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Noetherian first integrals

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Abstract

From time to time one finds claims in the literature that first integrals/invariants of Lagrangian systems are nonnoetherian. Such claims diminish the contribution of Noether in the topic of integrability. We provide an explicit demonstration of noetherian symmetries associated with the integrals which have been termed nonnoetherian. To further emphasise our point we construct the noetherian first integrals/invariants, which are associated with symmetries linear in the velocities, for the two-dimensional autonomous isotropic harmonic oscillator and the autonomous anisotropic oscillator and illustrate the roles which the invariants can play in the description of the classical motion. We relate these symmetries to the corresponding problem in quantum mechanics. Further we show that the complete symmetry group of this anisotropic harmonic oscillator has the same representation as that of the corresponding isotropic oscillator. As a concluding example we show that a symmetry claimed to be nonnoetherian is trivially Noetherian.

1 Introduction

In a series of papers [5, 9, 10, 11, 12, 13] various authors have developed a method to study what have been called nonnoetherian constants of motion and their associated symmetries. In [5] the above procedure, which is known in the relevant literature [9, 10, 11, 12, 13] as the method of s- and g-symmetries, has been applied to the two-dimensional autonomous isotropic harmonic oscillator with equations of motion

$$\begin{aligned} &\ddot{x} + x = 0 \\ &\ddot{y} + y = 0 \end{aligned} \tag{1.1}$$

and Lagrangian

$$L = \tfrac{1}{2}\left(\dot{x}^2 + \dot{y}^2 - x^2 - y^2\right) \tag{1.2}$$

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and to the autonomous anisotropic oscillator with equations of motion

$$\begin{aligned}&\ddot{x}+x=0\\&\ddot{y}+\omega^2y=0,\quad \omega^2\neq 1,\end{aligned}\tag{1.3}$$

and Lagrangian

$$L=\tfrac{1}{2}\left(\dot{x}^2+\dot{y}^2-x^2-\omega^2y^2\right).\tag{1.4}$$

In the case of the isotropic oscillator the formalism of the s- and g-symmetries [9, 10, 11, 12, 13] has been used [5] to obtain the off-diagonal component of the conserved Jauch-Hill-Fradkin tensor [4, 14]

$$A_{12}=\dot{x}\dot{y}+xy\tag{1.5}$$

and in the case of the anisotropic oscillator the corresponding conserved quantities,

$$I=(\dot{x}+ix)^{\omega}(\dot{y}-i\omega y)\tag{1.6}$$

and its complex conjugate. In both cases the first integrals were described [5] as nonnoetherian.

It is the purpose of this paper to demonstrate that these first integrals are in fact noetherian first integrals (Section 2) thereby restoring the credit to Noether which is her due, to outline how all of the quadratic first integrals/invariants for the harmonic oscillator can be obtained by the use of Noether's Theorem (Section 3) and to show how the nonquadratic first integrals of the anisotropic oscillator of the type (1.6) can be derived by means of the Lie theory of extended groups (Section 4). We illustrate the role which the explicitly time-dependent invariants play in the description of the classical motion. The excellent review of Noether's Theorem by Sarlet and Cantrijn [25] is the basis for Section 2 of the paper.

The papers to which particular attention is here drawn are but part of an inexplicable phenomenon of denial associated with Noether's Theorem for more than sixty years. In Noether's original work [23] her theorem is developed in the context of field theory, *ie* more than one independent variable, and at the level of generalised transformations. The latter marked a distinct departure from the point, subsequently contact, transformations used by Lie in his theory of infinitesimal transformations which was based upon geometrical principles. Scarcely a decade later Courant and Hilbert [2] provided a bowdlerised version of the then not so recently late Emmy Noether's Theorem. This treatment admitted only point transformations. Courant and Hilbert were not the only luminaries to short-sell Noether. In the seventies Lovelock and Rund [22] repeated the same minimalist treatment. Even later Dresner [3] persisted in the same despite earnest entreaty to give the woman a fair hearing. In this Dresner cited the authority of Courant and Hilbert! Doubtless there are other texts acting as purveyors of inadequate truth. The wonder of it all is that Noether's text is clear even to one unfamiliar with German and the translation by Tavel [24] into the *lingua franca* of our day does dot the *i*s and cross the *t*s for those who need that done.

