then the spectrum is only equispaced if $\bar{x} = 0$ (i. e., if $c = 0$, see (4.12d)): in this case, the spacing is $2\,\hbar\,\omega$, but each state is doubly degenerate, as stated above. On the other hand, if $\bar{x} \neq 0$, then the spectrum only becomes equispaced asymptotically, as $n \to \infty$, again with spacing $2\,\hbar\,\omega$.

Moreover, if the constant c vanishes, $c = 0$ (entailing $\bar{x} = 0$, see (4.12d)) – in which case the time evolution of the classical problem *always* runs into a singularity, for any initial data – the quantal problem nevertheless admits a well defined normalizable eigensolution in the interval $0 \le z < \infty$ provided j is an *odd integer* and $\lambda > -1/4$ (see (4.12b) and (4.12c)), to which there corresponds, up to a common shift of all the eigenvalues, an equispaced spectrum with spacing $2\,\hbar\,\omega$. Note that this spacing is just what one expects from the classical behavior, see (1.1) and the treatment of this *example 3.4* in the preceding section.

4.2 Three examples

Let us now turn to the quantization of the three models detailed in the preceding section as *examples 3.5*, *3.6* and *3.7*; while the quantization of the model of *example 3.8* is sufficiently complicated to require an *ad hoc* treatment, to be given in a separate paper.

To quantize the Hamiltonian (3.9b) of *example 3.5* we now set

$$q \Longrightarrow i\hbar\frac{d}{dz}\ , \quad p \Longrightarrow z\ , \tag{4.13}$$

which corresponds to the usual rule up to a Fourier transformation of the eigenfunctions. We thereby transform the Hamiltonian (3.9b) into the Hermitian *first-order* differential operator (in momentum space)

$$\begin{aligned} \check{H}_Q = \frac{b}{2}\left[\exp(az)\left(i\hbar\frac{d}{dz}+c\right)\exp(az)+\left(\frac{\omega}{ab}\right)^2\exp(-az)\left(i\hbar\frac{d}{dz}+c\right)\exp(-az)\right] \\ = \frac{b}{2}\left\{\exp(2az)\left[i\hbar\left(\frac{d}{dz}+a\right)+c\right]+\left(\frac{\omega}{ab}\right)^2\exp(-2az)\left[i\hbar\left(\frac{d}{dz}-a\right)+c\right]\right\}\ . \end{aligned} \tag{4.14a}$$

The corresponding stationary Schrödinger equation (in momentum space) reads

$$\begin{aligned} b\left\{\left[\exp(2az)+\left(\frac{\omega}{ab}\right)^2\exp(-2az)\right]\left[i\hbar\psi'(z)+c\psi(z)\right]\right. \\ \left.+i\hbar a\left[\exp(2az)-\left(\frac{\omega}{ab}\right)^2\exp(-2az)\right]\psi(z)\right\} = 2E\psi(z)\ , \end{aligned} \tag{4.14b}$$

with the energy eigenvalue E to be determined by the requirement that the corresponding eigenfunction $\psi(z)$ be *normalizable* and *univalent* in the interval $-\infty < z < \infty$. This *first-order* ODE is easily integrated, yielding, up to a normalization constant,

$$\begin{aligned} \psi_E(z) = \exp\left(\frac{icz}{\hbar}\right)\left[\exp(2az)+\left(\frac{\omega}{ab}\right)^2\exp(-2az)\right]^{-1/2}\cdot \\ \cdot\exp\left\{\frac{E}{i\omega\hbar}\arctan\left[\frac{ab}{\omega}\exp(2az)\right]\right\}\ . \end{aligned} \tag{4.14c}$$

This eigenfunction is clearly *normalizable* (on the entire *real* axis, $-\infty < z < +\infty$; recall that now the variable z corresponds to the classical canonical momentum p, see (4.13)), since it is *nonsingular* in that entire interval and its modulus vanishes asymptotically, at both ends, proportionally to $\exp(-|az|)$. Note that this happens without entailing any restriction on the energy eigenvalues E. But such restrictions – yielding a discrete *equispaced* energy spectrum – are caused by the additional requirements implied by the following discussion.

