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On the (2+1)-Dimensional Extension of 1-Dimensional Toda Lattice Hierarchy

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Abstract

We introduce a (2+1)-dimensional extension of the 1-dimensional Toda lattice hierarchy. The hierarchy is given by two different formulations. For the first formulation, we obtain the bilinear identity for the τ-functions and construct explicit solutions expressed by Wronski determinants. For the second formulation, we obtain the slightly different bilinear identity and clarify the relation to the hierarchy arising from the homogeneous vertex representation of the 2-toroidal Lie algebra $\mathfrak{sl}_2^{\mathrm{tor}}$, obtained by Kakei *et al.*

1 Introduction

There has been a great deal of studies on nonlinear integrable equations in higher dimensions. The typical example of such equations is a (2+1)-dimensional extension of the Korteweg de-Vries (KdV) equation

$$u_z = \frac{1}{4}u_{xxy} + uu_y + \frac{1}{2}\,u_x \int^x u_y dx. \tag{1.1}$$

This equation was first discovered by Calogero [3] and re-discovered by Bogoyavlensky [2]. Billig [1] and Iohara *et al* [6] obtained a hierarchy of Hirota bilinear equations associated with this equation from the homogeneous vertex representation of the 2-toroidal Lie algebra $\mathfrak{sl}_2^{\mathrm{tor}}$. Recently Ikeda and Takasaki [5] gave a Lax formalism of this hierarchy on the basis of the 2-reduced KP (KdV) hierarchy and discussed the bilinearlization of auxiliary linear equations, special solutions, a relation to the toroidal Lie algebra, etc. in the framework of the Sato theory of the KP hierarchy [10].

On these backgrounds, in [8], Kakei and two authors in [5] developed their previous work to a hierarchy associated with the (2+1)-dimensional nonlinear Schrödinger (NLS) equation [2, 11, 12]

$$iu_x + u_{xy} + 2u \int^x (|u|^2)_y dx = 0 \tag{1.2}$$

on the basis of the two-component KP theory. Their work strongly suggests that the KP theory provides a powerful tool to analyze such (2+1)-dimensional soliton equations.

The aim of the present paper, motivated by the work of Ikeda, Takasaki and Kakei, is to develop the theory of the Toda lattice (TL) hierarchy to treat (2+1)-dimensional soliton equations. Namely we consider a (2+1)-dimensional extension of the so-called *1-dimensional Toda lattice (1-D TL) hierarchy* [13] of nonlinear differential-difference equations associated with the Toda lattice equation

$$\partial_{t_1}^2 \phi(s) = e^{\phi(s-1)-\phi(s)} - e^{\phi(s)-\phi(s+1)}. \tag{1.3}$$

The resulting is a system of nonlinear differential-differential-difference equations, which we call the *(2+1)-dimensional Toda lattice ((2+1)-D TL) hierarchy*.

Our formulation of the (2+1)-D TL hierarchy is to add the toroidal-like evolution equations to the 1-D TL hierarchy. We present two different formulations, according to two types of reductions to the 1-D TL hierarchy from the TL hierarchy. For the first formulation, we obtain the bilinear identity for the τ-functions and construct explicit solutions expressed by Wronski determinants. For the second formulation, we obtain the bilinear identity slightly different from the previous one. This bilinear identity can be identified with that of the $\mathfrak{sl}_2^{\mathrm{tor}}$-hierarchy of [8] arising from the homogeneous vertex representation of $\mathfrak{sl}_2^{\mathrm{tor}}$.

This paper is organized as follows. In Sec.2, we first review on Lax and Sato formalisms of the TL hierarchy in the language of formal difference operators. The 1-D TL hierarchy is formulated as a special reduction from the TL hierarchy, and then we introduce the (2+1)-D TL hierarchy; this is the first version of our system. In Sec.3, from the linear equations for the Baker-Akhiezer functions, we obtain the bilinear identity for the τ-functions. In Sec.4, by using Wronskian technique, we construct explicit Wronskian-type solutions of the (2+1)-D TL hierarchy. In Sec.5, we present the second version of the (2+1)-D TL hierarchy. For this second version, a slightly different bilinear identity is obtained. We then clarify the relation to the $\mathfrak{sl}_2^{\mathrm{tor}}$-hierarchy of Kakei *et al.* Sec.6 is devoted to a conclusion and some remarks.

Notation. A (formal) difference operator is a linear combination, $A(s; e^{\partial_s}) = \sum_n a_n e^{n\partial_s}$, of integer powers of a shift operator e^{∂_s}, i.e., $e^{\partial_s} f(s) = f(s+1)$, with coefficients $a_n = a_n(s)$ that depend on a discrete variable s. The index n ranges over all integers with an upper bound N or a lower bound M. We call $A = \sum_{n=N}^{M} a_n e^{n\partial_s}$ a difference operator of type $[N, M]$. It is convenient to use the following notations:

$$(A)_{\geq 0} = \sum_{n\geq 0} a_n e^{n\partial_s}, \quad (A)_{<0} = \sum_{n<0} a_n e^{n\partial_s}, \quad (A)_k = a_k.$$

Throughout this paper, we shall use the abbreviations for differentiations

$$\partial_x A = \frac{\partial A}{\partial x}.$$

2 (2+1)-dimensional Toda lattice hierarchy

2.1 Toda lattice hierarchy

In this subsection, we recall basic notions about Lax and Sato formalisms of the Toda lattice hierarchy following [13] in order to fix our notations used throughout this paper.

We start with two types of difference operators called *Lax operators*

$$L = e^{\partial_s} + \sum_{n=0}^{\infty} u_{n+1}(s) e^{-n\partial_s}, \quad \bar{L} = \bar{u}_0(s) e^{-\partial_s} + \sum_{n=0}^{\infty} \bar{u}_{n+1}(s) e^{n\partial_s} \tag{2.1}$$

with $\bar{u}_0(s) \not\equiv 0$, where the coefficients of L, $\bar{L}$ are functions depending on two series of "time" variables $\boldsymbol{x} = (x_1, x_2, \dots)$, $\boldsymbol{y} = (y_1, y_2, \dots)$ along with a discrete variable s. Using the Lax operators, we define *Zakharov-Shabat operators*

$$B_n = (L^n)_{\geq 0}, \quad \bar{B}_n = (\bar{L}^n)_{<0}, \quad n \geq 1. \tag{2.2}$$

The Toda lattice hierarchy (TL hierarchy) then arises as the compatibility condition for the following linear equations for $\Psi(\lambda)$ and $\bar{\Psi}(\lambda)$:

$$\begin{aligned} L\Psi(\lambda) = \lambda\Psi(\lambda), \quad \bar{L}\bar{\Psi}(\lambda) = \lambda^{-1}\bar{\Psi}(\lambda), \\ \partial_{x_n}\widetilde{\Psi}(\lambda) = B_n\widetilde{\Psi}(\lambda), \quad \partial_{y_n}\widetilde{\Psi}(\lambda) = \bar{B}_n\widetilde{\Psi}(\lambda), \end{aligned} \tag{2.3}$$

where $\widetilde{\Psi}(\lambda) = \Psi(\lambda)$ or $\bar{\Psi}(\lambda)$, and λ is a spectral parameter. More explicitly, the TL hierarchy is a system of evolution equations defined by *Lax equations*

$$\begin{aligned} \partial_{x_n} L = [B_n, L], \quad \partial_{y_n} L = [\bar{B}_n, L], \\ \partial_{x_n} \bar{L} = [B_n, \bar{L}], \quad \partial_{y_n} \bar{L} = [\bar{B}_n, \bar{L}], \quad n \geq 1. \end{aligned} \tag{2.4}$$

Formal solutions of the linear equations (2.3) of the form

$$\begin{aligned} \Psi(s,\lambda) &= \left(1 + \sum_{n=1}^{\infty} v_n(s)\lambda^{-n}\right) \lambda^s e^{\xi(\boldsymbol{x},\lambda)+\xi(\boldsymbol{y},\lambda^{-1})}, \\ \bar{\Psi}(s,\lambda) &= \left(\sum_{n=0}^{\infty} \bar{v}_n(s)\lambda^{n}\right) \lambda^s e^{\xi(\boldsymbol{x},\lambda)+\xi(\boldsymbol{y},\lambda^{-1})} \quad \text{with} \quad \bar{v}_0(s) \not\equiv 0 \end{aligned} \tag{2.5}$$

are called *(formal) Baker-Akhiezer functions*. Here $\xi(\boldsymbol{x}, \lambda)$ denotes a formal power series

$$\xi(\boldsymbol{x}, \lambda) = \sum_{n=1}^{\infty} x_n \lambda^n.$$

If we define difference operators called *Sato-Wilson operators*

$$V = 1 + \sum_{n=1}^{\infty} v_n(s) e^{-n\partial_s}, \quad \bar{V} = \sum_{n=0}^{\infty} \bar{v}_n(s) e^{n\partial_s}, \tag{2.6}$$

then the Baker-Akhiezer functions can be written as

$$\Psi(s,\lambda) = V\lambda^s e^{\xi(\boldsymbol{x},\lambda)+\xi(\boldsymbol{y},\lambda^{-1})}, \quad \bar{\Psi}(s,\lambda) = \bar{V}\lambda^s e^{\xi(\boldsymbol{x},\lambda)+\xi(\boldsymbol{y},\lambda^{-1})}. \tag{2.7}$$