2 Noether Symmetries

Following Sarlet and Cantrijn [25] we have that a Lagrangian, L, possesses a first integral/invariant, I, associated with a symmetry, Γ, where

$$\Gamma = \tau(t,x,\dot{x})\partial_t + \eta_i(t,x,\dot{x}), \tag{2.1}$$

given by

$$I = f - \left[\tau L + (\eta_i - \dot{x}_i\tau)\frac{\partial L}{\partial \dot{x}_i}\right], \tag{2.2}$$

where the function, f, is commonly regarded as the gauge function which expresses the equivalence of the Lagrangian in the Action Integral up to a total time derivative of f. In fact its provenance is completely different as can be seen by reference to the original paper of Noether [23] [page 241], where its origin is demonstrated to be in the boundary contribution to the value of the Action Integral occasioned by the infinitesimal transformation which in contrast to the infinitesimal transformations for the Variational Principle does not have to be zero at the boundary. For the velocity-dependent transformation generated by the symmetry, Γ, we have

$$\frac{\partial f}{\partial \dot{x}_j} = L\frac{\partial \tau}{\partial \dot{x}_j} + \frac{\partial L}{\partial \dot{x}_i}\left(\frac{\partial \eta_i}{\partial \dot{x}_j} - \dot{x}_i\frac{\partial \tau}{\partial \dot{x}_j}\right) + \tau\frac{\partial L}{\partial x_j},\ j = 1, n, \tag{2.3}$$

and

$$\begin{aligned}\frac{\partial f}{\partial t} + \dot{x}_i\frac{\partial f}{\partial x_i} &= L\left(\frac{\partial \tau}{\partial t} + \dot{x}_i\frac{\partial \tau}{\partial x_i}\right) + \tau\frac{\partial L}{\partial t} + (\eta_i - \dot{x}_i)\frac{\partial L}{\partial x_i}\\ &\quad + \left(\frac{\partial \eta_i}{\partial t} + \dot{x}_j\frac{\partial \eta_i}{\partial x_j} - \dot{x}_i\frac{\partial \tau}{\partial t} - \dot{x}_i\dot{x}_j\frac{\partial \tau}{\partial x_j}\right)\frac{\partial L}{\partial \dot{x}_i}.\end{aligned} \tag{2.4}$$

In the case of a regular Lagrangian τ, η_i and I are related by [25]

$$\eta_i - \dot{x}_i\tau = -g^{ij}\frac{\partial I}{\partial \dot{x}_j}, \tag{2.5}$$

where g^{ij} is given by

$$\frac{\partial^2 L}{\partial \dot{x}_i\partial \dot{x}_k}g^{kj} = \delta_i^j. \tag{2.6}$$

When the velocity-dependent transformations are used, there is no loss of generality in setting $\tau = 0$ whereby (2.2)-(2.4) are considerably simplified. Both isotropic and anisotropic oscillators have $g^{ij} = \delta^{ij}$. Thus for the former we obtain from (2.5) and the integral (1.5) that

$$\eta = -\dot{y}, \quad \xi = -\dot{x}, \tag{2.7}$$

where the notation $x_1 = x$, $x_2 = y$, $\eta_1 = \eta$ and $\eta_2 = \xi$ has been used. The symmetry Γ defined by (2.1) follows now from (2.7) and is

$$\Gamma = \dot{y}\partial_x + \dot{x}\partial_y, \tag{2.8}$$

in which the common minus sign has been dropped. For the latter (1.6) yields

$$\Gamma = (\dot{x} + ix)^{\omega - 1} \left[\omega\left(\dot{y} - i\omega y\right)\partial_x + \left(\dot{x} + ix\right)\partial_y\right]. \tag{2.9}$$

The generator for the complex conjugate invariant is obvious.

Thus we have, by explicit demonstration, showed that (1.5) and (1.6) are noetherian constants of the motion.

3 Quadratic first integrals/invariants for the isotropic oscillator

Given that there appears to be some confusion in the literature concerning the nature of Noether symmetries and noetherian first integrals/invariants we determine those for the isotropic oscillator which are quadratic in the velocities. From (1.2), (2.5) and (2.6) it is evident that the noetherian symmetry which generates a first integral/invariant quadratic in the velocities is itself linear in the velocity. Hence from (2.1) it follows that

$$\eta = a\dot{x} + b\dot{y} + d, \quad \xi = a_1\dot{x} + c\dot{y} + e, \tag{3.1}$$

where a, b, d, a_1, c and e are functions of x, y and t only.

From (2.3) and due to (1.2) we obtain that ($\tau = 0$, see (2.6) and following)

$$\begin{aligned} \frac{\partial f}{\partial \dot{x}} &= \dot{x}\frac{\partial \eta}{\partial \dot{x}} + \dot{y}\frac{\partial \xi}{\partial \dot{x}} \\ \frac{\partial f}{\partial \dot{y}} &= \dot{x}\frac{\partial \eta}{\partial \dot{y}} + \dot{y}\frac{\partial \xi}{\partial \dot{y}}. \end{aligned} \tag{3.2}$$

The requirement of consistency between the expressions for $\partial f/\partial\dot{x}$ and $\partial f/\partial\dot{y}$ yields

$$\frac{\partial \xi}{\partial \dot{x}} = \frac{\partial \eta}{\partial \dot{y}}. \tag{3.3}$$

From equations (2.1), (3.1)-(3.3) we readily deduce that

$$\Gamma = \left(a\dot{x} + b\dot{y} + d\right)\partial_x + \left(b\dot{x} + c\dot{y} + e\right)\partial_y \tag{3.4}$$

and

$$f = \tfrac{1}{2}a\dot{x}^2 + b\dot{x}\dot{y} + \tfrac{1}{2}c\dot{y}^2 + g, \tag{3.5}$$

where g depends upon x, y and t only.