A somewhat heuristic approach might be based on revisiting the corresponding classical problem and recalling that, in that context, one concluded that the momentum variable $p(t)$ has a time evolution – see (3.8c) – entailing that it travels over and over across the *entire real* axis from $+\infty$ to $-\infty$, implying that the expression $\exp(2ap)$ travel, during the time evolution, over and over throughout the *positive real* axis, from $+\infty$ to 0. This suggests that, in the quantal context, one should correspondingly require the eigenfunction, see (4.14c), to be *univalent* when the quantity $\exp(2az)$ goes through an analogous ordeal, namely it traverses over and over the *positive real* axis an arbitrary number of times. Because of the corresponding $\bmod(\pi/2)$ shift of the arctangent function (see the right-hand side of (4.14c)), such a requirement would force the energy eigenvalues E_n to belong to the *equispaced* spectrum

$$E_n = 4\hbar\omega n \ , \quad n = 0, \pm 1, \pm 2, \dots \ . \tag{4.14d}$$

Note that this energy spectrum is indeed consistent via (1.1) with the periodicity $\tilde{T} = T/4$ of the solution of this model in the classical context, see *example 3.5* in the preceding section.

Another, more rigorous, argument – actually corresponding to a more rigorous treatment of this problem in the quantal context – assumes the variable z to span just once the *real* z axis, so that one can limit consideration to the principal determination of the arctangent function (when its argument ranges between infinity and zero, see (4.14c)), eliminating thereby any ambiguity about the univalence of these eigenfunctions. But one then notes that the requirement that must be imposed on this quantal problem is not only that its eigenfunctions be *normalizable*: they must belong to a space within which the operator $\check{H}_Q$ is *self-adjoint*.

The simplest way to enforce this requirement is to require that the Hilbert-space scalar product of two of these eigenfunctions vanish when the corresponding energy eigenvalues are different, namely that

$$\int_0^\infty dz \, [\psi_E(z)]^* \, \psi_{E'}(z) = 0 \quad \text{whenever} \quad E \neq E' \ , \tag{4.15a}$$

i. e., via (4.14c),

$$\int_0^\infty dz \left[\exp(2az) + \left(\frac{\omega}{ab}\right)^2 \exp(-2az) \right]^{-1} \exp\left\{ \frac{i\,(E-E')}{\omega\hbar} \arctan\left[\frac{ab}{\omega} \exp(2az) \right] \right\}$$
$$= \ 0 \quad \text{whenever} \quad E \neq E' \ . \tag{4.15b}$$

The integral in the left-hand side of this formula can be easily performed, yielding the formula

$$\frac{b\,[\exp(i\Delta\pi) - 1]}{4i\omega\Delta} = 0 \quad \text{whenever} \quad E \neq E' \ , \tag{4.15c}$$

with

$$\Delta = \frac{E - E'}{2\hbar\omega} \ . \tag{4.15d}$$

And this yields clearly the energy spectrum

$$E_n = 4n\hbar\omega + E_0 \ , \quad n = 0, \pm 1, \pm 2, \dots \ . \tag{4.15e}$$

And let us also mention that the same outcome obtains if a *self-adjoint* extension of the Hamiltonian operator $\check{H}_Q$, see (4.14a), were performed via the standard "deficiency indices" procedure (see for instance [12]). Indeed the freedom in the choice of E_0 corresponds to the fact that a one-parameter class of *self-adjoint* extensions of $\check{H}_Q$ exists.

Note that – somewhat surprisingly – the singularities of the *classical* motion do not show up disturbingly in the *quantal* context; and let us also re-emphasize the consistency (via (2.6)) of this result with (1.1).

Finally note that the solution (4.14c) can be rewritten in the following rather neat manner:

$$\psi_n(z) = \exp\left(\frac{icz}{\hbar}\right) v\left[T_{2n}(u) - iv\exp(az)\, U_{2n-1}(u)\right] \ , \tag{4.15f}$$

$$v = \left[\exp(2az) + \left(\frac{\omega}{ab}\right)^2 \exp(-2az)\right]^{-1/2} \ , \quad u = \frac{\omega v \exp(-az)}{ab} \ , \tag{4.15g}$$

where $T_n(u)$ respectively $U_n(u)$ are Chebyshev polynomials of the first respectively second kind (see for instance [13]).