Note that our definition of V, $\bar{V}$ is the same as that of "modified" Sato-Wilson operators in [13]. Substituting the expressions (2.7) into (2.3), we obtain "dressing relations" for the Lax operators

$$L = V e^{\partial_s} V^{-1}, \quad \bar{L} = \bar{V} e^{-\partial_s} \bar{V}^{-1}, \tag{2.8}$$

and evolution equations for the Sato-Wilson operators

$$\partial_{x_n} \widetilde{V} = B_n \widetilde{V} - \widetilde{V} e^{n\partial_s}, \quad \partial_{y_n} \widetilde{V} = \bar{B}_n \widetilde{V} - \widetilde{V} e^{-n\partial_s}, \tag{2.9}$$

where we have used the abbreviation $\widetilde{V} = V$ or $\bar{V}$. Since

$$B_n = (V e^{n\partial_s} V^{-1})_{\geq 0}, \quad \bar{B}_n = (\bar{V} e^{-n\partial_s} \bar{V}^{-1})_{<0}, \tag{2.10}$$

(2.9) represents a closed system of evolution equations for V and $\bar{V}$, which we call *Sato equations* hereafter. The Sato-Wilson operators V and $\bar{V}$ are uniquely determined from the dressing relations (2.8) up to arbitrariness of multiplication from the right by operators of the form $1 + \sum_{n=1}^{\infty} c_n e^{-n\partial_s}$ and $\sum_{n=0}^{\infty} \bar{c}_n e^{n\partial_s}$ respectively, where c_n ($n \geq 1$) and $\bar{c}_n$ ($n \geq 0$) are constants.

2.2 Reduction to 1-dimensional Toda lattice hierarchy

We formulate the 1-dimensional Toda lattice (1-D TL) hierarchy by considering special type of reduction from the TL hierarchy.

Let $\boldsymbol{t} = (t_1, t_2, \dots)$ and $\bar{\boldsymbol{t}} = (\bar{t}_1, \bar{t}_2, \dots)$ be variables defined by

$$t_n = \frac{x_n - y_n}{2}, \quad \bar{t}_n = \frac{x_n + y_n}{2}, \quad n \geq 1. \tag{2.11}$$

We impose the following equivalent constraints on the TL hierarchy:

$$L + L^{-1} = \bar{L} + \bar{L}^{-1} \stackrel{\text{def}}{=} \mathcal{L}, \tag{2.12}$$

$$\partial_{\bar{t}_n} V = \partial_{\bar{t}_n} \bar{V} = 0, \quad n \geq 1. \tag{2.13}$$

Eq.(2.12) implies that $\mathcal{L}$ takes the form

$$\mathcal{L} = e^{\partial_s} + u_1(s, \boldsymbol{t}) + \bar{u}_0(s, \boldsymbol{t}) e^{-\partial_s},$$

where the coefficients, $u_1(s, \boldsymbol{t})$ and $\bar{u}_0(s, \boldsymbol{t})$, are functions not depending on $\bar{\boldsymbol{t}}$, because of (2.13). The Lax equations (2.4) are now reduced to the Lax representation of the 1-D TL hierarchy [13]

$$\partial_{t_n} \mathcal{L} = [B_n - \bar{B}_n, \mathcal{L}], \quad n \geq 1. \tag{2.14}$$

The lowest equation in (2.14) is equivalent to the Toda lattice equation (1.3) where the $\phi(s)$ is defined through the relations $d\phi(s)/dt_1 = -2u_1(s)$ and $e^{\phi(s-1)-\phi(s)} = 4\bar{u}_0(s)$.

Remark 1. If we set $\mathcal{B}_n = (\mathcal{L}^n)_{\geq 0}$ and $\bar{\mathcal{B}}_n = (\mathcal{L}^n)_{<0}$ for each $n \geq 1$, then they are related to the Zakharov-Shabat operators through the linear transformation of the form

$$(\mathcal{B}_1, \mathcal{B}_2, \dots) = (B_1, B_2, \dots) A, \quad (\bar{\mathcal{B}}_1, \bar{\mathcal{B}}_2, \dots) = (\bar{B}_1, \bar{B}_2, \dots) A$$

with A being a certain semi-infinite upper-triangular constant matrix. Define new time variables T_n $(n \geq 1)$ through

$$(\partial_{T_1}, \partial_{T_2}, \dots) = (\partial_{t_1}, \partial_{t_2}, \dots)A.$$

Then the Lax equations (2.14) can be converted to a closed system for $\mathcal{L}$

$$\partial_{T_n}\mathcal{L} = [\mathcal{B}_n - \bar{\mathcal{B}}_n, \mathcal{L}], \quad n \geq 1.$$

2.3 Definition of (2+1)-dimensional Toda lattice hierarchy

We now introduce the (2+1)-dimensional Toda lattice ((2+1)-D TL) hierarchy by introducing additional "toroidal-like" evolution equations to the 1-D TL hierarchy.

2.3.1 Toroidal-like part of evolution equations

Let V and $\bar{V}$ be the Sato-Wilson operators of the 1-D TL hierarchy, i.e., V and $\bar{V}$ satisfy (2.13). Introduce a new set of infinite "time" variables $\check{\boldsymbol{z}} = (z_1, z_2, \dots)$ and a continuous "spatial" variable z_0. We define evolution equations for V, $\bar{V}$ with respect to $\check{\boldsymbol{z}}$ by

$$\partial_{z_n}\widetilde{V} = (B_n + \bar{B}_n)\,\partial_{z_0}\widetilde{V} + (K_n + \bar{K}_n)\widetilde{V}, \quad n \geq 1, \tag{2.15}$$

where

$$K_n = -\left(B_n\,\partial_{z_0}V \cdot V^{-1}\right)_{\geq 0}, \quad \bar{K}_n = -\left(\bar{B}_n\,\partial_{z_0}\bar{V} \cdot \bar{V}^{-1}\right)_{<0}. \tag{2.16}$$

These equations together with the Sato equations of the 1-D TL hierarchy

$$\partial_{t_n}\widetilde{V} = (B_n - \bar{B}_n)\widetilde{V} - \widetilde{V}(e^{n\partial_s} - e^{-n\partial_s}), \quad n \geq 1 \tag{2.17}$$

define the system of evolution equations for the Sato-Wilson operators in (2+1)-dimension, which we call the *(2+1)-D TL hierarchy* hereafter.

2.3.2 Lax equations

The evolution equations (2.15) induce linear equations for the Baker-Akhiezer functions

$$\partial_{z_n}\widetilde{\Psi}(s,\lambda) = (B_n + \bar{B}_n)\partial_{z_0}\widetilde{\Psi}(s,\lambda) + (K_n + \bar{K}_n)\widetilde{\Psi}(s,\lambda). \tag{2.18}$$

We now put

$$\mathcal{M} = \partial_{z_0}V \cdot V^{-1}, \quad \bar{\mathcal{M}} = \partial_{z_0}\bar{V} \cdot \bar{V}^{-1}. \tag{2.19}$$

Then we can express K_n (resp. $\bar{K}_n$) in terms of $L, \mathcal{M}$ (resp. $\bar{L}, \bar{\mathcal{M}}$) as follows:

$$K_n = -(L^n\mathcal{M})_{\geq 0}, \quad \bar{K}_n = -(\bar{L}^n\bar{\mathcal{M}})_{<0}. \tag{2.20}$$

Indeed, since $B_n = L^n - (L^n)_{<0}$, we have

$$K_n = -(B_n\mathcal{M})_{\geq 0} = -(L^n\mathcal{M})_{\geq 0} + [(L^n)_{<0}\mathcal{M}]_{\geq 0} = -(L^n\mathcal{M})_{\geq 0}$$

where in the last equality, the term $[(L^n)_{<0}\mathcal{M}]_{\geq 0}$ has vanished because the difference operator $(L^n)_{<0}\mathcal{M}$ has only negative powers of e^{∂_s}. The expression of $\bar{K}_n$ is obtained in the similar way.