When (3.4) and (3.5) are substituted into (2.4) and the coefficients of linearly independent combinations of powers of $\dot{x}$ and $\dot{y}$ are set equal to zero, we obtain

$$\frac{\partial a}{\partial x} = \tfrac{1}{2}\frac{\partial a}{\partial x} \tag{3.6}$$

$$2\frac{\partial b}{\partial x} + \frac{\partial a}{\partial y} = \frac{\partial b}{\partial x} + \tfrac{1}{2}\frac{\partial a}{\partial y} \tag{3.7}$$

$$2\frac{\partial b}{\partial y} + \frac{\partial c}{\partial x} = \frac{\partial b}{\partial y} + \tfrac{1}{2}\frac{\partial c}{\partial x} \tag{3.8}$$

$$\frac{\partial c}{\partial y} = \tfrac{1}{2}\frac{\partial c}{\partial y} \tag{3.9}$$

$$\frac{\partial a}{\partial t} + \frac{\partial d}{\partial x} = \tfrac{1}{2}\frac{\partial a}{\partial t} \tag{3.10}$$

$$2\frac{\partial b}{\partial t} + \frac{\partial d}{\partial y} + \frac{\partial e}{\partial x} = \frac{\partial b}{\partial t} \tag{3.11}$$

$$\frac{\partial c}{\partial t} + \frac{\partial e}{\partial y} = \tfrac{1}{2}\frac{\partial c}{\partial t} \tag{3.12}$$

$$-ax - by + \frac{\partial d}{\partial t} = \frac{\partial g}{\partial x} \tag{3.13}$$

$$-bx - cy + \frac{\partial e}{\partial t} = \frac{\partial g}{\partial y} \tag{3.14}$$

$$-dx - ey = \frac{\partial g}{\partial t}. \tag{3.15}$$

The solution of the system (3.6)-(3.15) is

$$a = A_0 + A_3 \sin 2t + A_4 \cos 2t + 2y\left(A_1 \sin t + A_2 \cos t\right) + D_0 y^2 \tag{3.16}$$

$$\begin{aligned} b = &-B_0 - E_3 \cos 2t + E_4 \sin 2t - x\left(A_1 \sin t + A_2 \cos t\right) \\ &-y\left(B_1 \sin t + B_2 \cos t\right) - D_0 xy \end{aligned} \tag{3.17}$$

$$c = C_0 + C_1 \sin 2t + C_2 \cos 2t + 2x\left(B_1 \sin t + B_2 \cos t\right) + D_0 x^2 \tag{3.18}$$

$$\begin{aligned} d = &E_1 \sin t + E_2 \cos t - x\left(A_3 \cos 2t - A_4 \sin 2t\right) + y\left(E_0 - E_3 \sin 2t - E_4 \cos 2t\right) \\ &-xy\left(A_1 \cos t - A_2 \sin t\right) + y^2\left(B_1 \cos t - B_2 \sin t\right) \end{aligned} \tag{3.19}$$

$$\begin{aligned} e = &-M_1 \sin t - M_2 \cos t + x\left(-E_0 - E_3 \sin 2t - E_4 \cos 2t\right) - y\left(C_1 \cos 2t - C_2 \sin 2t\right) \\ &+x^2\left(A_1 \cos t - A_2 \sin t\right) - xy\left(B_1 \cos t - B_2 \sin t\right) \end{aligned} \tag{3.20}$$

$$\begin{aligned} g = &-N_0 + x\left(E_1 \cos t - E_2 \sin t\right) - y\left(M_1 \cos t - M_2 \sin t\right) \\ &-\tfrac{1}{2}x^2\left(A_0 - A_3 \sin 2t - A_4 \cos 2t\right) + xy\left(B_0 - E_3 \cos 2t + E_4 \sin 2t\right) \\ &-\tfrac{1}{2}y^2\left(C_0 - C_1 \sin 2t - C_2 \cos 2t\right), \end{aligned} \tag{3.21}$$

where all upper case letters are constants.

