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Conservations laws for critical Kohn-Laplace equations on the Heisenberg group

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Abstract

Using the complete group classification of semilinear differential equations on the three-dimensional Heisenberg group $\mathbb{H}$, carried out in a preceding work, we establish the conservation laws for the critical Kohn-Laplace equations via the Noether's Theorem.

1 Introduction

In the last few decades a great number of papers treat semilinear partial differential equations on the $(2n+1)-$dimensional Heisenberg group H^n. Recall that H^n is a Lie group, topologically equivalent to the real vector space $\mathbb{R}^{2n+1}$ endowed with a product ϕ defined by

$$\phi((x, y, t), (x_0, y_0, t_0)) := (x + x_0, y + y_0, t + t_0 + 2(y \cdot x_0 - x \cdot y_0)),$$

where $x, x_0, y, y_0 \in \mathbb{R}^n$ and $t, t_0 \in \mathbb{R}$. Its name reflects the fact that it represents in an abstract form the commutation relations for the quantum-mechanical position and momentum operators in higher-dimensional configuration space.

The Heisenberg group is a representative of more general structures called *Carnot groups*. In the latter, there exists a natural second order differential operator in the form of sum of squares of certain vector fields and hence quite similar to the usual Laplacian. For the Heisenberg group this is the Kohn-Laplace operator Δ_{H^n} and the studied equations are of the form

$$\Delta_{H^n} u + f(x, u) = 0. \qquad (1.1)$$

(For the corresponding definitions of these notions, see section 2.)

It is a big tentation to extend to H^n known results concerning semilinear partial differential equations involving the Laplace operator in the Euclidean case. This however is not trivial and straightforward task due to the fact that the Kohn-Laplace operator is not a strongly elliptic operator and thus the analytical techniques do not always apply. In this regard there is a lot of existence and nonexistence results for solutions of (1.1). We shall briefly recall a few of them.

Existence results for weak solutions of the semilinear Kohn-Laplace equation (1.1) in open bounded or unbounded subsets of H^n with homogeneous Dirichlet boundary conditions are obtained by Garofalo and Lanconelli in [23] for $f(x,u)$ having growth of the form:

$$f(x,u) = f(u) = o(|u|^{\frac{Q+2}{Q-2}}) \text{ when } u \to \infty,$$

where $Q = 2n+2$ is the homogenous dimension of H^n. (For the definition see section 2.) These authors establish remarkable Pokhozhaev Identities which enabled them to prove nonexistence results for solutions of (1.1). They also prove regularity results. See [23]. Further Biagini [7] proved the existence of nonnegative classical nontrivial solutions u, assuming some hypothesis on the function $f(x,u)$, e.g. $f(x,u) = f(u)$ and $\frac{f(s)}{s^{p-1}} \to 0$, when $|s| \to \infty$, for $p < 2 + 2/n$. If $f(x,u)|_{u=0} = f_u(x,u)|_{u=0} = 0$, Birindelli *et al.* showed in [8] the existence of positive solutions for the Dirichlet problem in a bounded domain of H^n.

Earlier results regarding the Dirichlet problem for the Kohn-Laplacian on H^n belong to Jerison [28, 29]. In the works of Jerison and Lee [30, 31] on the Cauchy-Riemann (CR) Yamabe problem the Kohn-Laplace equation arises as Euler-Lagrange equation of a variational problem on CR manifolds. For further results we direct the interested reader to [6, 9, 11, 21, 22, 25, 26, 33, 34, 36, 5] and the references therein.

From the cited results it is clear that the critical Sobolev-Stein exponent for H^n

$$q_n^\star := \frac{Q+2}{Q-2} = \frac{n+2}{n}$$

plays an important role in the analysis of the Kohn-Laplace equation. We recall that the so-called critical exponents in the Euclidean case are found as critical powers for embedding theorems of Sobolev type. They can be viewed as numbers which divide the existence and nonexistence of solutions for semilinear differential equations with power nonlinearities. Such equations appear as Euler-Lagrange equations of functionals involving the Frechét derivatives of Sobolev and L_{p+1} norms. E.g. the equation $\Delta\varphi + \varphi^p = 0$, where Δ is the Laplacian in $\mathbb{R}^n$, $n \geq 3$, admits positive solutions if and only if $p \geq (n+2)/(n-2)$−the well-known Sobolev exponent (see [24, 16]).

