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Fourth-order recursion operators for third-order evolution equations

Marianna EULER

Department of Mathematics, Luleå University of Technology
SE-971 87 Luleå, Sweden

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Abstract We report the recursion operators for a class of symmetry integrable evolution equations of third order which admit fourth-order recursion operators. Under the given assumptions we obtain the complete list of equations, one of which is the well-known Krichever-Novikov equation.

We consider the third-order evolution equation

$$u_t = u_{xxx} + F(u, u_x, u_{xx}) \tag{1}$$

and require that (1) admits a recursion operator of the form

$$R[u] = D_x^4 + G_3 D_x^3 + G_2 D_x^2 + G_1 D_x + G_0 + I_1 D_x^{-1} \circ J_1 + I_2 D_x^{-1} \circ J_2, \tag{2}$$

where $G_j = G_j(u, u_x, u_{xx}, \ldots)$, I_i are Lie point symmetries and J_i integrating factors for (1), with

$$\frac{\partial J_i}{\partial u_{6x}} \neq 0 \tag{3}$$

for J_1 and/or J_2 such that

$$J_i = \hat{E}_u \Phi_i^t.$$

Here $\hat{E}_u$ is the Euler operator

$$\hat{E}_u = \frac{\partial}{\partial u} - D_x \circ \frac{\partial}{\partial u_x} + D_x^2 \circ \frac{\partial}{\partial u_{2x}} - \cdots$$

and Φ^t a conserved density for the evolution equation. We recall that, if (1) admits a recursion operator and an infinite hierarchy of conservation laws,

$$D_t \Phi_i^t + D_x \Phi_i^x = 0,$$

then (1) is said to be symmetry integrable [3].

Proposition: *Equations of the form (1) which admit recursion operators of the form (2) under condition (3) are exhausted by the following two cases:*

1. *The equation*

$$u_t = u_{3x} - \frac{3}{2}u_x^{-1}u_{2x}^2 + P(u)u_x^{-1} + Q(u)u_x^3, \tag{4}$$

where P and Q satisfy the relation

$$P^{(5)} + 10P^{(3)}Q + 15P''Q' + P'\left(9Q'' + 16Q^2\right) + 2P\left(Q^{(3)} + 8Q'Q\right) = 0 \tag{5}$$

2. *The equation*

$$u_t = u_{3x} - \frac{3}{2}\left(\frac{u_x}{u_x^2 - c}\right)u_{2x}^2 - \frac{3}{2c}\left(u_x^2 - c\right)u_x P(u), \tag{6}$$

where c is an arbitrary but nonzero constant and P satisfies the equation

$$\left(P'\right)^2 = \frac{4}{c}P^3 + a_1 P + a_2 \tag{7}$$

with a_1 and a_2 arbitrary constants.

Remarks:

1. *For $Q = 0$, equation (4) reduces to the well-known Krichever-Novikov equation [4]. The condition on P then reduces to*

$$P^{(5)} = 0. \tag{8}$$

A fourth-order recursion operator for the Krichever-Novikov equation was reported in [7]. Note further that (4) can be obtained from the Krichever-Novikov equation by a change of variables of the form $u \mapsto G(u)$.

2. *The equation (6) with condition (7) and $c = -1$ has been reported in ([5]). As far as we know, no recursion operator has been reported before for this equation.*

3. *We stress that in our current classification we require the equation of the form (1) to admit a recursion operator of order four and an integrating factor of order six. If these conditions are relaxed, a much larger class of equations emerges such as linearizable equations (which have first-order recursion operators and zero-order integrating factors [2]) and the class of semilinear third-order evolution equations which includes the the well-known Korteweg-de Vries equation (with second order recursion operators and second-order integrating factors [6]), as well as the three equations reported in [1] which admit second-order recursion operators with fourth-order integrating factors.*

4. Equations (4) and (6) do not admit second-order recursion operators of the form

$$R[u] = D_x^2 + G_1 D_x + G_0 + I_1 D_x^{-1} \circ J_1 + I_2 D_x^{-1} \circ J_2 \tag{9}$$

for any order of J_1 and J_2. The equations do, however, admit local Lie-Bäcklund symmetries of order five, seven, nine etc., which indicates that there exists a second-order nonlocal recursion operator.