We now apply (1.2), (3.1) and (3.5) to (2.2) (with $\tau = 0$) to obtain

$$I = -\tfrac{1}{2}ax^2 - bxy - \tfrac{1}{2}cy^2 - dx - ey + g. \tag{3.22}$$

When we insert (3.16) – (3.21) into (3.22), we find that the constant N_0 is a trivial constant of the motion since it is additive, but the other constants give rise to nineteen noetherian first integrals/invariants, *videlicet*

$$
\begin{array}{llcl}
(A_0) & I_1 & = & \frac{1}{2}\left(\dot{x}^2 + x^2\right) \\
(B_0) & I_2 & = & \dot{x}\dot{y} + xy \\
(C_0) & I_3 & = & \frac{1}{2}\left(\dot{y}^2 + y^2\right) \\
(A_3, A_4) & I_{4\pm} & = & \left[\frac{1}{2}\left(\dot{x}^2 - x^2\right) \mp ix\dot{x}\right] \exp\left[\pm 2it\right] \\
(C_1, C_2) & I_{5\pm} & = & \left[\frac{1}{2}\left(\dot{y}^2 - y^2\right) \mp iy\dot{y}\right] \exp\left[\pm 2it\right] \\
(D_0) & I_6 & = & \left(x\dot{y} - \dot{x}y\right)^2 \\
(A_1, A_2) & I_{7\pm} & = & \left(x\dot{y} - \dot{x}y\right)\left(x \pm i\dot{x}\right) \exp\left[\pm it\right] \\
(B_1, B_2) & I_{8\pm} & = & \left(x\dot{y} - \dot{x}y\right)\left(y \pm i\dot{y}\right) \exp\left[\pm it\right] \\
(E_3, E_4) & I_{9\pm} & = & \left[\left(\dot{x}\dot{y} - xy\right) \mp i\left(x\dot{y} + \dot{x}y\right)\right] \exp\left[\pm 2it\right] \\
(E_0) & I_{10} & = & x\dot{y} - \dot{x}y \\
(E_1, E_2) & I_{11\pm} & = & \left[x \pm i\dot{x}\right] \exp\left[\pm it\right] \\
(M_1, M_2) & I_{12\pm} & = & \left[y \pm i\dot{y}\right] \exp\left[\pm it\right],
\end{array}
\tag{3.23}
$$

where the constant(s) in parentheses indicate(s) which constants with nonzero values lead(s) to that/those integral(s). We know that in obtaining the quadratic integrals we would also obtain $I_{10} - I_{12\pm}$, *ie* those first integrals/invariants which are linear in the velocities. Needless to remark the integrals $I_1 - I_{10}$ can be expressed as functions of the fundamental integrals $I_{11\pm}$ and $I_{12\pm}$.

The integrals, I_1, I_2 and I_3, are essentially the elements of the Jauch-Hill-Fradkin tensor for the two-dimensional isotropic oscillator [14, 4]. When written in terms of canonically conjugate coordinates these integrals along with I_6 constitute the elements of the invariance algebra, $su(2)$, of the Hamiltonian for the two-dimensional isotropic oscillator under the operation of taking the Poisson Bracket. These integrals may be used in a quadratic form to give the orbit of the oscillator. The integrals, $I_{4\pm}$, $I_{5\pm}$ and $I_{9\pm}$, provide the elements of the noninvariance algebra of the isotropic oscillator [15, 16] and have been shown [19] also to provide information about the orbit in configuration space. If one defines

$$B_{ij} = \left(\dot{x}_i\dot{x}_j - x_ix_j\right)\sin 2t - \left(x_i\dot{x}_j + \dot{x}_ix_j\right)\cos 2t \tag{3.24}$$

$$C_{ij} = \left(\dot{x}_i\dot{x}_j - x_ix_j\right)\cos 2t + \left(x_i\dot{x}_j + \dot{x}_ix_j\right)\sin 2t \tag{3.25}$$

$$J = \tfrac{1}{2}\mathrm{Tr}(B_{ij}) \tag{3.26}$$

$$K = \tfrac{1}{2}\mathrm{Tr}(C_{ij}), \tag{3.27}$$

it is a simple matter to determine that the quadratic forms constructed from B_{ij} and C_{ij} by double contraction with the position vector, $\mathbf{r}$, lead to the relations

$$\mathbf{r}^T\left(JI - B\right)\mathbf{r} = L^2\sin 2t \tag{3.28}$$

$$\mathbf{r}^T\left(KI - C\right)\mathbf{r} = L^2\cos 2t, \tag{3.29}$$

where L is the magnitude of the angular momentum (written as $I_6 = I_{10}^2$ in the list of integrals/invariants above). Both quadratic forms, (3.28) and (3.29), describe hyperbolæ of time-dependent size. The hyperbolæ pulsate periodically and the trajectory of the oscillator is given by the common intersection of hyperbola and the ellipse given by the quadratic forms constructed from the elements of the Jauch-Hill-Fradkin tensor[2].

