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Harmonic maps between quaternionic Kähler manifolds

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Abstract

In this note we introduce the concept of (σ, σ')-holomorphic map between two almost quaternionic Hermitian manifolds. We prove that a (σ, σ')-holomorphic map between two quaternionic Kähler manifolds with a certain property is a harmonic map and give some conditions for the stability of such a map.

1 Introduction

The quaternionic structures generalize much relevant properties for 4-dimensional semi-Riemannian manifolds to higher $4n$-dimensional manifolds, some of them being relevant for mathematical physics. We consider the quaternionic Kähler manifolds which have the property to be Einstein manifolds and other remarkable properties [1, 4, 5, 16, 20]. R. Penrose founded out a twistor programme [19] using twistor correspondence for transforming conformal invariant fields given on Minkowski complex space into objects of complex geometry that are defined on the twistor space.

It is well known that the twistor theory is closely connected with the existence of canonical quaternionic structures on 4-dimensional oriented semi-Riemannian manifolds. L. Berard-Bergery [3], S. Salamon [20] and others authors extended the theory to $4n$-dimensional quaternionic manifolds. On other hand, an interesting mechanism for space-time compactification in the theory of Kaluza-Klein type is proposed in the form of a nonlinear sigma model, i.e. harmonic map. The general solutions of this model can be expressed in terms of harmonic maps satisfying the Einstein equations. The idea of coupling the Einstein field equation to harmonic map seems to appear firstly in the paper of V. de Alfaro, S. Fubini and G. Furlan [9].

Roughly speaking, a quaternionic Kähler manifold is an oriented $4n$-dimensional Riemannian manifold whose restricted holonomy group is contained in the subgroup $Sp(n)Sp(1)$

of $SO(4n)$. These manifolds are of special interest because $Sp(n)Sp(1)$ is included in the list of Berger of possible holonomy groups of locally irreducible Riemannian manifolds that are not locally symmetric [4]. On the other hand, the study of harmonic maps was initiated by J. Eells and J.H. Sampson [10] and this topic has been intensively studied later by several authors. In Section 2 we recall the definitions of quaternionic manifolds and harmonic maps.

There exists now a rich literature concerning the holomorphic maps between almost hermitian manifolds and also between almost contact metric manifolds. In Section 3 we extend this concept in quaternionic setting. Thus, we introduce the notion of (σ, σ')-holomorphic map between two almost quaternionic hermitian manifolds and study this kind of maps. In particular, we prove that they are harmonic maps under some hypothesis.

A harmonic map is said to be stable if the second variation of the energy is non-negative for any smooth variation of the map. The stability of harmonic maps it is of great interest in geometry and mathematical physics [2, 13, 21] and it has been studied in Riemannian geometry [17, 22], in complex geometry [6, 18], and in contact geometry [14, 15]. Motivated by these considerations, in Section 4 we give some conditions for the stability of a harmonic map between two quaternionic Kähler manifolds.

Owing to the remarkable properties of quaternionic Kähler manifolds, the results obtained can have important applications in string theory, solitons, theory of liquid crystals, gravity and general relativity.

2 Preliminaries

Let M be a differentiable manifold of dimension n and assume that there is a rank 3-subbundle σ of $End(TM)$ such that a local basis $\{J_1, J_2, J_3\}$ exists of sections of σ satisfying for all $\alpha \in \{1, 2, 3\}$:

$$J_\alpha^2 = -Id, J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}, \tag{2.1}$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Then the bundle σ is called an almost quaternionic structure on M and $\{J_1, J_2, J_3\}$ is called a canonical local basis of σ. Moreover, (M, σ) is said to be an almost quaternionic manifold. It is easy to see that any almost quaternionic manifold is of dimension $n = 4m$ and orientable.

A Riemannian metric g on M is said to be adapted to σ if it satisfies:

$$g(J_\alpha X, J_\alpha Y) = g(X, Y), \forall \alpha \in \{1, 2, 3\} \tag{2.2}$$

for all vector fields X,Y on M and any canonical local basis $\{J_1, J_2, J_3\}$ of σ. Moreover, (M, σ, g) is said to be an almost quaternionic Hermitian manifold.