The operators $\mathcal{M}$ and $\bar{\mathcal{M}}$ satisfy the following linear equations for the Baker-Akhiezer functions:

$$\mathcal{M}\Psi(s,\lambda) = \partial_{z_0}\Psi(s,\lambda), \quad \bar{\mathcal{M}}\bar{\Psi}(s,\lambda) = \partial_{z_0}\bar{\Psi}(s,\lambda). \tag{2.21}$$

The compatibility condition for the linear equations (2.3), (2.18) and (2.21), gives rise to Lax equations of the (2+1)-D TL hierarchy.

Proposition 1. *The operators $\mathcal{L}$, $\mathcal{M}$ and $\bar{\mathcal{M}}$ satisfy the Lax equations*

$$\begin{aligned}
\partial_{t_n}\mathcal{L} &= [B_n - \bar{B}_n, \mathcal{L}], \\
\partial_{z_n}\mathcal{L} &= \left[(B_n + \bar{B}_n)\partial_{z_0} + K_n + \bar{K}_n,\ \mathcal{L}\right], \\
\partial_{t_n}\widetilde{\mathcal{M}} &= \left[B_n - \bar{B}_n,\ \widetilde{\mathcal{M}} - \partial_{z_0}\right], \\
\partial_{z_n}\widetilde{\mathcal{M}} &= \left[(B_n + \bar{B}_n)\widetilde{\mathcal{M}} + K_n + \bar{K}_n,\ \widetilde{\mathcal{M}} - \partial_{z_0}\right], \quad n \geq 1,
\end{aligned} \tag{2.22}$$

where $\widetilde{\mathcal{M}} = \mathcal{M}$ or $\bar{\mathcal{M}}$.

Proof. Straightforward calculation. ■

Remark 2. To confirm the closedness of the above system of Lax equations, set $\mathcal{K}_n = -(\mathcal{L}^n\mathcal{M})_{\geq 0}$ and $\bar{\mathcal{K}}_n = -(\mathcal{L}^n\bar{\mathcal{M}})_{<0}$ for each $n \geq 1$. Then they are related to K_n and $\bar{K}_n$ $(n \geq 1)$ as

$$(\mathcal{K}_1, \mathcal{K}_2, \dots) = (K_1, K_2, \dots)A, \quad (\bar{\mathcal{K}}_1, \bar{\mathcal{K}}_2, \dots) = (\bar{K}_1, \bar{K}_2, \dots)A$$

for A being the same matrix as the one discussed in Remark 1. In a similar way as in Remark 1, if we introduce new time variables Z_n $(n \geq 1)$ through

$$(\partial_{Z_1}, \partial_{Z_2}, \dots) = (\partial_{z_1}, \partial_{z_2}, \dots)A,$$

then the Lax equations (2.22) can be converted to a closed system for $\mathcal{L}$, $\mathcal{M}$ and $\bar{\mathcal{M}}$

$$\begin{aligned}
\partial_{T_n}\mathcal{L} &= [\mathcal{B}_n - \bar{\mathcal{B}}_n, \mathcal{L}], \\
\partial_{Z_n}\mathcal{L} &= \left[(\mathcal{B}_n + \bar{\mathcal{B}}_n)\partial_{z_0} + \mathcal{K}_n + \bar{\mathcal{K}}_n,\ \mathcal{L}\right], \\
\partial_{T_n}\widetilde{\mathcal{M}} &= \left[\mathcal{B}_n - \bar{\mathcal{B}}_n,\ \widetilde{\mathcal{M}} - \partial_{z_0}\right], \\
\partial_{Z_n}\widetilde{\mathcal{M}} &= \left[(\mathcal{B}_n + \bar{\mathcal{B}}_n)\widetilde{\mathcal{M}} + \mathcal{K}_n + \bar{\mathcal{K}}_n,\ \widetilde{\mathcal{M}} - \partial_{z_0}\right], \quad n \geq 1.
\end{aligned} \tag{2.23}$$

In what follows, however, we will work with the system for the time variables t_n, z_n $(n \geq 1)$ rather than T_n, Z_n $(n \geq 1)$ for avoiding some technical difficulties.

Example. We shall now demonstrate the Lax equation for $\mathcal{L}$ with the first order flow z_1. We can write down K_1 and $\bar{K}_1$ in the following form:

$$K_1 = -\partial_{z_0}v_1(s+1), \quad \bar{K}_1 = -\bar{u}_0(s)\partial_{z_0}\log\bar{v}_0(s-1)e^{-\partial_s}.$$

The coefficients of the Lax equation for $\mathcal{L}$ with z_1-flow can be calculated as

$$\begin{aligned}\partial_{z_1} u_1(s) &= u_1(s)\partial_{z_0} u_1(s) + \partial_{z_0}\bar{u}_0(s+1) + \bar{u}_0(s+1)\partial_{z_0}\log\bar{v}_0(s) \\ &\quad - \bar{u}_0(s)\partial_{z_0}\log\bar{v}_0(s-1), \\ \partial_{z_1}\bar{u}_0(s) &= u_1(s)\partial_{z_0}\bar{u}_0(s) + \bar{u}_0(s)\partial_{z_0}u_1(s-1) + \bar{u}_0(s)[v_1(s) - v_1(s-1)] \\ &\quad + \bar{u}_0(s)[u_1(s) - u_1(s-1)]\partial_{z_0}\log\bar{v}_0(s-1).\end{aligned}$$

By the relations $u_1(s) = v_1(s) - v_1(s+1)$ and $\bar{u}_0(s) = \bar{v}_0(s)\bar{v}_0(s-1)^{-1}$ which are obtained from the dressing relations, the above equations can be cast into the following (2+1)-dimensional equation for $u_1(s)$ and $\bar{u}_0(s)$:

$$\partial_{z_1} u_1(s) = u_1(s)\partial_{z_0} u_1(s) + \partial_{z_0}\bar{u}_0(s+1) + \bar{u}_0(s+1)Q(s) - \bar{u}_0(s)Q(s-1), \qquad (2.24)$$

where $Q(s) \stackrel{\text{def}}{=} \partial_{z_0}\log\bar{v}_0(s)$ is given by

$$Q(s) = \frac{\partial_{z_1}\bar{u}_0(s+1) - u_1(s+1)\partial_{z_0}\bar{u}_0(s+1) - \bar{u}_0(s+1)\partial_{z_0}[u_1(s) + u_1(s+1)]}{\bar{u}_0(s+1)[u_1(s+1) - u_1(s)]}.$$

3 Bilinear identity and τ-function

In this section, we present a bilinear formulation of the (2+1)-D TL hierarchy.

3.1 Dual Baker-Akhiezer functions

We introduce *dual* Baker-Akhiezer functions

$$\widetilde{\Psi}^*(s+1,\lambda) = (\widetilde{V}^*)^{-1}\lambda^{-s-1}e^{-\xi(\boldsymbol{x},\lambda)-\xi(\boldsymbol{y},\lambda^{-1})}, \qquad (3.1)$$

where $\widetilde{\Psi}^*(s,\lambda) = \Psi^*(s,\lambda)$ or $\bar{\Psi}^*(s,\lambda)$, and $\widetilde{V} = V$ or $\bar{V}$. Here $\widetilde{V}^*$ denotes the *formal adjoint* of $\widetilde{V}$, which is, in general, defined by

$$A = \sum_n a_n(s)e^{n\partial_s} \longmapsto A^* = \sum_n e^{-n\partial_s}\circ a_n(s).$$

We note that $(AB)^* = B^*A^*$ that holds for two difference operators A and B such that the product AB is algebraically well-defined. Of particular importance is the following linear equations for the dual Baker-Akhiezer functions.

Lemma 1. *The dual Baker-Akhiezer functions satisfy the linear equations*

$$\left[\partial_{z_n} - (\lambda^n + \lambda^{-n})\partial_{z_0}\right]\widetilde{\Psi}^*(s+1,\lambda) = -(K_n^* + \bar{K}_n^*)\widetilde{\Psi}^*(s+1,\lambda). \qquad (3.2)$$

Proof. Noticing that $B_n + \bar{B}_n = \widetilde{V}(e^{n\partial_s} + e^{-n\partial_s})\widetilde{V}^{-1}$, we have

$$\begin{aligned}\partial_{z_n}(\widetilde{V}^{-1}) &= -\widetilde{V}^{-1}\left[(B_n + \bar{B}_n)\partial_{z_0}\widetilde{V} + (K_n + \bar{K}_n)\widetilde{V}\right]\widetilde{V}^{-1} \\ &= (e^{n\partial_s} + e^{-n\partial_s})\partial_{z_0}(\widetilde{V}^{-1}) - \widetilde{V}^{-1}(K_n + \bar{K}_n).\end{aligned}$$

Taking the formal adjoint of both sides of this equation, we obtain

$$\partial_{z_n}(\widetilde{V}^*)^{-1} = \partial_{z_0}(\widetilde{V}^*)^{-1}(e^{n\partial_s} + e^{-n\partial_s}) - (K_n^* + \bar{K}_n^*)(\widetilde{V}^*)^{-1},$$

from which the lemma follows immediately. ■

3.2 Bilinear identity

The aim of this subsection is to prove Theorem 1 below. We shall write in what follows $\boldsymbol{z} = (z_0, \check{\boldsymbol{z}}) = (z_0, z_1, z_2, \dots)$.