[2]Note that we use the exponential form for the listing of the integrals/invariants in (3.23) since these correspond to the forms of the operators of use in quantum mechanics. The trigonometric forms, being real, are more convenient in classical mechanics.

4 Anharmonic oscillator

When one looks for the noetherian first integrals associated the symmetries linear in the velocities, the integrals obtained are quadratics or biquadratics in the four fundamental integrals

$$\begin{aligned} I_1 &= x\cos t - \dot{x}\sin t \\ I_2 &= x\sin t + \dot{x}\cos t \\ I_3 &= y\cos\omega t - \dot{y}\sin\omega t/\omega \\ I_4 &= \omega y\sin\omega t + \dot{y}\cos\omega t. \end{aligned} \tag{4.1}$$

The only autonomous quadratic integrals which can be obtained from (4.1) are those corresponding to I_1 and I_3 for the isotropic oscillator. The integral (1.6) can be obtained as follows: From the combinations

$$J_1 = I_2 + iI_1 = (\dot{x} + ix)\exp[-it] \tag{4.2}$$

$$J_2 = I_4 - i\omega I_3 = (\dot{y} - i\omega y)\exp[i\omega t] \tag{4.3}$$

we find that (1.6) is given by

$$I = {J_1}^{\omega} J_2. \tag{4.4}$$

The integral complex conjugate to (1.6) follows similarly.

However, a more direct route to (1.6) comes by an application of the Lie theory of extended groups to the differential equations (1.3). In general a system of second-order differential equations

$$N(t, x, \dot{x}, \ddot{x}) = 0 \tag{4.5}$$

possesses a Lie point symmetry

$$\Gamma = \tau(t, x)\partial_t + \eta_i(t, x)\partial_{x_i} \tag{4.6}$$

if

$$\Gamma^{[2]} N_{|N=0} = 0, \tag{4.7}$$

where $\Gamma^{[2]}$ is the second extension of Γ given by

$$\Gamma^{[2]} = \Gamma + (\dot{\eta}_i - \dot{\tau}\dot{x}_i)\,\partial_{\dot{x}_i} + (\ddot{\eta}_i - 2\dot{\tau}\ddot{x}_i - \ddot{\tau}\dot{x}_i)\,\partial_{\ddot{x}_i}. \tag{4.8}$$

We can show, the computation proceeds essentially along the lines of Section 3, that equations (1.3) possess the Lie point symmetries

$$\begin{array}{ll} \Gamma_1 = \partial_t & \\ \Gamma_2 = \sin t\partial_x & \Gamma_4 = \sin\omega t\partial_y \\ \Gamma_3 = \cos t\partial_x & \Gamma_5 = \cos\omega t\partial_y \\ \Gamma_6 = x\partial_x & \Gamma_7 = y\partial_y \end{array} \tag{4.9}$$

which is the minimum number of Lie point symmetries which a two-dimensional autonomous system of second-order equations can have [6, 7, 26]. If, for example, we take the symmetry

$$X = \Gamma_3 - i\Gamma_2 = \exp\left[-it\right]\partial_x, \tag{4.10}$$

the first integral associated with X, $I(t, x, \dot{x})$, must satisfy the two conditions

$$X^{[1]} I = 0 \tag{4.11}$$

and

$$\frac{\mathrm{d}I}{\mathrm{d}t}_{(1.3)} = 0. \tag{4.12}$$

The associated Lagrange's system for (4.11) is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{\exp\left[-it\right]} = \frac{\mathrm{d}y}{0} = \frac{i\mathrm{d}\dot{x}}{\exp\left[-it\right]} = \frac{\mathrm{d}\dot{y}}{0} \tag{4.13}$$

for which the characteristics may be taken to be

$$\begin{array}{ll} u_1 = t & u_2 = \dot{x} + ix \\ u_3 = y & u_4 = \dot{y}. \end{array} \tag{4.14}$$

In terms of the characteristics in (4.14) the associated Lagrange's system of (4.12) is

$$\frac{\mathrm{d}u_1}{1} = \frac{\mathrm{d}u_2}{iu_2} = \frac{\mathrm{d}u_3}{u_4} = -\frac{\mathrm{d}u_4}{\omega^2 u_3}. \tag{4.15}$$

The combinations of, respectively, the first and second and the third and fourth trivially give the two characteristics

$$v_1 = u_2 \exp\left[-iu_1\right] \quad \text{and} \quad v_2 = u_4^2 + \omega^2 u_3^2. \tag{4.16}$$