Next, to quantize the Hamiltonian (3.11b) of *example 3.6* of the preceding section, we use again the prescription (4.13) with the specific ordering assignment yielding the following Hermitian operator (again of course in momentum space):

$$\check{H}_Q = \frac{1}{2}\left[\Omega^2 \exp(az)\left(c + i\hbar\frac{d}{dz}\right)^2 \exp(az) + \left(\frac{\omega}{a\Omega}\right)^2 \exp(-2az)\right] \ . \tag{4.16a}$$

It can then be verified that the corresponding Schrödinger equation (in momentum space),

$$\Omega^2 \exp(2az)\left[-\hbar^2\psi''(z) + 2i\hbar(c + i\hbar a)\psi'(z) + (c + i\hbar a)^2\psi(z)\right] + \left(\frac{\omega}{a\Omega}\right)^2 \exp(-2az)\,\psi(z) = 2E\psi(z) \ , \tag{4.16b}$$

can be reduced via the position

$$\psi(z) = \exp\left[\left(-a + i\frac{c}{\hbar}\right)z\right]\exp\left(-\frac{x}{2}\right)\varphi(x) \ , \tag{4.17a}$$

$$x = \left(\frac{\omega}{\hbar a^2\Omega^2}\right)\exp(-2az) \ , \tag{4.17b}$$

to the confluent hypergeometric equation [14]

$$x\varphi''(x) + (\beta - x)\varphi'(x) - \alpha\varphi(x) = 0 \ , \tag{4.17c}$$

with

$$\alpha = \frac{1}{2}(1 - \eta) \ , \quad \beta = 1 \ , \quad E = \hbar\omega\eta \ . \tag{4.17d}$$

It is now easily seen that the requirement that the eigenfunction (4.17) be *normalizable* is *not* sufficient to identify uniquely a corresponding solution of the confluent hypergeometric equation; indeed it is easily seen [14] that *any* solution of the confluent hypergeometric equation (4.17c) with $\beta = 1$, when inserted in the *ansatz* (4.17), yields a function $\psi(z)$ which is *normalizable* over the entire real axis, $-\infty < z < \infty$ (for any sign of the *real* constant a) if

$$\eta = \eta_n = 2n + 1 \ , \tag{4.18}$$

while for *arbitrary* η the solution

$$\varphi(x)=\exp\left(-\frac{x}{2}\right)U\left(\frac{1-\eta}{2},1,x\right) \tag{4.19}$$

is also *normalizable.* Here $U(\alpha,1,x)$ is the second ("logarithmic") solution of the confluent hypergeometric equation ODE with $\beta=1$, see [14]. The point is that one must again introduce the stronger requirement that the eigenfunction $\psi(z)$ belong to a functional space in which the operator (4.16a) is *self-adjoint.* This amounts to the requirement that, when the norm of the operator $\hat{H}_Q$, see (4.16a), is written out,

$$\left(\psi,\hat{H}_Q\psi\right)=\int_{-\infty}^{+\infty}dz\,\psi^*(z)\,\hat{H}_Q\psi(z)\ , \tag{4.20}$$

in the integral appearing in the right-hand side integrations by parts should be allowed without running into divergences. It is then easily seen (using standard properties of the confluent hypergeometric function [14]; or inferring them directly from (4.17c)) that this requirement excludes the logarithmic solution $U(\alpha,1,x)$ of the confluent hypergeometric equation and restrict the other solution via the requirement (4.18), implying that

$$\varphi_n(x)=L_n(x)\ , \tag{4.21a}$$

so that the eigenfunctions read

$$\psi_n(z)=\exp\left[\left(-a+i\frac{c}{\hbar}\right)z\right]\exp\left(-\frac{x}{2}\right)L_n(x)\ ,\quad n=0,1,2,...\ , \tag{4.22a}$$

where $L_n(x)$ is the standard Laguerre polynomial [13] of order n. The fact that these eigenfunctions are *nonsingular*, *univalent* and *normalizable* over the entire real axis, $-\infty<z<\infty$ (for any sign of the *real* constant a) is plain (although the requirement that these properties hold was in fact not sufficient to identify these eigenfunctions). And of course the corresponding energy spectrum is (see the last of the formulas (4.17d) and (4.18))

$$E_n=(2n+1)\,\hbar\omega\ ,\quad n=0,1,2,...\ . \tag{4.22b}$$

It is therefore *equispaced* with spacing $2\hbar\omega$, consistently via (1.1) with the periodicity with period $\tilde{T}=T/2$ of the corresponding *classical* problem, see *example 3.6* in the preceding section. And note – once more – that the *quantal* problem does not seem to be affected at all by the singular character of the motions in the corresponding *classical* problem.