If the bundle σ is parallel with respect to the Levi-Civita connection ∇ of g, then (M, σ, g) is said to be a quaternionic Kähler manifold. Equivalently, locally defined 1-forms $\omega_1, \omega_2, \omega_3$ exist such that we have:

$$\nabla_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}, \tag{2.3}$$

for any vector field X on M, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

The second fundamental form α_f of a map $f : (M, g) \to (N, g')$ between two Riemannian manifolds is defined by: $\alpha_f(X, Y) = \widetilde{\nabla}_X f_*Y - f_*\nabla_X Y$, for any vector fields X, Y on M, where ∇ is the Levi-Civita connection of M and $\widetilde{\nabla}$ is the pullback of the connection ∇' of N to the induced vector bundle $f^{-1}(TN)$: $\widetilde{\nabla}_X f_*Y = \nabla'_{f_*X} f_*Y$. The tension field $\tau(f)$ of f is defined as the trace of α_f, i.e.:

$$\tau(f)_x = \sum_{i=1}^{m} \alpha_f(e_i, e_i), \tag{2.4}$$

where $\{e_1, e_2, ..., e_m\}$ is a local orthonormal frame of T_xM, $x \in M$. We say that f is a harmonic map if and only if $\tau(f)$ vanishes at each point $x \in M$.

3 Harmonic maps between almost quaternionic Hermitian manifolds

Definition 1. Let (M, σ, g) and (N, σ', g') be two almost quaternionic Hermitian manifolds. A map $f : M \to N$ is called a (σ, σ')-holomorphic map at a point $x \in M$ if for any $J \in \sigma_x$ exists $J' \in \sigma'_{f(x)}$ such that $f_* \circ J = J' \circ f_*$. Moreover, we say that f is a (σ, σ')-holomorphic map if f is a (σ, σ')-holomorphic map at each point $x \in M$.

Theorem 1. *Let (M, σ, g) and (N, σ', g') be two almost quaternionic Hermitian manifolds. If $f : M \to N$ is a (σ, σ')-holomorphic map, then we have:*

$$J'_\alpha(\tau(f)) = f_*(divJ_\alpha) - trace_g f^*\nabla' J'_\alpha, \tag{3.1}$$

for all $\alpha \in \{1, 2, 3\}$, for any canonical local basis $\{J_1, J_2, J_3\}$ of σ and corresponding local basis $\{J'_1, J'_2, J'_3\}$ of σ'.

Proof. Let $\{e_1, ..., e_m, J_1e_1, ..., J_1e_m, J_2e_1, ..., J_2e_m, J_3e_1, ..., J_3e_m\}$ be an orthonormal basis of T_xM, $x \in M$. It is easy to obtain:

$$\begin{aligned} f_*(divJ_\alpha) - J'_\alpha(\tau(f)) \;&=\; \sum_{i=1}^{m} f_*\nabla_{e_i} J_\alpha e_i - \sum_{i=1}^{m} f_*\nabla_{J_\alpha e_i} e_i \\ &\quad - \sum_{i=1}^{m} J'_\alpha \widetilde{\nabla}_{e_i} f_*e_i + \sum_{i=1}^{m} \sum_{\beta\neq\alpha} f_*\nabla_{J_\beta e_i} J_\alpha J_\beta e_i \\ &\quad - \sum_{i=1}^{m} \sum_{\beta=1}^{3} J'_\alpha \widetilde{\nabla}_{J_\beta e_i} f_*J_\beta e_i. \end{aligned} \tag{3.2}$$

Also, we have:

$$trace_g f^*\nabla' J'_\alpha = \sum_{i=1}^{m} (\nabla'_{f_*e_i} J'_\alpha)(f_*e_i) + \sum_{i=1}^{m} \sum_{\beta=1}^{3} (\nabla'_{f_*J_\beta e_i} J'_\alpha)(f_*J_\beta e_i),$$

and since f is a (σ, σ')-holomorphic map, we obtain:

$$\begin{aligned} trace_g f^*\nabla' J'_\alpha &= \sum_{i=1}^m \widetilde{\nabla}_{e_i} f_* J_\alpha e_i - \sum_{i=1}^m J'_\alpha \widetilde{\nabla}_{e_i} f_* e_i - \sum_{i=1}^m \widetilde{\nabla}_{J_\alpha e_i} f_* e_i \\ &\quad + \sum_{i=1}^m \sum_{\beta\neq\alpha} \widetilde{\nabla}_{J_\beta e_i} f_* J_\alpha J_\beta e_i - \sum_{i=1}^m \sum_{\beta=1}^3 J'_\alpha \widetilde{\nabla}_{J_\beta e_i} f_* J_\beta e_i. \end{aligned} \tag{3.3}$$

On the other hand, from:

$$\widetilde{\nabla}_X f_* Y - \widetilde{\nabla}_Y f_* X = f_*[X, Y]$$

we derive:

$$\sum_{i=1}^m \widetilde{\nabla}_{e_i} f_* J_\alpha e_i - \sum_{i=1}^m \widetilde{\nabla}_{J_\alpha e_i} f_* e_i = \sum_{i=1}^m f_*[e_i, J_\alpha e_i]. \tag{3.4}$$