The third characteristic is obtained from the second and third of (4.15) with the aid of the second of (4.16). It is

$$v_3 = \omega \log u_2 + \log\left[i\left(u_4 - i\omega u_3\right) v_2^{-1/2}\right] \tag{4.17}$$

and (1.6) is recovered from

$$I = \exp\left[v_3 - \log\left(iv_2^{-1/2}\right)\right]. \tag{4.18}$$

We noted that the symmetry, X, in fact gives three first integrals [18] which is a point to bear in mind when comparing Noether's Theorem with the Lie method. Given a Noether symmetry, Noether's Theorem gives a single first integral by the application of the formula (2.2) whereas the Lie method gives $(2n - 1)$ first integrals/invariants, where n is the dimension of the system, provided closed -form solutions to the associated Lagrange's system can be obtained. The 'provided' covers some possibly very difficult practical mathematics in seeking the solutions of the associated Lagrange's system as has been evidenced in the determination of the first integrals for the Kepler and related problems [8, 17]. An attractive approach is to use Noether's Theorem to determine those integrals/invariants with 'obvious' Noether symmetries and to supplement those results with the Lie method[3] [1].

5 Conclusion

A prime purpose of this paper was to emphasise that the integrals possessed by Lagrangian systems are necessarily noetherian. In particular we showed that the integrals constructed by Garcia [5] using the method of s- and g-symmetries can be obtained using Noether's Theorem. This is not to naysay the development or employment of other methods for the calculation of first integrals/invariants. Any method which leads to the determination of first integrals/invariants for systems of differential equations has its validity and may play a useful role in a particular context. Our major point is that such first integrals/invariants for nondegenerate Lagrangian systems must necessarily be associated with a Noether symmetry as a direct consequence of the relationship (2.5). It is most unfortunate that many texts, both classical and modern, fail to do justice to the Theorem of Noether as she originally presented it. Much ingenuity, as in the development of the method of s- and g-symmetries, has been employed to determine first integrals/invariants for systems in which the emasculated version of Noether's Theorem has been found wanting. These developments are a timely reminder that we should look carefully at the statement of the theorem and not to its latter-day expositions. The acceptance of symmetries other than point symmetries is necessary if one is to obtain the full benefits of the analysis of systems of differential equations and their associated Lagrangians using the symmetric approach. The origin of symmetry analysis is found in geometry and the transformations which make sense there. The generalised symmetries of Noether's Theorem have departed from the geometric origins of transformation theory. What has been lost in the geometric connection is gained in a wider variety of problems susceptible to analysis using Noether's Theorem and the unification of the concept of the relationship between invariance under infinitesimal transformation and the existence of first integrals/invariants. As an example of this we consider an example discussed by Hojman which he claims is not only not a Noether symmetry but is not even an s-equivalence symmetry and therefore belongs to a third kind of Lagrangian symmetry [12] [p 2408]. The example is based upon a two-dimensional simple harmonic oscillator, but the number of dimensions is immaterial. The Lagrangian and equations of motion are, respectively,

$$L = \tfrac{1}{2}\sum_{i=1}^{n}\left(\dot{q}_i^2 - q_i^2\right), \quad \ddot{q}_i + q_i = 0, \quad i = 1, n, \tag{5.1}$$

[3] A noetherian integral can have a Lie symmetry which differs from the generating Noether symmetry, *eg* the rescaling symmetry, $\Gamma = 3t\partial_t + 2r\partial_r$, is a Lie point symmetry of the components of the Laplace-Runge-Lenz vector which are obtained by Noether's Theorem from generalised symmetries [21] and so those components are two of the invariants of the associated Lagrange's system. This facilitates the reduction of the number of active variables in what are somewhat complicated equations, thereby making them more manageable.

in an usual notation. The so-claimed nonnoetherian symmetry is

$$\Gamma = \tfrac{1}{2}\sum_{i=1}^{n}\left(\dot{q}_i^2+q_i^2\right)\sum_{j=1}^{n} q_j\partial_{q_j}. \tag{5.2}$$

In the absence of a transformation of the independent variable the formula for the calculation of a Noetherian symmetry is

$$\dot{f} = \sum_{i=1}^{n}\left[\eta_i\frac{\partial L}{\partial q_i}+\dot{\eta}_i\frac{\partial L}{\partial \dot{q}_i}\right]. \tag{5.3}$$

When (5.1) and (5.2) are substituted into (5.3), the result is an exact differential so that it is a trivial matter to find that

$$f = \tfrac{1}{2}\sum_{i=1}^{n}\left(\dot{q}_i^2+q_i^2\right)\sum_{j=1}^{n} q_j\dot{q}_j. \tag{5.4}$$

Consequently (5.2) is a mainstream Noetherian symmetry.