We finally turn to *example 3.7* of the preceding section, which has some peculiar features. Firstly we treat it using again the quantization prescription (4.13). It then yields the following Hermitian operator (in momentum space)

$$\check{H}_Q=\frac{\omega}{\alpha}\left\{\cosh(\chi)\left(c+i\hbar\frac{d}{dz}\right)\right.$$
$$\left.-\frac{\sinh(\chi)}{2}\left[\left(c+i\hbar\frac{d}{dz}\right)\cos(2\alpha z)+\cos(2\alpha z)\left(c+i\hbar\frac{d}{dz}\right)\right]\right\}\ , \tag{4.23a}$$

where, for simplicity, we restricted consideration to just one specific ordering. This gives as stationary Schrödinger equation (in momentum space) the following first-order linear

ODE:

$$\frac{\omega}{\alpha}\left[\cosh\left(\chi\right)-\sinh\left(\chi\right)\cos\left(2\alpha z\right)\right]\left(c+i\hbar\frac{d}{dz}\right)\psi\left(z\right)$$
$$+i\hbar\omega\sinh\left(\chi\right)\sin\left(2\alpha z\right)\psi\left(z\right)=E\psi\left(z\right)\ . \tag{4.23b}$$

Proceeding formally, that is, disregarding issues concerning the range of z and the possible multivaluedness of $\psi(z)$, one gets

$$\begin{aligned}\psi(z) &= \left[\cosh\left(\chi\right)-\sinh\left(\chi\right)\cos\left(2\alpha z\right)\right]^{-1/2}\cdot\\ &\cdot\exp\left(-\frac{cz}{i\hbar}\right)\exp\left\{\frac{E}{i\hbar\omega}\arctan\left[\exp\left(\chi\right)\tan\left(\alpha z\right)\right]\right\}\ .\end{aligned} \tag{4.23c}$$

Let us now consider the restrictions to be imposed on E by the requirement that $\psi(z)$ be *univalent.* Here we must distinguish two cases, as usual for systems where one of the variables (in this case z) has a discrete translation symmetry.

Firstly, we may assume that z runs over the entire real line. In this case E can take arbitrary values and one obtains from (4.23c) the relation

$$\psi\left(z+\frac{\pi}{\alpha}\right)=\exp\left[-\frac{\pi}{i\hbar}\left(\frac{c}{\alpha}+\frac{E}{\omega}\right)\right]\psi(z)\ . \tag{4.23d}$$

Therefore the eigenfunction $\psi(z)$ behaves as a Bloch wave function under translations by π/α.

If, on the other hand, z is assumed to be an *angular* variable with range π/α, then the requirement $\psi\left(z+\pi/\alpha\right)=\psi(z)$ yields the quantization condition

$$E_n=2n\hbar\omega-\frac{c\omega}{\alpha}\ ,\quad n=0,\pm1,\pm2,...\ , \tag{4.23e}$$

so that the corresponding eigenfunctions read

$$\begin{aligned}\psi_n(z) &= \left[\cosh\left(\chi\right)-\sinh\left(\chi\right)\cos\left(2\alpha z\right)\right]^{-1/2}\cdot\\ &\cdot\exp\left\{\left(-\frac{c}{i\hbar}\right)\left(z-\frac{\arctan\left[\exp\left(\chi\right)\tan\left(\alpha z\right)\right]}{\alpha}\right)\right\}\cdot\\ &\cdot\exp\left\{-2in\arctan\left[\exp\left(\chi\right)\tan\left(\alpha z\right)\right]\right\}\ .\end{aligned} \tag{4.23f}$$

Note that in this case, which, as pointed out above (see *example 3.7* in the preceding section), is the only one that, in the *classical* context, is truly *isochronous*, one has the expected *equispaced* spectrum, with the correct spacing $2\hbar\omega$, see (1.1), since the period $\tilde{T}$ of the corresponding *classical* motion is $T/2=\pi/\omega$.