Theorem 1. *For any $k \geq 0$, we have*

$$\begin{aligned}&\oint (\lambda+\lambda^{-1})^k \Psi(s', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}+\boldsymbol{b}_\lambda)\Psi^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}+\boldsymbol{c}_\lambda)\,\frac{d\lambda}{2\pi i}\\ &\quad = \oint (\lambda+\lambda^{-1})^k \bar{\Psi}(s', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}+\boldsymbol{b}_\lambda)\bar{\Psi}^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}+\boldsymbol{c}_\lambda)\,\frac{d\lambda}{2\pi i}\end{aligned} \tag{3.3}$$

where $\boldsymbol{b} = (b_1, b_2, \dots)$, $\boldsymbol{c} = (c_1, c_2, \dots)$ are indeterminates, and we have set $\boldsymbol{b}_\lambda = (-\xi(\boldsymbol{b}, \lambda) - \xi(\boldsymbol{b}, \lambda^{-1}), b_1, b_2, \dots)$ and similarly for $\boldsymbol{c}_\lambda$. Each of the contour integrals is understood to be an algebraic operation extracting the coefficient of λ^{-1}, i.e., $\oint \lambda^n d\lambda/(2\pi i) = \delta_{n,-1}$.

For the proof, we use the following lemma.

Lemma 2. *It holds that*

$$\begin{aligned}&\oint \Psi(s', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}, \lambda)\,\Psi^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda)\,\frac{d\lambda}{2\pi i}\\ &\quad = \oint \bar{\Psi}(s', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}, \lambda)\,\bar{\Psi}^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda)\,\frac{d\lambda}{2\pi i}.\end{aligned} \tag{3.4}$$

Proof. See Appendix. ■

Since $\mathcal{L}$ acts on the Baker-Akhiezer functions by $\mathcal{L}\widetilde{\Psi}(s, \lambda) = (\lambda + \lambda^{-1})\widetilde{\Psi}(s, \lambda)$, multiplying $\mathcal{L}^k$ ($k \geq 0$) on both sides of the above equation as a difference operator with respect to s', we obtain

$$\begin{aligned}&\oint (\lambda+\lambda^{-1})^k \Psi(s', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}, \lambda)\,\Psi^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda)\,\frac{d\lambda}{2\pi i}\\ &\quad = \oint (\lambda+\lambda^{-1})^k \bar{\Psi}(s', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}, \lambda)\,\bar{\Psi}^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda)\,\frac{d\lambda}{2\pi i}.\end{aligned} \tag{3.5}$$

Proof of Theorem 1. Eq.(2.18) is equivalently written as

$$\left[\partial_{z_n} - (\lambda^n + \lambda^{-n})\partial_{z_0}\right]\widetilde{\Psi}(s, \lambda) = \left[K_n + \bar{K}_n - \partial_{z_0}(B_n + \bar{B}_n)\right]\widetilde{\Psi}(s, \lambda). \tag{3.6}$$

Iteration of the linear equations (3.2) and (3.6), yields higher order evolution equations of the form

$$\begin{aligned}&[\partial_{z_1} - (\lambda+\lambda^{-1})\partial_{z_0}]^{\beta_1}[\partial_{z_2} - (\lambda^2+\lambda^{-2})\partial_{z_0}]^{\beta_2}\cdots\widetilde{\Psi}(s, \lambda) = K^{(1)}_{\beta_1,\beta_2,\dots}\widetilde{\Psi}(s, \lambda),\\ &[\partial_{z_1} - (\lambda+\lambda^{-1})\partial_{z_0}]^{\beta_1}[\partial_{z_2} - (\lambda^2+\lambda^{-2})\partial_{z_0}]^{\beta_2}\cdots\widetilde{\Psi}^*(s, \lambda) = K^{(2)}_{\beta_1,\beta_2,\dots}\widetilde{\Psi}^*(s, \lambda),\end{aligned} \tag{3.7}$$

where $\beta_1, \beta_2, \dots \geq 0$, and $K^{(n)}_{\beta_1,\beta_2,\dots}$ ($n = 1, 2$) are some difference operators. We now multiply $K^{(1)}_{\beta_1,\beta_2,\dots}$ on (3.5) as a difference operator with respect to s'. Then from the first equation of (3.7), we obtain

$$\begin{aligned}&\oint (\lambda+\lambda^{-1})^k \prod_{n\geq 1}[\partial_{z_n} - (\lambda^n+\lambda^{-n})\partial_{z_0}]^{\beta_n}\Psi(s', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}, \lambda)\cdot\Psi^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda)\,\frac{d\lambda}{2\pi i}\\ &\quad = \oint (\lambda+\lambda^{-1})^k \prod_{n\geq 1}[\partial_{z_n} - (\lambda^n+\lambda^{-n})\partial_{z_0}]^{\beta_n}\bar{\Psi}(s', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}, \lambda)\cdot\bar{\Psi}^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda)\,\frac{d\lambda}{2\pi i}\end{aligned}$$

for $\beta_1, \beta_2, \dots \geq 0$, which can be cast into a generating function

$$\oint (\lambda + \lambda^{-1})^k \Psi(s', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z} + \boldsymbol{b}_\lambda) \Psi^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda) \frac{d\lambda}{2\pi i}$$
$$= \oint (\lambda + \lambda^{-1})^k \bar{\Psi}(s', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z} + \boldsymbol{b}_\lambda) \bar{\Psi}^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda) \frac{d\lambda}{2\pi i}$$

where $\boldsymbol{b} = (b_1, b_2, \dots)$ are new indeterminates. Similarly, if we multiply $K^{(2)}_{\gamma_1,\gamma_2,\dots}$ on this equation as a difference operator with respect to s, the result can be converted to the desirous formula. ■

3.3 τ-function

We now introduce an important notion of τ-functions. The τ-function can be consistently defined in the same way as the case of the TL hierarchy [13]:

$$\begin{aligned}
\Psi(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda) &= \frac{\tau(s, \boldsymbol{x} - [\lambda^{-1}], \boldsymbol{y}, \boldsymbol{z})}{\tau(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})} \lambda^s e^{\xi(\boldsymbol{x},\lambda) + \xi(\boldsymbol{y},\lambda^{-1})}, \\
\bar{\Psi}(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda) &= \frac{\tau(s+1, \boldsymbol{x}, \boldsymbol{y} - [\lambda], \boldsymbol{z})}{\tau(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})} \lambda^s e^{\xi(\boldsymbol{x},\lambda) + \xi(\boldsymbol{y},\lambda^{-1})}, \\
\Psi^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda) &= \frac{\tau(s, \boldsymbol{x} + [\lambda^{-1}], \boldsymbol{y}, \boldsymbol{z})}{\tau(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})} \lambda^{-s} e^{-\xi(\boldsymbol{x},\lambda) - \xi(\boldsymbol{y},\lambda^{-1})}, \\
\bar{\Psi}^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda) &= \frac{\tau(s-1, \boldsymbol{x}, \boldsymbol{y} + [\lambda], \boldsymbol{z})}{\tau(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})} \lambda^{-s} e^{-\xi(\boldsymbol{x},\lambda) - \xi(\boldsymbol{y},\lambda^{-1})},
\end{aligned} \tag{3.8}$$

where $[\lambda]$ stands for $[\lambda] = (\lambda, \lambda^2/2, \lambda^3/3, \dots)$. Note that this definition of the τ-function is the same as that of the "modified" τ-function τ' in [13]. From the reduction condition (2.13), one easily see that derivatives of the logarithm of $\tau = \tau(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ with respect to $\bar{\boldsymbol{t}}$ are independent of s, $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\partial_{\bar{t}_n} \log \tau = c_n, \quad n \geq 1.$$

If we set $\hat{\tau} = \tau \exp(-\sum_{n \geq 1} c_n \bar{t}_n)$, then $\hat{\tau}$ also gives a τ-function and satisfies

$$\partial_{\bar{t}_n} \hat{\tau} = 0, \quad n \geq 1. \tag{3.9}$$

Hence the τ-function can always be re-taken so that (3.9) is fulfilled. We may thus assume, without loss of generality, that the τ-function is independent of $\bar{\boldsymbol{t}}$. (With regards such "arbitrariness" of the τ-function, see [13].)