By the Folland-Stein embedding theorem (see [19, 34]), the Sobolev space $S_k^p(\Omega) \subseteq L^q(\Omega)$, where $\Omega \subset H^1$ is a bounded domain, $q \leq \frac{Qp}{Q-kp}$, $1 < p, q < \infty$ and $k \geq 1$. The embedding is compact if $q < \frac{Qp}{Q-kp}$.

We note that the value of the critical Sobolev-Stein exponent for the Kohn-Laplace equation can be computed using the symmetry approach to criticality of semilinear differential equations proposed in [15].

The existence of solutions of (1.1) with critical nonlinearity $f(u) = u^{\frac{Q+2}{Q-2}}$, in H^n, is ensured by a result of Citti and Uguzzoni in [17]. For a survey of results regarding the critical Kohn-Laplace equations on the Heisenberg group see [32].

In a previous work [13], we obtained the complete group classification of the following semilinear equation on the three-dimensional Heisenberg group $\mathbb{H} := H^1$:

$$\Delta_{\mathbb{H}} u + f(u) = 0, \tag{1.2}$$

(Here $\Delta_{\mathbb{H}}$ is the Kohn-Laplace operator.). Further, we showed in [14] that all Lie point symmetries of (1.2) in the critical Sobolev-Stein case $f(u) = u^3$ are variational or divergence symmetries.

The purpose of this paper is to establish the corresponding conservation laws via the Noether's Theorem ([10, 35]). As it is well known, the latter provides an algorithimic procedure for construction of conservation laws. Namely, let

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u}$$

be the generator of an infinitesimal transformation admitted by certain Euler-Lagrange equation $E(\mathcal{L}) = 0$ of order $2k$, whose Lagrangian is denoted by $\mathcal{L}$. If X is a divergence symmetry of $E(\mathcal{L}) = 0$, that is, if there exists a vector valued function $\varphi = (\varphi^i)$ such that

$$X^{(k)}\mathcal{L} + \mathcal{L} D_i \xi^i = D_i \varphi^i, \tag{1.3}$$

then the Noether's Theorem states that the following conservation law holds

$$D_i(\xi^i \mathcal{L} + W^i[u, \eta - \xi^j u_j] - \varphi^i) = 0 \tag{1.4}$$

for all solutions u of $E(\mathcal{L}) = 0$. Above we have used the same notations and conventions as in [10]. (For the definition of W^i see [10], pp. 254-255.) Therefore, as pointed out in [10], to apply this theorem one must

(i) find all transformations admitted by $E(\mathcal{L}) = 0$ and

(ii) check which infinitesimal generators X satisfy the condition (1.3).

Hence it is clear that the major difficulty in applying the Noether's Theorem is that usually there is no explicit formula for the potential φ.

As it was already mentioned in the begining, the first step (i) is done in [13] and the second (ii) - in the work [14]. Further we observe that in [14] we found explicity the potentials φ associated to the corresponding divergence symmetries of the equation

$$\Delta_{\mathbb{H}} u + u^3 = 0. \tag{1.5}$$

Thus we have at our disposal all ingredients which will enable us to apply *directly* the Noether's Theorem by a straightforward calculation.

An alternative possibility is to apply modern versions of the Noether's Theorem, namely, the so-called Direct Construction Method, devised and developed by George Bluman and Stephen Anco [2, 3, 4, 1]. In this way one can by-pass the need of potentials φ since the full form of the admitted conservation laws can be computed directly from the admitted symmetries [2]. Moreover, since the equation (1.2) with power nonlinearity $f(u) = u^p, p \neq 0, p \neq 1$ possesses the scaling symmetry

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} + \frac{2}{1-p} u \frac{\partial}{\partial u}$$

(see [13]), all conservation laws with non-zero scaling weight can be obtained from an explicit formula of Anco [1] which uses the full admitted symmetry group of the equation. However, in our specific case, this will result in a calculation lengthier than here presented. It will contain a big number of divergence free expressions. To recognize the latter fact, one needs to perform another lengthy computation. Actually, this is the price of generality of the modern method when applied to the considered particular equation. On the other hand, we have already granted from [13, 14] the potentials φ and this is the reason to apply the classical approach to conservation laws.