The coefficients G_{1j}, integrating factors J_{1j} and symmetries I_{1j} of the recursion operator

$$R_1[u] = D_x^4 + G_{13} D_x^3 + G_{12} D_x^2 + G_{11} D_x + G_{10} + I_{11} D_x^{-1} \circ J_{11} + I_{12} D_x^{-1} \circ J_{12} \tag{10}$$

for equation (4) under condition (5) take the form

$$G_{13} = -4\frac{u_{2x}}{u_x} \tag{11a}$$

$$G_{12} = -2\frac{u_{3x}}{u_x} + 6\frac{u_{2x}^2}{u_x^2} + 4Qu_x^2 - \frac{4}{3}\frac{P}{u_x^2} \tag{11b}$$

$$G_{11} = -2\frac{u_{4x}}{u_x} + 8\frac{u_{3x}u_{2x}}{u_x^2} - 6\frac{u_{2x}^3}{u_x^3} + 4P\frac{u_{2x}}{u_x^3} - 4Qu_x u_{2x} + 2Q' u_x^3 - \frac{2}{3}\frac{P'}{u_x} \tag{11c}$$

$$G_{10} = \frac{u_{5x}}{u_x} - 4\frac{u_{4x}u_{2x}}{u_x^2} - 2\frac{u_{3x}^2}{u_x^2} + \left(8\frac{u_{2x}^2}{u_x^3} + 8Qu_x\right) u_{3x} - 3\frac{u_{2x}^4}{u_x^4} - \frac{4}{3}\left(3Qu_x^4 - P\right)\frac{u_{2x}^2}{u_x^4}$$
$$+8\left(Q'u_x^2 - \frac{1}{3}\frac{P'}{u_x^2}\right) u_{2x} + 4Q^2 u_x^4 + 2Q'' u_x^4 + \frac{4}{9}\frac{P^2}{u_x^4} + \frac{8}{9}PQ + \frac{10}{9}P'' \tag{11d}$$

$$J_{11} = \frac{u_{6x}}{u_x^2} - 6\frac{u_{2x}u_{5x}}{u_x^3} + \left(-10\frac{u_{3x}}{u_x^3} + \frac{45}{2}\frac{u_{xx}^2}{u_x^4} + 5Q - \frac{5}{3}\frac{P}{u_x^4}\right) u_{4x} + 30\frac{u_{3x}^2 u_{2x}}{u_x^4}$$
$$+\left(-60\frac{u_{2x}^3}{u_x^5} + \frac{40}{3}\frac{P u_{2x}}{u_x^5} + 10Q' u_x - \frac{10}{3}\frac{P'}{u_x^3}\right) u_{3x} + \left(\frac{9}{4}u_{2x}^5 - \frac{5}{3}P u_{2x}^3\right)\frac{10}{u_x^6}$$
$$+\frac{15}{2}\left(\frac{P'}{u_x^4} + Q'\right) u_{2x}^2 + \left(9Q'' u_x^2 + 6Q^2 u_x^2 - \frac{5}{3}\frac{P''}{u_x^2} + \frac{10}{9}\frac{P^2}{u_x^6}\right) u_{2x}$$
$$+\frac{5}{9}\left(P''' - PQ' - P'Q\right) - \frac{5}{9}\frac{PP'}{u_x^4} + \left(Q''' + 3Q'Q\right) u_x^4 \tag{12a}$$

$$J_{12} = \frac{u_{4x}}{u_x^2} - 4\frac{u_{2x}u_{3x}}{u_x^3} + 3\frac{u_{2x}^3}{u_x^4} + \left(2Q - \frac{2P}{u_x^4}\right) u_{2x} + Q' u_x^2 + \frac{P'}{u_x^2} \tag{12b}$$

$$I_{11} = -u_x \tag{13a}$$

$$I_{12} = -\left(u_{3x} - \frac{3}{2}u_x^{-1} u_{2x}^2 + P(u) u_x^{-1} + Q(u) u_x^3\right). \tag{13b}$$