If we substitute the expressions (3.8) into (3.3), then the outcome is

$$\begin{aligned}
&\oint F(\lambda)^{-1} (\lambda + \lambda^{-1})^k \lambda^{s'-s} e^{\xi(\boldsymbol{t}'-\boldsymbol{t},\lambda) - \xi(\boldsymbol{t}'-\boldsymbol{t},\lambda^{-1})} \\
&\quad \times \tau(s', \boldsymbol{t}' - [\lambda^{-1}]/2, \boldsymbol{z} + \boldsymbol{b}_\lambda)\, \tau(s, \boldsymbol{t} + [\lambda^{-1}]/2, \boldsymbol{z} + \boldsymbol{c}_\lambda) \frac{d\lambda}{2\pi i} \\
&= \oint F(\lambda)^{-1} (\lambda + \lambda^{-1})^k \lambda^{s'-s} e^{\xi(\boldsymbol{t}'-\boldsymbol{t},\lambda) - \xi(\boldsymbol{t}'-\boldsymbol{t},\lambda^{-1})} \\
&\quad \times \tau(s'+1, \boldsymbol{t}' + [\lambda]/2, \boldsymbol{z} + \boldsymbol{b}_\lambda)\, \tau(s-1, \boldsymbol{t} - [\lambda]/2, \boldsymbol{z} + \boldsymbol{c}_\lambda) \frac{d\lambda}{2\pi i},
\end{aligned} \tag{3.10}$$

where the factor $F(\lambda)$ is given as follows:

$$F(\lambda) = e^{-\xi(\bar{\boldsymbol{t}}'-\bar{\boldsymbol{t}},\lambda)-\xi(\bar{\boldsymbol{t}}'-\bar{\boldsymbol{t}},\lambda^{-1})}\tau(s,\boldsymbol{t},\boldsymbol{z}+\boldsymbol{b}_\lambda)\tau(s,\boldsymbol{t},\boldsymbol{z}+\boldsymbol{c}_\lambda)$$

which is a power series of $\lambda+\lambda^{-1}$ by the Taylor expansion. Inside both integrals of (3.10), the terms $F(\lambda)^{-1}$ can actually be dropped out. Indeed, according to Theorem 1, one can insert any power series of $\lambda+\lambda^{-1}$ in both integrals of the bilinear identity. If we insert a series $F(\lambda)$ in the integrals of (3.10), then the terms $F(\lambda)^{-1}$ can be cancelled out.

As a consequence, we obtain the following bilinear identity.

Theorem 2. *For any $k \geq 0$, the τ-function satisfies the bilinear identity*

$$\begin{aligned} &\oint (\lambda+\lambda^{-1})^k \lambda^{s'-s} e^{\xi(\boldsymbol{t}'-\boldsymbol{t},\lambda)-\xi(\boldsymbol{t}'-\boldsymbol{t},\lambda^{-1})} \\ &\quad \times \tau(s',\boldsymbol{t}'-[\lambda^{-1}]/2,\boldsymbol{z}+\boldsymbol{b}_\lambda)\,\tau(s,\boldsymbol{t}+[\lambda^{-1}]/2,\boldsymbol{z}+\boldsymbol{c}_\lambda)\,\frac{d\lambda}{2\pi i} \\ &= \oint (\lambda+\lambda^{-1})^k \lambda^{s'-s} e^{\xi(\boldsymbol{t}'-\boldsymbol{t},\lambda)-\xi(\boldsymbol{t}'-\boldsymbol{t},\lambda^{-1})} \\ &\quad \times \tau(s'+1,\boldsymbol{t}'+[\lambda]/2,\boldsymbol{z}+\boldsymbol{b}_\lambda)\,\tau(s-1,\boldsymbol{t}-[\lambda]/2,\boldsymbol{z}+\boldsymbol{c}_\lambda)\,\frac{d\lambda}{2\pi i}. \end{aligned} \tag{3.11}$$

We can further rewrite (3.11) into a generating function of an infinite number of Hirota bilinear equations by applying the standard technique (see for example [5]). The simplest examples among them are

$$D_{t_1}^2\tau(s)\cdot\tau(s) - 8\tau(s+1)\tau(s-1) + 8\tau(s)^2 = 0, \tag{3.12}$$

$$(D_{t_1}^2 - 2D_{t_2})\tau(s-1)\cdot\tau(s) = 0, \tag{3.13}$$

$$(D_{t_1}D_{z_0} + 2D_{z_1})\tau(s-1)\cdot\tau(s) = 0. \tag{3.14}$$

Eq.(3.12) is known as a bilinear form of the Toda lattice equation. As for others, (3.12) and (3.13) give rise to the nonlinear Schrödinger (NLS) equation, while (3.12) and (3.14) give the (2+1)-dimensional NLS equation (see [8, 9, 11, 12]).

4 Special solutions of the (2+1)-D TL hierarchy

In this section, we construct special solutions of the (2+1)-D TL hierarchy. Our method for the construction is an application of Date's construction [4] (see also [5, 8, 13]).

4.1 Construction of special solutions

Let $\Xi_N(s)$ be an $N\times N$-matrix defined by

$$\Xi_N(s) = (\boldsymbol{f}(s), \boldsymbol{f}(s+1), \ldots, \boldsymbol{f}(s+N-1))$$

where $\boldsymbol{f}(s)$ denotes a column vector ${}^{\mathrm{t}}(f_1(s), f_2(s), \ldots, f_N(s))$ and is assumed to be subject to the following "dispersion relations":

$$\begin{aligned} &\partial_{x_n}\boldsymbol{f}(s) = \boldsymbol{f}(s+n), \quad \partial_{y_n}\boldsymbol{f}(s) = \boldsymbol{f}(s-n), \\ &\partial_{\bar{t}_n}\boldsymbol{f}(s) = C_n\boldsymbol{f}(s), \quad \partial_{z_n}\boldsymbol{f}(s) = (e^{n\partial_s}+e^{-n\partial_s})\partial_{z_0}\boldsymbol{f}(s), \quad n\geq 1. \end{aligned} \tag{4.1}$$

Here C_n $(n \geq 1)$ are constant diagonal matrices. We furthermore assume that $\Xi_N(s)$ is nonsingular, i.e., $\det \Xi_N(s) \not\equiv 0$. Then there exists uniquely a difference operator of type $[-N, 0]$

$$V_N(s) \;=\; 1 + \sum_{n=1}^{N} v_n(s) e^{-n\partial_s} \tag{4.2}$$

such that

$$V_N(s)\, \boldsymbol{f}(s+N) = \boldsymbol{0}. \tag{4.3}$$

Indeed one can solve (4.3) by using Cramer's formula to determine each $v_n(s)$ as follows (see [13] for details):

$$v_{N-k}(s) = -\frac{|s, \ldots, s+k-1, s+N, s+k+1, \ldots, s+N-1|}{\det \Xi_N(s)} \tag{4.4}$$

for $k = 0, 1, \ldots, N-1$, where $|s, s+1, \ldots, s+N-1| \overset{\text{def}}{=} \det(\boldsymbol{f}(s), \boldsymbol{f}(s+1), \ldots, \boldsymbol{f}(s+N-1))$. We set $\bar{V}_N(s) \;=\; V_N(s) e^{N\partial_s}$, which satisfies

$$\bar{V}_N(s) \boldsymbol{f}(s) = \boldsymbol{0}. \tag{4.5}$$

The main theorem of this section is the following.