For a discussion and comparison of four approaches to the calculation of conservation laws see [38].

In this paper we are interested in the critical Kohn-Laplace equation (1.5) since it possesses the widest symmetry group among the *nonlinear* equations of form (1.2). See [13]. Actually our main motivation to write [13, 14] as well as the present paper is the general property, established and discussed in [12] (and in some of the references therein), relating the variational/divergence symmetries of critical differential equations and the Sobolev Theorem. Namely, the Lie point symmetry group of large classes of ordinary and partial differential equations and systems, involving power nonlinearities, coincides with their Noether symmetry group if and only if the corresponding exponent assumes the critical value.

The Noether symmetries and the corresponding conservation laws in the non-critical cases (e.g. the *linear* cases $f(u) = 0$ and $f(u) = u$ as well as the exponential case $f(u) = Exp(u)$) are studied in [20].

From a general point of view, conservation laws for partial differential equations (PDEs) provide a priori estimates which can be used for proving existence and uniqueness results. Following Wolf [38], "the knowledge of conservation laws is useful in the numerical integration of PDEs. The existence of a large number of conservation laws of a PDE is a strong indication of its integrability." In this regard, as Bluman and Kumei pointed out in [10], p. 252, "finding the conservation laws of a system is often the first step towards finding its solution: the more conservation laws one finds the closer one gents to the complete solution."

The next steps in this research are to study the invariant solutions of (1.2) and to construct nonlocal symmetries and the corresponding to them nonlocal conservation laws for the solutions of critical semilinear Kohn-Laplace equations on the Heisenberg Group using the above mentioned recent methods of Bluman, Anco *et al.* These problems will be treated elsewhere. Here we merely point out that the obtained (local) conservation laws will be used for that purpose.

The paper is organized as follows. In section 2 we recall briefly some of the main aspects of the Heisenberg group and the Kohn-Laplace equations. In section 3 we present parts of the results, obtained in [13, 14], which will be used later. The conservation laws are stated in section 4, in the form of Theorem 1.

2 The Heisenberg group and the Kohn-Laplace equations

Let g be a Lie algebra. g is said to be a nilpotent Lie algebra if there exist $k \in \mathbb{N}$ such that $g^k = 0$, where

$$g^1 := g, \quad g^2 := [g,g], \quad \cdots \quad g^k := [g, g^{k-1}],$$

$[g,g] := \{[X,Y],\ X,Y \in g\}$ and the others are defined by the same way. In this case, we say that g is a k-step nilpotent Lie algebra, where $k := \min_{n\in\mathbb{N}}\{n \in \mathbb{N} \mid g^n = 0\}$.

A Lie algebra g is called graded if it can be written as a direct sum

$$g = \oplus_{j=1}^{r} g^j$$

with the following property: $[g^j, g^k] \subseteq g^{j+k}$, if $j+k \leq r$. A homogeneous structure of a graded nilpotent Lie algebra is the pair $(\mathbb{R}^n, \delta_\lambda)$, where $\delta_\lambda(X_j) := \lambda^{a_j} X_j,\ X_j \in g^j,\ \lambda > 0,\ a_j \in \mathbb{R}$, and each g^j is identified with $\mathbb{R}^{n_j}$, $n_1 + \cdots + n_r = n$.