From (3.3) and (3.4) we deduce:

$$\begin{aligned} trace_g f^*\nabla' J'_\alpha &= \sum_{i=1}^m f_*[e_i, J_\alpha e_i] - \sum_{i=1}^m J'_\alpha \widetilde{\nabla}_{e_i} f_* e_i \\ &\quad + \sum_{i=1}^m \sum_{\beta\neq\alpha} \widetilde{\nabla}_{J_\beta e_i} f_* J_\alpha J_\beta e_i - \sum_{i=1}^m \sum_{\beta=1}^3 J'_\alpha \widetilde{\nabla}_{J_\beta e_i} f_* J_\beta e_i. \end{aligned} \tag{3.5}$$

Now, from (3.2) and (3.5) we obtain:

$$f_*(divJ_\alpha) - J'_\alpha(\tau(f)) = trace_g f^*\nabla' J'_\alpha - \sum_{i=1}^m \sum_{\beta\neq\alpha} \alpha_f(J_\beta e_i, J_\alpha J_\beta e_i). \tag{3.6}$$

But, since the second fundamental form α_f of f is a symmetric form, we remark that:

$$\sum_{i=1}^m \sum_{\beta\neq\alpha} \alpha_f(J_\beta e_i, J_\alpha J_\beta e_i) = 0. \tag{3.7}$$

The proof is now complete from (3.6) and (3.7). ∎

Theorem 2. *Let (M, σ, g) and (N, σ', g') be two quaternionic Kähler manifolds. If $f : M \to N$ is a (σ, σ')-holomorphic map such that, for any local section $J \in \Gamma(\sigma)$ and corresponding $J' \in \Gamma(\sigma')$ one has $(\nabla'_{f_*X} J') \circ f_* = f_* \circ (\nabla_X J)$, for any local vector field X on M, then f is a harmonic map.*

Proof. Since M is a quaternionic Kähler manifold, we obtain:

$$\begin{aligned} divJ_\alpha &= \sum_{i=1}^{m}[\omega_{\alpha+1}(J_{\alpha+2}e_i) - \omega_{\alpha+2}(J_{\alpha+1}e_i)]e_i \\ &+ \sum_{i=1}^{m}[\omega_{\alpha+1}(J_{\alpha+1}e_i) + \omega_{\alpha+2}(J_{\alpha+2}e_i)](J_\alpha e_i) \\ &+ \sum_{i=1}^{m}[\omega_{\alpha+2}(e_i) - \omega_{\alpha+1}(J_\alpha e_i)](J_{\alpha+1}e_i) \\ &- \sum_{i=1}^{m}[\omega_{\alpha+1}(e_i) + \omega_{\alpha+2}(J_\alpha e_i)](J_{\alpha+2}e_i). \end{aligned} \tag{3.8}$$

Similarly we find:

$$\begin{aligned} trace_g f^*\nabla' J'_\alpha &= \sum_{i=1}^{m}[\omega'_{\alpha+1}(f_*J_{\alpha+2}e_i) - \omega'_{\alpha+2}(f_*J_{\alpha+1}e_i)]f_*e_i \\ &+ \sum_{i=1}^{m}[\omega'_{\alpha+1}(f_*J_{\alpha+1}e_i) + \omega'_{\alpha+2}(f_*J_{\alpha+2}e_i)](f_*J_\alpha e_i) \\ &+ \sum_{i=1}^{m}[\omega'_{\alpha+2}(f_*e_i) - \omega'_{\alpha+1}(f_*J_\alpha e_i)](f_*J_{\alpha+1}e_i) \\ &- \sum_{i=1}^{m}[\omega'_{\alpha+1}(f_*e_i) + \omega'_{\alpha+2}(f_*J_\alpha e_i)](f_*J_{\alpha+2}e_i). \end{aligned} \tag{3.9}$$

On another hand, from $(\nabla'_{f_*X}J'_\alpha)\circ f_* = f_*\circ(\nabla_X J_\alpha)$, we obtain $\omega'_\alpha \circ f_* = \omega_\alpha$, $\forall \alpha \in \{1, 2, 3\}$, and from (3.8) and (3.9) we derive:

$$f_*(divJ_\alpha) - trace_g f^*\nabla' J'_\alpha = 0. \tag{3.10}$$

Now, from (3.1) and (3.10), we deduce $\tau(f) = 0$ and thus we conclude that f is a harmonic map. ■

4 On the stability of the (σ, σ')-holomorphic maps

Let (M, g) be a compact Riemannian manifold and let $f : (M, g) \to (N, h)$ be a harmonic map. The energy of f is defined by $E(f) = \int_M e(f)\vartheta_g$, where ϑ_g is the canonical measure associated with the metric g and $e(f)_x = \frac{1}{2}trace(f^*h)_x$, $\forall x \in M$.