Theorem 3. *$V_N(s)$ and $\bar{V}_N(s)$ are Sato-Wilson operators of the (2+1)-D TL hierarchy.*

Proof. See Sec.4.2. ■

Moreover we have

Corollary 1. *$\tau_N(s) = \det \Xi_N(s)$ is a τ-function corresponding to $V_N(s)$ and $\bar{V}_N(s)$.*

Proof. It is easy to see that $\partial_{\bar{t}_n} \tau_N(s) = \text{constant}$ for $n \geq 1$. We have to verify

$$1 + \sum_{n=1}^{N} v_n(s) \lambda^{-n} = \frac{\tau_N(s, \boldsymbol{x} - [\lambda^{-1}], \boldsymbol{y}, \boldsymbol{z})}{\tau_N(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})}, \tag{4.6}$$

$$\lambda^N + \sum_{n=1}^{N} v_n(s) \lambda^{N-n} = \frac{\tau_N(s+1, \boldsymbol{x}, \boldsymbol{y} - [\lambda], \boldsymbol{z})}{\tau_N(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})}. \tag{4.7}$$

Using the dispersion relations (4.1), one finds

$$\boldsymbol{f}_s(\boldsymbol{x} - [\lambda^{-1}], \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{f}_s - \lambda^{-1} \boldsymbol{f}_{s+1}, \quad \boldsymbol{f}_{s+1}(\boldsymbol{x}, \boldsymbol{y} - [\lambda], \boldsymbol{z}) = \boldsymbol{f}_{s+1} - \lambda \boldsymbol{f}_s$$

where we have denoted $\boldsymbol{f}_s = \boldsymbol{f}_s(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{f}(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ for simplicity. It then follows that $\tau_N(s, \boldsymbol{x} - [\lambda^{-1}], \boldsymbol{y}, \boldsymbol{z})$ is equal to

$$\begin{aligned} &\det(\boldsymbol{f}_s - \lambda^{-1} \boldsymbol{f}_{s+1}, \boldsymbol{f}_{s+1} - \lambda^{-1} \boldsymbol{f}_{s+2}, \ldots, \boldsymbol{f}_{s+N-1} - \lambda^{-1} \boldsymbol{f}_{s+N}) \\ &= \tau_N(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) + \sum_{n=1}^{N} (-1)^{N-n+1} \, |s, \ldots, s \widehat{+ n -} 1, \ldots, s+N| \lambda^{-N+n-1}, \end{aligned}$$

where the hat denotes the absence of the corresponding factor. This formula implies (4.6) (see (4.4)). Eq.(4.7) can be verified in a similar way. ■

Let c_j and p_j $(j = 1, 2, \dots, N)$ be distinct complex numbers. We take

$$f_j(s) = c_j\, p_j^s \exp[\eta_{r_j}(p_j)] + p_j^{-s} \exp[\eta_{r'_j}(p_j^{-1})] \tag{4.8}$$

for $j = 1, 2, \dots, N$, where we have set

$$\eta_r(p) = \xi(\boldsymbol{x}, p) + \xi(\boldsymbol{y}, p^{-1}) + r z_0 + r\xi(\check{\boldsymbol{z}}, p) + r\xi(\check{\boldsymbol{z}}, p^{-1})$$

for some constant $r \in \mathbb{C}$. It is easy to see that these $f_j(s)$ indeed satisfy the condition (4.1), and $\det \Xi_N(s) \not\equiv 0$. The solution $\tau_N(s)$ for this functional data represents an N-soliton solution of the (2+1)-D TL hierarchy.

Corollary 2. *$\tau_N(s)$ for the above functional data can be expressed as*

$$\begin{aligned}\tau_N(s) = {} & \prod_{j>k} (p_j^{-1} - p_k^{-1}) \prod_{j=1}^{N} p_j^{-s} \exp\left[\eta_{r'_j}(p_j^{-1})\right] \\ & \times \sum_{J \subset I} \left\{ \prod_{\substack{j, j' \in J \\ j < j'}} \frac{(p_j - p_{j'})^2}{(p_j p_{j'} - 1)^2} \prod_{j \in J} \frac{c_j\, p_j^{2s}}{p_j^2 - 1} \exp\left[\eta_{r_j}(p_j) - \eta_{r'_j}(p_j^{-1})\right] \right\}\end{aligned} \tag{4.9}$$

where the summation is taken over any subset J of $I = \{1, 2, \cdots, N\}$.

4.2 Verification of Theorem 3

This subsection is devoted to the proof of Theorem 3. We first recall the following lemma, which plays an important role in the proof of the theorem.

Lemma 3 ([13]). *For $l \geq N$, we have the following.*

1j Let A be a difference operator of type $[\,0,\, l\,]$. Then A is uniquely expressed as $A = B\bar{V}_N + R$ for some difference operators B of type $[\,0,\, l - N]$ and R of type $[\,0,\, N-1]$.

2j Let $\bar{A}$ be a difference operator of type $[\,-l,\, 0\,]$. Then $\bar{A}$ is uniquely expressed as $\bar{A} = \bar{B} V_N + \bar{R}$ for some difference operators $\bar{B}$ of type $[\,-l + N,\, 0\,]$ and R of type $[\,-N + 1,\, 0\,]$.

Proof of Theorem 3. For the proof, it will be helpful to divide it in three steps.

1) First of all, we differentiate (4.5) with respect to x_n and obtain

$$\left[\partial_{x_n} \bar{V}_N(s) + \bar{V}_N(s) e^{n\partial_s}\right] \boldsymbol{f}(s) = \boldsymbol{0}.$$

Since the difference operator in parentheses is of $[\,0, n + N\,]$-type, we can use Lemma 3 to express it as

$$\partial_{x_n} \bar{V}_N(s) + \bar{V}_N(s) e^{n\partial_s} \;=\; B_n \bar{V}_N(s) + R_n$$

for some difference operators B_n and R_n. Combining this formula with the above equation, we have $R_n \boldsymbol{f}(s) = \boldsymbol{0}$, which implies $R_n = 0$ because R_n is a difference operator of $[0, N-1]$-type and $\det \Xi_N(s) \not\equiv 0$. We can thereby write B_n as

$$\begin{aligned} B_n &= \partial_{x_n} \bar{V}_N(s) \cdot \bar{V}_N(s)^{-1} + \bar{V}_N(s) e^{n\partial_s} \bar{V}_N(s)^{-1} \\ &= \partial_{x_n} V_N(s) \cdot V_N(s)^{-1} + V_N(s) e^{n\partial_s} V_N(s)^{-1}. \end{aligned}$$

Taking a positive part $(\cdot)_{\geq 0}$ of this equation, we obtain $B_n = [V_N(s)e^{n\partial_s}V_N(s)^{-1}]_{\geq 0}$.

By the same argument as above, we can also show that there exists a difference operator $\bar{B}_n$ such that

$$\bar{B}_n = \partial_{y_n}\widetilde{V}_N(s)\cdot\widetilde{V}_N(s)^{-1} + \widetilde{V}_N(s)e^{-n\partial_s}\widetilde{V}_N(s)^{-1}, \quad \widetilde{V}_N(s) = V_N(s) \ \text{ or } \ \bar{V}_N(s),$$

and $\bar{B}_n = [\bar{V}_N(s)e^{-n\partial_s}\bar{V}_N(s)^{-1}]_{<0}$. Hence $V_N(s)$ and $\bar{V}_N(s)$ satisfy the Sato equations of the TL hierarchy.

2) Our next task is to show that $V_N(s)$ and $\bar{V}_N(s)$ satisfy the reduction condition (2.13). To this end, we differentiate (4.5) with respect to $\bar{t}_n$ and obtain

$$\mathbf{0} = \partial_{\bar{t}_n}\bar{V}_N(s)\cdot\boldsymbol{f}(s) + C_n\bar{V}_N(s)\boldsymbol{f}(s) = \partial_{\bar{t}_n}\bar{V}_N(s)\cdot\boldsymbol{f}(s).$$

By the same reason as above, this equation implies $\partial_{\bar{t}_n}\bar{V}_N(s) = \partial_{\bar{t}_n}V_N(s) = 0$, and consequently $V_N(s)$ and $\bar{V}_N(s)$ are independent of $\bar{\boldsymbol{t}}$, as required.

3) The last step to complete the proof is to show that $V_N(s)$ and $\bar{V}_N(s)$ satisfy the Sato equations with $\check{\boldsymbol{z}}$-flows. We differentiate (4.5) with respect to z_n and obtain the equation

$$\partial_{z_n}\bar{V}_N(s)\cdot\boldsymbol{f}(s) + \bar{V}_N(s)\left(e^{n\partial_s} + e^{-n\partial_s}\right)\partial_{z_0}\boldsymbol{f}(s) = \mathbf{0}.$$

Noticing that $B_n+\bar{B}_n = \bar{V}_N(s)(e^{n\partial_s}+e^{-n\partial_s})\bar{V}_N(s)^{-1}$ and $\bar{V}_N(s)\partial_{z_0}\boldsymbol{f}(s) = -\partial_{z_0}\bar{V}_N(s)\cdot\boldsymbol{f}(s)$, we can rewrite the above equation into

$$\left[\partial_{z_n}\bar{V}_N(s) - B_n\partial_{z_0}\bar{V}_N(s)\right]\boldsymbol{f}(s) - \bar{B}_n\partial_{z_0}\bar{V}_N(s)\cdot\boldsymbol{f}(s) = \mathbf{0}. \tag{4.10}$$

Since the difference operator appearing in the first term is of $[\,0,\,N+n-1\,]$-type, using Lemma 3, we can express it as

$$\partial_{z_n}\bar{V}_N(s) - B_n\partial_{z_0}\bar{V}_N(s) = K_n\bar{V}_N(s) + r_n \tag{4.11}$$

for some difference operators K_n and r_n. Plugging this formula into (4.10) gives

$$\left[r_n e^{-N\partial_s} - \bar{B}_n\partial_{z_0}V_N(s)\right]\boldsymbol{f}(s+N) = \mathbf{0}. \tag{4.12}$$