Now consider the space $\mathbb{R}^n$ as a Lie group with a product ϕ. Let g be its associated Lie algebra. Further, suppose $\mathbb{R}^n$ endowed with a homogeneous structure $\{\delta_\lambda\}_{\lambda>0}$ of the form

$$\delta_\lambda(x) := (\lambda x^{(1)}, \lambda^2 x^{(2)}, \cdots, \lambda^r x^{(r)}), \tag{2.1}$$

where $x^{(i)} \in \mathbb{R}^{n_i},\ 1 \leq i \leq r$ and $n_1 + \cdots + n_r = n$. For $i = 1, \cdots, n_1$, define X_i as the vector fields of the Lie algebra g which agree at the origin with the canonical basis of $\mathbb{R}^{n_i}$. Assume that the Lie algebra generated by $X_1, \cdots, X_{n_1}$ is the same of g. The triple $\mathbb{G} := (\mathbb{R}^n, \phi, \delta_\lambda)$ is called Carnot Group. We say that $\mathbb{G}$ is r-step nilpotent Lie group; it has n_1 generators.

The number

$$Q := \sum_{j=1}^{r} j n_j \tag{2.2}$$

is called homogeneous dimension of $\mathbb{G}$.

Definition 1. We define the linear-homogeneous space of a Carnot group $(\mathbb{R}^n, \phi, \delta_\lambda)$, as the space genereted by $\{X_1, \cdots, X_{n_1}\}$.

In the linear-homogeneous space L_{hom} of a Carnot group $\mathbb{G}$ there exists a kind of second-order operator, called sub-Laplacian, defined by a sum of square of vector fields of L_{hom}, that is,

$$\Delta_{\mathbb{G}} := \sum_{i=1}^{n_1} X_i^2.$$

These operators are subelliptic operators. See [37].

The three-dimensional Heisenberg group $\mathbb{H}$ $(= H^1)$ is defined as follows: let $\phi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, defined by $\phi((x,y,t),(x_0,y_0,t_0)) := (x+x_0, y+y_0, t+t_0+2(yx_0 - xy_0))$, be the composition law of $\mathbb{H}$ determining its Lie group structure. Then the vector fields

$$\begin{aligned} X &= \frac{d}{ds}\phi((x,y,t),(s,0,0))\Big|_{s=0} = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \\ Y &= \frac{d}{ds}\phi((x,y,t),(0,s,0))\Big|_{s=0} = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \\ T &= \frac{d}{ds}\phi((x,y,t),(0,0,s))\Big|_{s=0} = \frac{\partial}{\partial t} \end{aligned} \tag{2.3}$$

form a basis of left invariant vectors fields on $\mathbb{H}$. The Heisenberg group $\mathbb{H}$ can be easily seen as the three-dimensional 2-step nilpotent Carnot group of homogeneous dimension $Q = 4$ and graded algebra $h = g_1 \oplus g_2$, with $g_1 = \{X, Y\}$ and $g_2 = \{T\}$, where the vector fields X, Y, T are defined by (2.3). For $\mathbb{H}$, the critical Sobolev-Stein exponent $q_1^\star = 3$.

For the Heisenberg group, a basis for the linear-homogeneous space is given by X, Y. In this space the subelliptic Kohn-Laplace operator $\Delta_{\mathbb{H}} := X^2 + Y^2$, where X and Y are defined in (2.3). For $u = u(x,y,t) : \mathbb{R}^3 \to \mathbb{R}$, we have

$$\Delta_{\mathbb{H}} u = u_{xx} + u_{yy} + 4(x^2+y^2)u_{tt} + 4yu_{xt} - 4xu_{yt}.$$

From (2.3) one can immediately construct the natural Riemannian metric of $\mathbb{H}$

$$ds^2 = dx^2 + dy^2 + (2ydx - 2xdy + dt)^2.$$

This metric is a left invariant metric. The Lie algebra generated by

$$T = \frac{\partial}{\partial t}, \quad R = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad \tilde{X} = \frac{\partial}{\partial x} - 2y\frac{\partial}{\partial t}, \quad \tilde{Y} = \frac{\partial}{\partial y} + 2x\frac{\partial}{\partial t} \tag{2.4}$$

is the Lie algebra of the infinitesimal isometries of $\mathbb{H}$. We note that the operator T corresponds to translations in t, the operator R - to rotations in the $x - y$ plane, $\tilde{X}$ and $\tilde{Y}$ are the generators of the right multiplication in $\mathbb{H}$.