We consider now $\{f_{s,t}\}_{s,t\in(-\epsilon,\epsilon)}$ a smooth two-parameter variation of f such that $f_{0,0} = f$ and let $V, W \in \Gamma(f^{-1}(TN))$ be the corresponding variational vector fields: $V = \frac{\partial}{\partial s}(f_{s,t})|_{(s,t)=(0,0)}$, $W = \frac{\partial}{\partial t}(f_{s,t})|_{(s,t)=(0,0)}$. The Hessian of a harmonic map f is defined by: $H_f(V, W) = \frac{\partial^2}{\partial s\partial t}(E(f_{s,t}))|_{(s,t)=(0,0)}$.

The index of a harmonic map $f : (M, g) \to (N, h)$ is defined as the dimension of the largest subspace of $\Gamma(f^{-1}(TN))$ on which the Hessian H_f is negative definite. A harmonic map f is said to be stable if the index of f is zero and otherwise, is said to be unstable. We recall the next second variation formula obtained by E. Mazet and R.T. Smith:

$$H_f(V,W) = \int_M h(J_f(V),W)\vartheta_g, \tag{4.1}$$

where J_f is the Jacobi operator of f (see [2]).

Let (M^{4m},σ,g) be a compact quaternionic Kähler manifold, (N^{4n},σ',h) be a quaternionic Kähler manifold with scalar curvature ρ' and $f: M \to N$ be a (σ,σ')-holomorphic map. After some long but straightforward computation, we obtain:

$$\begin{aligned}\int_M h(J_f(V),V)\vartheta_g &= \frac{1}{2}\int_M h(DV,DV)\vartheta_g - \int_M h(\widetilde{\nabla}_{divJ_\beta}V, J'_\beta V)\vartheta_g \\ &-\frac{2\rho'}{n+2}\sum_{i=1}^m [\int_M h(f_*e_i,V)^2\vartheta_g + \sum_{\alpha=1}^3\int_M h(f_*J_\alpha e_i,V)^2\vartheta_g] \\ &+\sum_{i=1}^m [h(\widetilde{\nabla}_{e_i}V,(\widetilde{\nabla}_{J_\beta e_i}J'_\beta)V) + \sum_{\alpha=1}^3 h(\widetilde{\nabla}_{J_\alpha e_i}V,(\widetilde{\nabla}_{J_\beta J_\alpha e_i}J'_\beta)V)]\vartheta_g,\end{aligned} \tag{4.2}$$

for all $\beta \in \{1,2,3\}$, where $DV : \Gamma(TM) \to \Gamma(f^{-1}(TN))$ is given by:

$$DV(X) = \widetilde{\nabla}_{J_\beta X}V - J'_\beta\widetilde{\nabla}_X V,\ \forall X \in \Gamma(TM).$$

Theorem 3. *Let (M^{4m},σ,g) and (N^{4n},σ',g') be two quaternionic Kähler manifolds such that M is compact, N has non positive scalar curvature and, in any point $p \in M$, exists a basis $\{J_1,J_2,J_3\}$ of σ_p such that one of J_1,J_2 or J_3 is parallel. If $f: M \to N$ is a (σ,σ')-holomorphic map such that, for any local section $J \in \Gamma(\sigma)$ and corresponding $J' \in \Gamma(\sigma')$ one has $(\nabla'_{f_*X}J')\circ f_* = f_*\circ(\nabla_X J)$, for any local vector field X on M, then f is stable.*

Proof. From (4.2) we obtain:

$$\begin{aligned}\int_M h(J_f(V),V)\vartheta_g &= \frac{1}{2}\int_M h(DV,DV)\vartheta_g \\ &-\frac{2\rho'}{n+2}\sum_{i=1}^m [\int_M h(f_*e_i,V)^2\vartheta_g + \sum_{\alpha=1}^3\int_M h(f_*J_\alpha e_i,V)^2\vartheta_g].\end{aligned} \tag{4.3}$$

From (4.1) and (4.3) we obtain:

$$H_f(V,V) \geq 0, \forall V \in \Gamma(f^{-1}(TN))$$

and then we deduce that f is a stable harmonic map. ■

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