The coefficients G_{2j}, integrating factors J_{2j} and symmetries I_{2j} of the recursion operator

$$R_2[u] = D_x^4 + G_{23}D_x^3 + G_{22}D_x^2 + G_{21}D_x + G_{20} + I_{21}D_x^{-1} \circ J_{21} + I_{22}D_x^{-1} \circ J_{22} \tag{14}$$

for equation (6) under condition (7) take the form

$$G_{23} = -4\frac{u_x u_{2x}}{u_x^2 - c} \tag{15a}$$

$$G_{22} = -2\frac{u_x u_{3x}}{u_x^2 - c} + 2\frac{(3u_x^2 + 2c)u_{2x}^2}{(u_x^2 - c)^2} - \frac{6}{c}Pu_x^2 + 2P \tag{15b}$$

$$G_{21} = -\frac{2u_x u_{4x}}{u_x^2 - c} + 4\frac{(2u_x^2 + c)u_{2x}u_{3x}}{(u_x - c)^2} - 2\frac{(3u_x^2 + 7c)u_x u_{2x}^3}{(u_x^2 - c)^3}$$
$$+\frac{2}{c}\frac{(3u_x^2 + c)u_x u_{2x}}{(u_x^2 - c)}P - 3\frac{(u_x^2 - c)u_x}{c}P' \tag{15c}$$

$$G_{20} = \frac{u_x u_{5x}}{u_x^2 - c} - 4\frac{u_x^2 u_{2x} u_{4x}}{(u_x^2 - c)^2} - \frac{(c + 2u_x^2)u_{3x}^2}{(u_x^2 - c)^2} + 4\frac{(2c - 3u_x^2)u_x u_{3x}}{c(u_x^2 - c)}P$$
$$+2\frac{(5c + 4u_x^2)u_x u_{2x}^2 u_{3x}}{(u_x^2 - c)^3} - 2\frac{(c - 2u_x^2)(c - 3u_x^2)u_{2x}}{c(u_x^2 - c)}P' - 2\frac{(c - 3u_x^2)u_x^2 u_{2x}^2}{c(u_x^2 - c)^2}P$$
$$-3\frac{(4c + u_x^2)u_x^2 u_{2x}^4}{(u_x^2 - c)^4} + \frac{(c - 3u_x^2)^2}{c^2}P^2 + \frac{(c - 3u_x^2)u_x^2}{c}P'' \tag{15d}$$

$$J_{21} = \frac{u_{6x}}{u_x^2 - c} - \frac{6u_x u_{2x} u_{5x}}{(u_x^2 - c)^2} - \left(\frac{10u_x u_{3x}}{(u_x^2 - c)^2} - \frac{23c + 45u_x^2}{2(u_x^2 - c)^3}u_{2x}^2 - \frac{7c - 15u_x^2}{2c(u_x^2 - c)}P\right)u_{4x}$$
$$+\frac{6(5u_x^2 + 3c)u_{2x}u_{3x}^2}{(u_x^2 - c)^3} + \frac{16u_x u_{2x} u_{3x}}{(u_x^2 - c)^2}P - \frac{(15u_x^2 - 7c)u_x u_{3x}}{c(u_x^2 - c)}P'$$
$$-\frac{12(9c + 5u_x^2)u_x u_{2x}^3 u_{3x}}{(u_x^2 - c)^4} - \frac{4(3u_x^2 + c)u_{2x}^3}{(u_x^2 - c)^3}P - \frac{3(-7c + 3u_x^2)(-c + 5u_x^2)u_{2x}^2}{4c(u_x^2 - c)^2}P'$$
$$+\frac{3(58cu_x^2 + 15u_x^4 + 7c^2)}{2(u_x^2 - c)^5}u_{2x}^5 + \left(\frac{c^2 + 18cu_x^2 - 27u_x^4}{2c(u_x^2 - c)}P'' - \frac{3(-9u_x^2 + c)}{2c^2}P^2\right)u_{2x}$$
$$-\frac{9u_x^4 - 2cu_x^2 + 3c^2}{6c}P''' + \frac{27u_x^4 - 34cu_x^2 + 23c^2}{4c^2}PP' \tag{16a}$$

$$J_{22} = -\frac{u_{4x}}{u_x^2 - c} + \frac{4u_x u_{2x} u_{3x}}{(u_x^2 - c)^2} - \frac{(3u_x^2 + c)u_{2x}^3}{(u_x^2 - c)^3} + \frac{3u_{2x}}{c}P + \frac{3u_x^2 + c}{2c}P' \tag{16b}$$

$$I_{21} = -u_x \tag{16c}$$

$$I_{22} = u_{3x} - \frac{3}{2}\left(\frac{u_x}{u_x^2 - c}\right)u_{2x}^2 - \frac{3}{2c}\left(u_x^2 - c\right)u_x P(u). \tag{16d}$$

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