3 The Noether symmetries of critical Kohn-Laplace equations

The equation (1.5) arises from the following Lagrangian

$$\mathcal{L} = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2+y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - \frac{u^4}{4}. \tag{3.1}$$

By the group classification [13], the symmetry algebra of (1.5) is generated by (2.4) and the following vectors fields:

$$Z = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u},$$

$$V_1 = (xt - x^2y - y^3)\frac{\partial}{\partial x} + (yt + x^3 + xy^2)\frac{\partial}{\partial y} + (t^2 - (x^2 + y^2)^2)\frac{\partial}{\partial t} - tu\frac{\partial}{\partial u},$$

$$V_2 = (t - 4xy)\frac{\partial}{\partial x} + (3x^2 - y^2)\frac{\partial}{\partial y} - (2yt + 2x^3 + 2xy^2)\frac{\partial}{\partial t} + 2yu\frac{\partial}{\partial u},$$

$$V_3 = (x^2 - 3y^2)\frac{\partial}{\partial x} + (t + 4xy)\frac{\partial}{\partial y} + (2xt - 2x^2y - 2y^3)\frac{\partial}{\partial t} - 2xu\frac{\partial}{\partial u}.$$

The operator Z determines a dilation while the operators V_1, V_2, V_3 correspond to transformations of more complicated structure.

In [14] we showed that (2.4) and Z are variational symmetries ($\varphi = 0$ in (1.3)) and V_1, V_2 and V_3 are divergence symmetries of (1.5). Hence all Lie point symmetries of the critical Kohn-Laplace equation (1.5) are Noether symmetries (variational or divergence symmetries). Moreover, in [14] we found explicitly the potentials φ in the conservation laws implied by the Noether's Theorem.

In the next section we state the main result of this paper.

4 The Conservation Laws

Theorem 1. *The conservations laws of the Noether symmetries are:*

1. *For the symmetry T, the conservation law is $Div(\tau) = 0$, where $\tau = (\tau_1, \tau_2, \tau_3)$ and*

$$\tau_1 = -2yu_t^2 - u_xu_t,$$

$$\tau_2 = 2xu_t^2 - u_yu_t,$$

$$\tau_3 = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 - 2(x^2 + y^2)u_t^2 - \frac{1}{4}u^4.$$

2. *For the symmetry R, the conservation law is $Div(\sigma) = 0$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and*

$$\sigma_1 = -\frac{1}{2}yu_x^2 + \frac{1}{2}yu_y^2 + 2y(x^2 + y^2)u_t^2 + xu_xu_y - \frac{1}{4}yu^4,$$

$$\sigma_2 = -\frac{1}{2}xu_x^2 - \frac{1}{2}xu_y^2 - 2x(x^2 + y^2)u_t^2 - yu_xu_y + \frac{1}{4}xu^4,$$

$$\sigma_3 = -2y^2u_x^2 - 2x^2u_y^2 + 4xyu_xu_y - 4y(x^2 + y^2)u_xu_t + 4x(x^2 + y^2)u_yu_t.$$

3. *For the symmetry $\tilde{X}$, the conservation law is $Div(\chi) = 0$, where $\chi = (\chi_1, \chi_2, \chi_3)$*

and

$$\chi_1=-\frac{1}{2}u_x^2+\frac{1}{2}u_y^2+2(x^2+3y^2)u_t^2+2yu_xu_t-2xu_yu_t-\frac{1}{4}u^4,$$

$$\chi_2=-4xyu_t^2-u_xu_y+2xu_xu_t+2yu_yu_t,$$

$$\chi_3=-3yu_x^2-yu_y^2+4y(x^2+y^2)u_t^2+2xu_xu_y-4(x^2+y^2)u_xu_t+\frac{1}{2}yu^4.$$

4. For the symmetry $\tilde{Y}$, the conservation law is $Div(\upsilon)=0$, where $\upsilon=(\upsilon_1,\upsilon_2,\upsilon_3)$ and

$$\upsilon_1=-4xyu_t^2-u_xu_y-2xu_xu_t-2yu_yu_t,$$

$$\frac{1}{2}u_x^2-\frac{1}{2}u_y^2+2(3x^2+y^2)u_t^2+2yu_xu_t-2xu_yu_t-\frac{1}{4}u^4,$$

$$\upsilon_3=xu_x^2+3xu_y^2-4x(x^2+y^2)u_t^2-2yu_xu_y-4(x^2+y^2)u_yu_t-\frac{1}{2}xu^4.$$