Note that the difference operator appearing in this formula is of $[\,-N-n,\,-1\,]$-type. So, using Lemma 3 again, we can express it as

$$r_n e^{-N\partial_s} - \bar{B}_n\partial_{z_0}V_N(s) = \bar{K}_nV_N(s) + r'_n \tag{4.13}$$

for some difference operators $\bar{K}_n$ and r'_n. From (4.12) and (4.13), it follows that $r'_n\boldsymbol{f}(s+N) = \mathbf{0}$, which implies $r'_n = 0$, so that (4.13) turns to

$$r_n = \bar{K}_n\bar{V}_N(s) + \bar{B}_n\partial_{z_0}\bar{V}_N(s).$$

Eliminate r_n in (4.11) by using this expression, then we obtain

$$K_n + \bar{K}_n = \partial_{z_n}\widetilde{V}_N(s)\cdot\widetilde{V}_N(s)^{-1} - (B_n + \bar{B}_n)\,\partial_{z_0}\widetilde{V}_N(s)\cdot\widetilde{V}_N(s)^{-1},$$

which is identical to the Sato equation with respect to z_n-flow, since taking the positive and negative parts of this equation yields the expressions of K_n and $\bar{K}_n$ as in (2.16). Hence the proof is completed. ■

5 Another formulation of the (2+1)-D TL hierarchy

In this section, we give another formulation of the (2+1)-D TL hierarchy. We obtain the bilinear identity slightly different from that of Theorem 2, which enables us to relate our system to the $\mathfrak{sl}_2^{\mathrm{tor}}$-hierarchy of [8].

5.1 Another reduction to the 1-D TL hierarchy

We start with usual Sato-Wilson operators of the TL hierarchy [13]:

$$W = V e^{\xi(\boldsymbol{y}, e^{-\partial_s})}, \quad \bar{W} = \bar{V} e^{\xi(\boldsymbol{x}, e^{\partial_s})}. \tag{5.1}$$

We impose the following equivalent constraints on the TL hierarchy:

$$L = \bar{L}, \tag{5.2}$$
$$\partial_{\bar{t}_n} W = \partial_{\bar{t}_n} \bar{W} = 0, \quad n \geq 1. \tag{5.3}$$

The Lax operator L now takes the form

$$L = e^{\partial_s} + u_1(s, \boldsymbol{t}) + \bar{u}_0(s, \boldsymbol{t}) e^{-\partial_s},$$

and satisfies Lax equations

$$\partial_{t_n} L = [B_n - \bar{B}_n, L], \quad n \geq 1. \tag{5.4}$$

These Lax equations are identical to the Lax equations of the 1-D TL hierarchy discussed in Remark 1, by the correspondence of the Lax operators $L \leftrightarrow \mathcal{L}$ and the time variables $t_n \leftrightarrow T_n$ $(n \geq 1)$. Note that the Sato equations for $W, \bar{W}$ have the following expressions:

$$\partial_{t_n} W = (B_n - \bar{B}_n) W - W e^{n\partial_s}, \quad \partial_{t_n} \bar{W} = (B_n - \bar{B}_n) \bar{W} + \bar{W} e^{-n\partial_s}. \tag{5.5}$$

5.2 The (2+1)-D TL hierarchy

The toroidal-like part of evolution equations is defined in the same way as in the previous case:

$$\partial_{z_n} \widetilde{W} = (B_n + \bar{B}_n)\, \partial_{z_0} \widetilde{W} + (K_n + \bar{K}_n) \widetilde{W}, \quad n \geq 1, \tag{5.6}$$

where $\widetilde{W} = W$ or $\bar{W}$, and

$$K_n = -\left(B_n\, \partial_{z_0} W \cdot W^{-1}\right)_{\geq 0}, \quad \bar{K}_n = -\left(\bar{B}_n\, \partial_{z_0} \bar{W} \cdot \bar{W}^{-1}\right)_{<0}. \tag{5.7}$$

If we set

$$M = \partial_{z_0} W \cdot W^{-1}, \quad \bar{M} = \partial_{z_0} \bar{W} \cdot \bar{W}^{-1}, \tag{5.8}$$

then K_n and $\bar{K}_n$ have the expressions

$$K_n = -(L^n M)_{\geq 0}, \quad \bar{K}_n = -(L^n \bar{M})_{<0}. \tag{5.9}$$

Lax equations of this system can be derived in the same way as in Sec.2.3.2,

$$
\begin{aligned}
&\partial_{t_n} L = [B_n - \bar{B}_n, L], \\
&\partial_{z_n} L = \left[(B_n + \bar{B}_n)\partial_{z_0} + K_n + \bar{K}_n,\ L\right], \\
&\partial_{t_n} \widetilde{M} = \left[B_n - \bar{B}_n,\ \widetilde{M} - \partial_{z_0}\right], \\
&\partial_{z_n} \widetilde{M} = \left[(B_n + \bar{B}_n)\widetilde{M} + K_n + \bar{K}_n,\ \widetilde{M} - \partial_{z_0}\right], \quad \widetilde{M} = M, \bar{M}, \quad n \geq 1,
\end{aligned}
\tag{5.10}
$$

which coincide with the Lax equations discussed in Remark 2 by the correspondence of the Lax operators $(L, M, \bar{M}) \leftrightarrow (\mathcal{L}, \mathcal{M}, \bar{\mathcal{M}})$ and the time variables $(t_n, z_n) \leftrightarrow (T_n, Z_n)$ $(n \geq 1)$. In this sense, the system just considered is equivalent to the (2+1)-D TL hierarchy previously discussed.

5.3 Bilinear identity

In the present setting of the formulation of the (2+1)-D TL hierarchy, auxiliary linear equations take the different form to the previous case. Namely spectral equations for the Lax operator

$$
L\Psi(s,\lambda) = \lambda\Psi(s,\lambda), \quad L\bar{\Psi}(s,\lambda) = \lambda^{-1}\bar{\Psi}(s,\lambda), \tag{5.11}
$$

and evolution equations for the Baker-Akhiezer functions

$$
(\partial_{z_n} - \lambda^n \partial_{z_0})\Psi(s,\lambda) = \left[K_n + \bar{K}_n - \partial_{z_0}(B_n + \bar{B}_n)\right]\Psi(s,\lambda), \tag{5.12}
$$

$$
(\partial_{z_n} - \lambda^{-n} \partial_{z_0})\bar{\Psi}(s,\lambda) = \left[K_n + \bar{K}_n - \partial_{z_0}(B_n + \bar{B}_n)\right]\bar{\Psi}(s,\lambda), \tag{5.13}
$$

$$
(\partial_{z_n} - \lambda^n \partial_{z_0})\Psi^*(s+1,\lambda) = -(K_n^* + \bar{K}_n^*)\Psi^*(s+1,\lambda), \tag{5.14}
$$

$$
(\partial_{z_n} - \lambda^{-n} \partial_{z_0})\bar{\Psi}^*(s+1,\lambda) = -(K_n^* + \bar{K}_n^*)\bar{\Psi}^*(s+1,\lambda). \tag{5.15}
$$

These linear equations give rise to the following bilinearlization of the (2+1)-D TL hierarchy:

$$
\begin{aligned}
&\oint \lambda^k \Psi(s', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z} + \boldsymbol{b}_\lambda^+)\Psi^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} + \boldsymbol{c}_\lambda^+)\,\frac{d\lambda}{2\pi i} \\
&= \oint \lambda^{-k} \bar{\Psi}(s', \boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z} + \boldsymbol{b}_\lambda^-)\bar{\Psi}^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} + \boldsymbol{c}_\lambda^-)\,\frac{d\lambda}{2\pi i}, \quad k \geq 0,
\end{aligned}
\tag{5.16}
$$

where we have set $\boldsymbol{b}_\lambda^\pm = (-\xi(\boldsymbol{b}, \lambda^{\pm 1}), b_1, b_2, \dots)$ and similarly for $\boldsymbol{c}_\lambda^\pm$.