As part of the realisation of the purpose of this paper we have presented the integrals of the anisotropic oscillator as a consequence of the existence of Noether symmetries. To emphasise further the relationship between generalised symmetries and Noether invariants we constructed the nineteen first integrals/invariants which have terms up to quadratic in the velocities for the autonomous isotropic harmonic oscillator in two dimensions. Furthermore we showed how the same could be done with the anisotropic oscillator. In this instance an *Ansatz* for the velocity dependence in the symmetry is not *a priori* obvious although it becomes obvious *a posteriori* because of the relationship (2.5). In such a case the value of the Lie method of extended groups in leading to the desired result by a prescribed procedure was illustrated.

The relationship between the Noether point symmetries of the classical Lagrangian and the Lie point symmetries of the corresponding time-dependent Schrödinger equation and the use of the latter in determining the wave functions for the quantal isotropic oscillator and related potentials was demonstrated by Lemmer *et al* [20]. It is interesting to see that the same features are observed for the quantal anisotropic oscillator. The time-dependent Schrödinger equation corresponding to the Lagrangian (1.4) is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\left(x^2+\omega^2y^2\right)+2i\frac{\partial u}{\partial t}=0 \tag{5.5}$$

and the normalised wave function, obtained by the usual method of separation of variables, is

$$\begin{aligned} u(x,y,t) = {} & \frac{\omega^{1/4}}{\left(\pi 2^{m+n}m!n!\right)^{1/2}}H_m(x)H_n(y\omega^{1/2}) \\ & \times\exp\left[-\tfrac{1}{2}x^2-\tfrac{1}{2}\omega y^2-i\left(m+\omega n+\tfrac{1}{2}\left(1+\omega\right)\right)t\right]. \end{aligned} \tag{5.6}$$

The Lie point symmetries of (5.5) are conveniently written as

$$Z_t = \partial_t$$

$$
\begin{aligned}
Z_{x\pm} &= \exp[\pm it]\left(\partial_x \mp xu\partial_u\right) \\
Z_{y\pm} &= \exp[\pm\omega it]\left(\omega^{-1/2}\partial_y \mp \omega^{1/2}yu\partial_u\right) \\
Z_u &= u\partial_u \\
Z_s &= f(t,x,y)\partial_u,
\end{aligned}
\tag{5.7}
$$

where $f(t,x,y)$ is a solution of (5.5). The symmetries Z_u and Z_s are generic for homogeneous linear partial differential equations and take no part in the generation of the similarity solutions. The Lie Brackets of the finite symmetries with the infinite number of solution symmetries are

$$
\begin{aligned}
&[Z_t, Z_s]_{LB} = f_t\partial_u; \qquad [Z_{x\pm}, Z_s]_{LB} = \exp[\pm it]\left(f_x \pm xf\right)\partial_u \\
&[Z_u, Z_s]_{LB} = -f\partial_u; \qquad [Z_{y\pm}, Z_s]_{LB} = \exp[\pm\omega it]\left(\omega^{1/2}f_y \pm \omega^{1/2}yf\right)\partial_u
\end{aligned}
\tag{5.8}
$$

and the coefficient of ∂_u on each right hand side is a solution of (5.5)

The corresponding Noether symmetries of the classical Lagrangian have as integrals the energy, the two integrals related to the initial conditions in the x coordinate and the two integrals related to the initial conditions in the y coordinate. The actions of the Lie point symmetries of (5.7) on solution surfaces defined by

$$
\begin{aligned}
\Sigma_{m,n} = u^{-1}&\frac{\omega^{1/4}}{\left(\pi 2^{m+n}m!n!\right)^{1/2}}H_m(x)H_n(y\omega^{1/2}) \\
&\exp\left[-\tfrac{1}{2}x^2 - \tfrac{1}{2}\omega y^2 - i\left(m+\omega n+\tfrac{1}{2}\left(1+\omega\right)\right)t\right]
\end{aligned}
\tag{5.9}
$$

are easily calculated to be

$$
\begin{aligned}
iZ_t\Sigma_{m,n} &= \left(m+\omega n+\tfrac{1}{2}(1+\omega)\right)\Sigma_{m,n} \\
iZ_{x+}\Sigma_{m,n} &= (2m)^{1/2}\,\Sigma_{m-1,n} \\
iZ_{x-}\Sigma_{m,n} &= -\left[2\left(m+1\right)\right)^{1/2}\Sigma_{m+1,n} \\
iZ_{y+}\Sigma_{m,n} &= (2n)^{1/2}\,\Sigma_{m,n-1} \\
iZ_{y-}\Sigma_{m,n} &= -\left[2\left(n+1\right)\right]^{1/2}\Sigma_{m,n+1}
\end{aligned}
\tag{5.10}
$$

so that iZ_t maps solutions into themselves with an eigenvalue equal to the energy and the other symmetries map solutions into other solutions. The symmetries, Z_{x+} and Z_{y+}, act as annihilation operators in the x and y coordinate respectively and the symmetries, Z_{x-} and Z_{y-}, as creation operators. Unlike the case of the isotropic oscillator there is no interaction between the two parts of the wave function [20].