5. For the symmetry Z, the conservation law is $Div(\zeta)=0$, where $\zeta=(\zeta_1,\zeta_2,\zeta_3)$ and

$$\begin{aligned}\zeta_1 &= -\frac{1}{2}xu_x^2+\frac{1}{2}xu_y^2+2(x^3-2ty+xy^2)u_t^2-yu_xu_y-2tu_xu_t\\ &\quad -2(x^2+y^2)u_yu_t-uu_x-2yuu_t-\frac{1}{4}xu^4,\\ \zeta_2 &= \frac{1}{2}yu_x^2-\frac{1}{2}yu_y^2+2(2tx+x^2y+y^3)u_t^2-xu_xu_y+2(x^2+y^2)u_xu_t\\ &\quad -2tu_yu_t-uu_y+2xuu_t-\frac{1}{4}yu^4,\\ \zeta_3 &= (t-2xy)u_x^2+(t+2xy)u_y^2-4t(x^2+y^2)u_t^2+2(x^2-y^2)u_xu_y-4x(x^2+y^2)u_xu_t\\ &\quad -4y(x^2+y^2)u_yu_t+2xuu_y-2yuu_x-4(x^2+y^2)uu_t-\frac{1}{2}tu^4.\end{aligned}$$

6. For the symmetry V_1, the conservation law is $Div(A)=0$, where $A=(A_1,A_2,A_3)$ and

$$\begin{aligned}A_1 &= -\frac{1}{2}(tx-x^2y-y^3)u_x^2+\frac{1}{2}(tx-x^2y-y^3)u_y^2+2t(x^3+xy^2-ty)u_t^2\\ &\quad -(x^3+xy^2+ty)u_xu_y-[t^2-(x^2+y^2)^2]u_xu_t-2t(x^2+y^2)u_yu_t\\ &\quad -tuu_x-2tyuu_t+yu^2-\frac{1}{4}(tx-x^2y-y^3)u^4,\end{aligned}$$

$$\begin{aligned}
A_2 &= \frac{1}{2}(x^3+ty+xy^2)u_x^2 - \frac{1}{2}(x^3+ty+xy^2)u_y^2 + 2t(x^2y+y^3+tx)u_t^2 \\
&-(tx-x^2y-y^3)u_xu_y + 2t(x^2+y^2)u_xu_t - [t^2-(x^2+y^2)^2]u_yu_t \\
&-tuu_y + 2txuu_t - xu^2 - \frac{1}{4}(x^3+ty+xy^2)u^4, \\
A_3 &= +\frac{1}{2}(t^2-x^4-4txy+2x^2y^2+3y^4)u_x^2 + \frac{1}{2}(t^2+3x^4+4txy+2x^2y^2-y^4)u_y^2 \\
&-2(x^2+y^2)[t^2-(x^2+y^2)^2]u_t^2 + 2[t(x^2-y^2)-2xy(x^2+y^2)]u_xu_y \\
&-4(x^2+y^2)(tx-x^2y-y^3)u_xu_t - 4(x^2+y^2)(x^3+ty+xy^2)u_yu_t \\
&-2tyuu_x + 2txuu_y - 4t(x^2+y^2)uu_t + 2(x^2+y^2)u^2 - \frac{1}{4}[t^2-(x^2+y^2)^2]u^4.
\end{aligned}$$