Let us finally point out that, if one uses the “standard” quantization rule

$$q\Longrightarrow x\ ,\quad p\Longrightarrow -i\hbar\frac{d}{dx}\ , \tag{4.24a}$$

(see (4.1), and notice the merely notational change of independent variable) instead of its momentum momentum space counterpart (4.13), then the Schrödinger equation, using the same ordering rule as before, reads as follows:

$$\frac{\omega}{\alpha}(x+c)\left\{\cosh\left(\chi\right)\varphi(x)-\frac{\sinh\chi}{2}\left[\varphi(x+2\alpha\hbar)+\varphi(x-2\alpha\hbar)\right]\right\}$$
$$-\frac{\hbar\omega}{2}\sinh\left(\chi\right)\left[\varphi(x+2\alpha\hbar)-\varphi(x-2\alpha\hbar)\right]=E\varphi(x)\ , \tag{4.24b}$$

where we now use the notation $\varphi(x)$ for the eigenfunction in x-space. And of course this three-term recursion relation, supplemented by the appropriate boundary conditions, shall yield the same energy spectrum (4.23e), with the eigenfunctions $\varphi_n(x)$ being just the Fourier transforms of the eigenfunctions $\psi_n(z)$, see (4.23f),

$$\varphi_n(x) = \int_0^{\pi/\alpha} dz \exp\left(\frac{ixz}{\hbar}\right) \psi_n(z) \ . \tag{4.24c}$$

4.3 Singularities of the classical motion that do not affect the corresponding quantal problem

Finally, let us make two heuristic comments concerning the fact we observed in some of the examples discussed above, namely that some systems yielding motions running into singularities in a finite time when considered in the *classical* context do not seem to be plagued by any corresponding difficulty when treated in a *quantal* context: in particular, no disturbing effects surfaced regarding the existence, smoothness and univalence of the corresponding quantum wave functions.

In the first place this can be attributed to the general – if admittedly vague – notion that PDEs tend to smooth out singularities more than ODEs; and to the fact that the transition from quantum mechanics to classical mechanics entails a limit ($\hbar \to 0$) that is notoriously singular, thereby explaining why problems featuring no singular phenomena when treated in a *quantal* context may instead give rise to such difficulties when treated in a *classical* context.

Secondly, let us consider the semiclassical approximation for the wave function at energy E of the type of systems we have been considering here. It is given by

$$\psi_E(x) = \left(\frac{dS_E(x)}{dx}\right)^{-1/2} \exp\left[\frac{i}{\hbar} S_E(x)\right] \ , \tag{4.25a}$$

where $S_E(x)$ is the action at the point x along an orbit having energy E, evaluated from some given initial point x_0:

$$S_E(x) = \int_{x_0}^{x} dx' \, p_E(x') \ , \tag{4.25b}$$

where $p_E(x)$ denotes the momentum corresponding to the energy E and the coordinate x, and the integration is of course along the orbit. The approximate WKB quantization condition then follows straightforwardly, for closed orbits, from the requirement of *univalence* of this approximate wave function $\psi_E(x)$. It is straightforward to check that in all the cases discussed above this integral (4.25b) converges. This is a consequence for *isochronous* systems of the general property

$$\frac{d}{dE} \oint S_E(x) dx = T(E) \ , \tag{4.25c}$$

where the integral is now extended over the (closed: forth and back) trajectory of a periodic orbit, $T(E)$ is the period of this orbit and E its energy. Since for an *isochronous* system $T(E)$ is always *finite* indeed *constant*, the action integral (4.25b) can never diverge. The semiclassical approximation of $\psi_E(x)$ is therefore smooth, apart from divergences of the prefactor, which are generally well-known to disappear in an *exact* (rather than *semiclassical*) treatment. This can therefore be seen as another heuristic justification of the fact that the exact wave functions of *isochronous* systems will not be singular even when singularities plague the corresponding classical motions, as indeed found in some of the examples treated above.

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