5.4 Relation to the (2+1)-D NLS hierarchy

The τ-function can be defined by the same formulas as in (3.8). But we use the following "$\widetilde{\tau}$-function":

$$
\widetilde{\tau}(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \exp\left(-\sum_{n=1}^{\infty} n x_n y_n\right)\tau(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}). \tag{5.17}
$$

The reason for this is that the $\widetilde{\tau}$-function can be chosen so that $\partial_{\bar{t}_n}\widetilde{\tau}(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = 0$ $(n \geq 1)$ from the reduction condition (5.3), while the ordinary τ-function does not so. We can verify that for any $k \geq 0$, the $\widetilde{\tau}$-function satisfies the following bilinear identity:

$$\oint \lambda^{k+s'-s} e^{\xi(\boldsymbol{t}'-\boldsymbol{t},\lambda)} \widetilde{\tau}(s', \boldsymbol{t}' - [\lambda^{-1}]/2, \boldsymbol{z} + \boldsymbol{b}_\lambda^+)\, \widetilde{\tau}(s, \boldsymbol{t} + [\lambda^{-1}]/2, \boldsymbol{z} + \boldsymbol{c}_\lambda^+)\, \frac{d\lambda}{2\pi i}$$
$$= \oint \lambda^{-k+s'-s} e^{-\xi(\boldsymbol{t}'-\boldsymbol{t},\lambda^{-1})} \widetilde{\tau}(s'+1, \boldsymbol{t}' + [\lambda]/2, \boldsymbol{z} + \boldsymbol{b}_\lambda^-)\, \widetilde{\tau}(s-1, \boldsymbol{t} - [\lambda]/2, \boldsymbol{z} + \boldsymbol{c}_\lambda^-)\, \frac{d\lambda}{2\pi i}. \tag{5.18}$$

This bilinear identity leads to a remarkable link to the $\mathfrak{sl}_2^{\mathrm{tor}}$-hierarchy of [8]. If we compare (5.18) with that of Proposition 5 of [8], then the both are identified with each other by the following correspondences of τ-functions and independent variables.

Theorem 4. *The following formula gives a correspondence between our $\widetilde{\tau}$-function and that of the $\mathfrak{sl}_2^{\mathrm{tor}}$-hierarchy in a homogeneous picture [8]:*

$$\widetilde{\tau}(s, \boldsymbol{t}, \boldsymbol{z}) = (-1)^{s(s-1)/2} \tau_{0,0}^{s,-s}(\boldsymbol{x}', \boldsymbol{y}'), \quad 2\boldsymbol{t} = \boldsymbol{x}', \quad \boldsymbol{z} = \boldsymbol{y}'. \tag{5.19}$$

In particular, if we set $F = \widetilde{\tau}(0, \boldsymbol{t}, \boldsymbol{z})$, $G = \widetilde{\tau}(1, \boldsymbol{t}, \boldsymbol{z})$ and $\widetilde{G} = \tau(-1, \boldsymbol{t}, \boldsymbol{z})$, then $(F, G, \widetilde{G})$ gives τ-functions of the (2+1)-D NLS hierarchy.

6 Concluding Remarks

We have introduced two versions of the (2+1)-D TL hierarchy. In each version, the hierarchy has been introduced by adding toroidal-like evolution equations to the Sato equations of the 1-D TL hierarchy. The difference has been originated from two different reductions to the 1-D TL hierarchy from the TL hierarchy; see Sec.2.2 and 5.1.

For the first hierarchy discussed in Sec.2–4, we have obtained the bilinear identity for the Baker-Akhiezer functions (Theorem 1). This has led to the bilinear identity for the τ-functions (Theorem 2), which produces infinitely many Hirota bilinear equations, including the bilinear forms of the (2+1)-D NLS equation ((3.12), (3.14)). By the Wronskian technique, we have obtained the explicit solutions that include N-soliton solutions (Theorem 3). The solutions constructed here will provide Wronskian-type solutions of the (2+1)-D NLS equation, which are substantially different from "double" Wronskian-type solutions of [8].

For the second hierarchy discussed in Sec.5, we have obtained the bilinear identity for the τ-functions (5.18). This bilinear identity has been identified with that of the $\mathfrak{sl}_2^{\mathrm{tor}}$-hierarchy of [8] (Theorem 4). We should remark that the second hierarchy is equivalent to the first hierarchy in the sense that two Lax equations (2.22) and (5.10) coincide (under the change of the time variables). However, it is *not* clear that the bilinear identity (5.18) for the second hierarchy can be transformed/identical to the first one (3.11) by suitable transformations. This is a non-trivial and important problem to be solved in the future. Moreover, the Wronskian technique that we have taken for the first hierarchy, does not work for the second hierarchy (precisely for the Sato equations (5.5) and (5.6)). It is also a future problem to construct explicit solutions for the second hierarchy.

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A Residue formula for difference operators

The bilinear identity for the KP hierarchy was derived by using a residue formula for the product of two pseudo-differential operators (see for example [5]). Here we derive an analogous formula for difference operators, and prove Lemma 2.

Lemma 4. *Let A and B be difference operators such that the product BA^* is algebraically well-defined. Then the following formula holds:*

$$(BA^*)_k = \oint (\, e^{k\partial_s} A\lambda^s) \cdot (B\lambda^{-s})\, \frac{d\lambda}{2\pi i\lambda}. \tag{A.1}$$

Proof. We prove the lemma in the case of $A = \sum_{n=0}^{\infty} a_n(s) e^{(N-n)\partial_s}$ and $B = \sum_{m=0}^{\infty} b_m(s) e^{(M+m)\partial_s}$. For these, the right hand side of (A.1) can be calculated as

$$\oint (e^{k\partial_s} A\,\lambda^s\,) \cdot (B\lambda^{-s})\, \frac{d\lambda}{2\pi i\lambda} = \oint \sum_{l=0}^{\infty} \sum_{\substack{n+m=l\\ n,\,m\geq 0}} a_n(s+k)\, b_m(s)\, \lambda^{k-l+N-M}\, \frac{d\lambda}{2\pi i\lambda}$$

$$= \sum_{\substack{n+m=k+N-M\\ n,\,m\geq 0}} a_n(s+k)\, b_m(s).$$

As for the left hand side, we have

$$\begin{aligned} BA^* &= \sum_{m=0}^{\infty} b_m(s)\, e^{(M+m)\partial_s} \sum_{n=0}^{\infty} e^{(-N+n)\partial_s} \circ a_n(s) \\ &= \sum_{l=0}^{\infty} \sum_{\substack{n+m=l\\ n,\,m\geq 0}} a_n(s+l-N+M)\, b_m(s)\, e^{(l-N+M)\,\partial_s}, \end{aligned}$$

from which it follows that

$$(BA^*)_k = \sum_{\substack{n+m=k+N-M\\ n,\,m\geq 0}} a_n(s+k)\, b_m(s).$$

Hence we have proved the lemma. ■

We now proceed to verify Lemma 2.

Proof of Lemma 2. Applying Lemma 4 to the case where $(A, B) = (W, (W^*)^{-1})$ and $(\bar{W}, (\bar{W}^*)^{-1})$ gives the equation

$$\begin{aligned} &\oint \Psi(s', \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda)\Psi^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda)\, \frac{d\lambda}{2\pi i} \\ &= \oint \bar{\Psi}(s', \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda)\bar{\Psi}^*(s, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \lambda)\, \frac{d\lambda}{2\pi i} = \delta_{s',\,s-1}. \end{aligned} \tag{A.2}$$

Iteration of the linear equations for the Baker-Akhiezer functions with respect to $\boldsymbol{x}$, $\boldsymbol{y}$-flows gives rise to higher order evolution equations of the form

$$\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \widetilde{\Psi}(s,\lambda) = B_{\alpha_1,\alpha_2,\ldots} \widetilde{\Psi}(s,\lambda), \quad \partial_{y_1}^{\alpha_1} \partial_{y_2}^{\alpha_2} \cdots \widetilde{\Psi}(s,\lambda) = \bar{B}_{\alpha_1,\alpha_2,\ldots} \widetilde{\Psi}(s,\lambda),$$

where $\alpha_1, \alpha_2, \dots \geq 0$, and $B_{\alpha_1,\alpha_2,\dots}$, $\bar{B}_{\alpha_1,\alpha_2,\dots}$ are some difference operators. Multiplying $B_{\alpha_1,\alpha_2,\dots}\bar{B}_{\beta_1,\beta_2,\dots}$ on both sides of (A.2) as a difference operator with respect to s', we obtain

$$\oint \prod_{n,m\geq 1} \partial_{x_n}^{\alpha_n}\partial_{y_m}^{\beta_m}\Psi(s',\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\lambda)\cdot\Psi^*(s,\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\lambda)\frac{d\lambda}{2\pi i}$$
$$= \oint \prod_{n,m\geq 1} \partial_{x_n}^{\alpha_n}\partial_{y_m}^{\beta_m}\bar{\Psi}(s',\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\lambda)\cdot\bar{\Psi}^*(s,\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\lambda)\frac{d\lambda}{2\pi i},$$

which is equivalent to the desirous formula. ■

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