7. For the symmetry V_2, the conservation law is $Div(B)=0$, where $B=(B_1,B_2,B_3)$ and

$$\begin{aligned}
B_1 &= -\frac{1}{2}(t-4xy)u_x^2 + \frac{1}{2}(t-4xy)u_y^2 + [2t(x^2+3y^2)-4xy(x^2+y^2)]u_t^2 \\
&+-(3x^2-y^2)u_xu_y + 2(x^3+ty+xy^2)u_xu_t - 2(tx-x^2y-y^3)u_yu_t \\
&+2yuu_x + 4y^2uu_t - \frac{1}{4}(t-4xy)u^4, \\
B_2 &= \frac{1}{2}(3x^2-y^2)u_x^2 - \frac{1}{2}(3x^2-y^2)u_y^2 + 2(x^4-2txy-y^4)u_t^2 - (t-4xy)u_xu_y \\
&+2(tx-x^2y-y^3)u_xu_t + 2(x^3+ty+xy^2)u_yu_t + 2yuu_y - 4xyuu_t - u^2 \\
&-\frac{1}{4}(3x^2-y^2)u^4, \\
B_3 &= (7xy^2-x^3-3ty)u_x^2 + (5x^3-3xy^2-ty)u_y^2 + 4(x^2+y^2)(x^3+ty+xy^2)u_t^2 \\
&+2(tx-7x^2y+y^3)u_xu_y - 4(t-4xy)(x^2+y^2)u_xu_t - 4(3x^4+2x^2y^2-y^4)u_yu_t \\
&+2xu^2 + 4y^2uu_x - 4xyuu_y + 8y(x^2+y^2)uu_t + \frac{1}{2}(x^3+ty+xy^2)u^4
\end{aligned}$$

8. For the symmetry V_3, the conservation law is $Div(C)=0$, where $C=(C_1,C_2,C_3)$

and

$$
\begin{aligned}
C_1 &= \frac{1}{2}(x^2y - tx + y^3)u_x^2 + \frac{1}{2}(tx - x^2y - y^3)u_y^2 + 2t(x^3 - ty + xy^2)u_t^2 \\
&\quad -(x^3 + ty + xy^2)u_xu_y - [t^2 - (x^2+y^2)^2]u_xu_t - 2t(x^2+y^2)u_yu_t \\
&\quad -tuu_x - 2tyuu_t - \frac{1}{4}(tx - x^2y - y^3)u^4, \\
C_2 &= \frac{1}{2}(x^3 + ty + xy^2)u_x^2 - \frac{1}{2}(x^3 + ty + xy^2)u_y^2 + 2t(tx + x^2y + y^3)u_t^2 \\
&\quad -(tx - x^2y - y^3)u_xu_y + 2t(x^2+y^2)u_xu_t - [t^2 - (x^2+y^2)^2]u_yu_t \\
&\quad -u^2 - tuu_y + 2txuu_t - \frac{1}{4}(x^3 + ty + xy^2)u^4,
\end{aligned}
$$

$$
\begin{aligned}
C_3 &= \frac{1}{2}(t^2 - x^4 - 4txy + 2x^2y^2 + 3y^4)u_x^2 + \frac{1}{2}(t^2 + 3x^4 + 4txy + 2x^2y^2 - y^4)u_y^2 \\
&\quad -2(x^2+y^2)[t^2 - (x^2+y^2)^2]u_t^2 + 2[t(x^2 - y^2) - 2xy(x^2+y^2)]u_xu_y \\
&\quad +4(x^2+y^2)(x^2y - tx + y^3)u_xu_t - 4(x^2+y^2)(x^3 + ty + xy^2)u_yu_t \\
&\quad +2txuu_y - 2tyuu_x - 4t(x^2+y^2)uu_t + 2yu^2 - \frac{1}{4}[t^2 - (x^2+y^2)^2]u^4.
\end{aligned}
$$

Proof. First, we observe that the potentials for the symmetries T, R, $\tilde{X}$, $\tilde{Y}$ and Z of (1.5) are 0, that is, these symmetries are variational [14]. Further, the potentials φ of V_1, V_2, V_3 are $(-yu^2, xu^2, -2(x^2+y^2)u^2)$, $(0, u^2, -2xu^2)$, $(-u^2, 0, -2yu^2)$ respectively. See [14].

Now, with these at hand, as mentioned in the introduction, the proof is by a tedious straightforward calculation, which we shall not present here for obvious reasons. However, a computer assisted proof can be obtained by two simple *Mathematica* programs. The first one calculates the components of the conservation laws, which appear in the equation (1.4). The second program verifies the conservation laws using the Noether Identity [27]. Both Mathematica notebooks can be obtained form the authors upon request. ■

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