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References

[1] Cotsakis S, Pantazi H & Leach PGL, Symmetries of homogeneous cosmologies, *Gravitation & Cosmology* **4** (1999), 314–325.

[2] Courant R & Hilbert D, *Methods of Mathematical Physics* (Wiley Interscience, New York, 1953).

[3] Dresner L, *Applications of Lie's Theory of Ordinary and Partial Differential Equations* (IOP Publishing, Philadelphia, 1998).

[4] Fradkin DM, Three-dimensional isotropic oscillator and SU_3, *American Journal of Physics* **33** (1965), 207–211.

[5] Garcia-Sucre M, Percoco U & Nuñez L, An example of a general class of symmetries of Lagrangians and their equations of motion, *Canadian Journal of Physics* **69** (1991), 1217–1220.

[6] Gonzalez-Gascon F & Gonzalez-Lopez A, Symmetries of differential equations IV, *Journal of Mathematical Physics* **24** (1983), 2006–2021.

[7] Gorringe VM & Leach PGL, Lie point symmetries for systems of second-order linear ordinary differential equations, *Quæstiones Mathematicæ* **14** (1988), 95–117.

[8] Gorringe VM & Leach PGL, Conservation laws for the monopole oscillator problem, *Quæstiones Mathematicæ* **20**, (1997), 71–83.

[9] Hojman S & Harleston H, Equivalent Lagrangians: Multidimensional case, *Journal of Mathematical Physics* **22** (1981), 1414–1419.

[10] Hojman S & Urrutia LF, On the inverse problem of the calculus of variations, *Journal of Mathematical Physics* **22** (1981), 1986–1903.

[11] Hojman S & Gomez J, First-order equivalent Lagrangians and conservation laws, *Journal of Mathematical Physics* **25** (1984), 1776–1779.

[12] Hojman S, Symmetries of Lagrangians and of their equations of motion, *Journal of Physics A: Mathematical and General* **17** (1984), 2399–2413.

[13] Hojman S, Nuñez L, Pantino A & Rago H, Symmetries and conserved quantities in geodesic motion, *Journal of Mathematical Physics* **27** (1986), 281–286.

[14] Jauch JM & Hill EL, On the problem of degeneracy in quantum mechanics, *Physical Review* **57** (1940), 641–645.

[15] Leach PGL, Quadratic Hamiltonians, quadratic invariants and the symmetry group, $SU(n)$, *Journal of Mathematical Physics* **19** (1978), 1608–1611.

[16] Leach PGL, Quadratic Hamiltonians: the four classes of invariants, their interrelations and symmetries, *Journal of Mathematical Physics* **21** (1980), 32–37.

[17] Leach PGL, Applications of the Lie theory of extended groups in Hamiltonian mechanics: the oscillator and the Kepler problem, *Journal of the Australian Mathematical Society Series B* **23** (1981), 173–186.

[18] Leach PGL & Mahomed FM, Maximal subalgebra associated with a first integral of a system possessing $sl(3, R)$ symmetry, *Journal of Mathematical Physics* **29** (1988), 1807–1813.

[19] Lemmer RL, *The Paradigms of Mechanics: A symmetry based approach* (dissertation: University of Natal, Durban 4041, Republic of South Africa, 1996)

[20] Lemmer RL & Leach PGL, A classical viewpoint on quantum chaos, *Arab Journal of Mathematical Sciences* **5** (1999), 1–17.

[21] Lévy-Leblond JM, Conservation laws for gauge-variant Lagrangians in classical mechanics, *American Journal of Physics* **39** (1971), 502–506.

[22] Lovelock D & Rund H, *Tensors, Differential Forms and Variational Principles* (Wiley, New York, 1975)

[23] Noether Emmy, Invariante Variationsprobleme, *Königlich Gesellschaft der Wissenschaften Göttingen Nachrichten Mathematik-physik Klasse* **2** (1918), 235–267.

[24] Noether E & Tavel MA, Invariant variation problems, *Transport Theory and Statistical Mechanics* **1** (1971), 183–207.

[25] Sarlet W & Cantrijn F, Generalisations of Noether's theorem in classical mechanics, *SIAM Review* **23** (1981), 467–494.

[26] Wafo Soh C & Mahomed FM, Symmetry breaking for a system of two linear second-order ordinary differential equations, *Nonlinear Dynamics* **22** (2000), 121–133.