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On bi-hamiltonian structure of some integrable systems on $so^*(4)$

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Abstract

We classify quadratic Poisson structures on $so^*(4)$ and $e^*(3)$, which have the same foliations by symplectic leaves as canonical Lie-Poisson tensors. The separated variables for some of the corresponding bi-integrable systems are constructed.

1 Introduction

Let M be a Poisson manifold endowed with a bivector P_0 fulfilling the Jacobi condition

$$[P_0, P_0] = 0$$

with respect to the Schouten bracket $[.,.]$ [15].

A bi-Hamiltonian manifold M is a smooth (or complex) manifold endowed with two compatible bivectors P_0, P_1 such that

$$[P_0, P_0] = [P_0, P_1] = [P_1, P_1] = 0. \tag{1.1}$$

Classification of compatible Poisson bivectors and the corresponding bi-integrable systems with integrals of motion

$$\{H_i, H_j\}_0 = \{H_i, H_j\}_1 = 0, \qquad i, j = 1, \ldots, n,$$

is nowadays a subject of intense research [3, 13, 14]. Of course, direct solution of the equations (1.1) is generally quite difficult. We can try to lighten this work by using properties of the given Poisson manifold (M, P_0).

Bivectors P_1 fulfilling the compatibility condition $[P_0, P_1] = 0$ are called 2-cocycles in the Poisson-Lichnerowicz cohomology defined by P_0 on M [7]. The Lie derivative of P_0 along any vector field X on M

$$P_1 = \mathcal{L}_X(P_0) \tag{1.2}$$

is 2-coboundary, more precisely it is 2-cocycle associated with the Liouville vector field X. For such bivectors P_1 the system of equations (1.1) is reduced to the single equation

$$[\mathcal{L}_X(P_0), \mathcal{L}_X(P_0)] = 0, \quad \Leftrightarrow \quad [\mathcal{L}^2_X(P_0), P_0] = 0. \tag{1.3}$$

The second Poisson-Lichnerowicz cohomology group $H^2_{P_0}(M)$ of M is precisely the set of bivectors P_1 solving $[P_0, P_1] = 0$ modulo the solutions of the form $P_1 = \mathcal{L}_X(P_0)$. We can interpret $H^2_{P_0}(M)$ as the space of infinitesimal deformations of the Poisson structure modulo trivial deformations. For regular Poisson manifolds this cohomology reflects the topology of the leaf space and the variation in the symplectic structure as one passes from one leaf to another [15].

The aim of this paper is to show some particular solutions of the equation (1.3) and to discuss separation of variables for the corresponding bi-integrable system on $so^*(4)$.

2 The Lie-Poisson bivectors on $so^*(4)$.

Let us consider the semisimple Lie algebra $so(4)$. The dual space $M = so^*(4)$ is a Poisson manifold [15]. Since $M = so^*(4) = so^*(3) \oplus so^*(3)$, we can introduce the following coordinates $z = (s, t)$ on M, where $s = (s_1, s_2, s_3)$ and $t = (t_1, t_2, t_3)$ are two vectors in $\mathbb{R}^3$. As usual we identify $(\mathbb{R}^3, \wedge)$ and $(so(3), [.,.])$ by using the well known isomorphism of the Lie algebras

$$z = (z_1, z_2, z_3) \to z_M = \begin{pmatrix} 0 & z_3 & -z_2 \\ -z_3 & 0 & z_1 \\ z_2 & -z_1 & 0 \end{pmatrix}, \tag{2.1}$$

where $\wedge$ is the cross product in $\mathbf{R}^3$ and $[.,.]$ is the matrix commutator in $so(3)$. In these coordinates the canonical Poisson bivector on $so^*(4)$ is equal to

$$P_0 = \begin{pmatrix} s_M & 0 \\ 0 & t_M \end{pmatrix}. \tag{2.2}$$

The generic symplectic leaves are the level sets of two globally defined Casimir functions

$$C_1 = \langle s, s \rangle \equiv |s|^2 \equiv \sum_{i=1}^{3} s_i^2, \qquad C_2 = \langle t, t \rangle, \qquad P_0 dC_{1,2} = 0. \tag{2.3}$$

In the paper [11] the overdetermined system of equations (1.1) on $so^*(4)$ has been directly solved in the class of the linear Lie-Poisson bivectors P_1, i.e using the anzats $P_1^{ij} = \sum a_m^{ij} z_m$ for the components of the second bivector.

According to [11] the Frahm-Schottky, Steklov and Poincaré systems are bi-integrable systems and the corresponding second bivectors $P_1 = \mathcal{L}_X(P_0)$ are generated by the Liouville vector fields $X_j = \sum X_j^i \partial_i$ with the following components

$$\begin{aligned} X_1 &= (a_1 t_1, \quad a_2 t_2, \quad a_3 t_3, \quad a_1 s_1, \quad a_2 s_2, \quad a_3 s_3), \\ X_2 &= (\mathcal{X}_1, a\mathcal{X}_2) \qquad\qquad a, a_1, a_2, a_3 \in \mathbb{C}, \\ X_3 &= (0, \quad 0, \quad a t_3, \quad 0, \quad 0, \quad -a s_3), \end{aligned}$$

where

$$\begin{aligned} \mathcal{X}_1 &= (a_1 t_1 + \tfrac{a_1(a_2^2+a_3^2)}{a_2 a_3} s_1, \quad a_2 t_2 + \tfrac{a_2(a_1^2+a_3^2)}{a_1 a_3} s_2, \quad a_3 t_3 + \tfrac{a_3(a_1^2+a_2^2)}{a_1 a_2} s_3), \\ \mathcal{X}_2 &= (-a_2 a_3 s_1 - \tfrac{a_2^2+a_3^2}{2} t_1, \quad -a_1 a_3 s_2 - \tfrac{a_1^2+a_3^2}{2} t_2, \quad -a_1 a_2 s_3 - \tfrac{a_1^2+a_2^2}{2} t_3), \end{aligned}$$

The remaining four solutions of the system (1.1) are associated with the following vector fields

$$X_4 = ((a_3 + a_2)s_1, (a_3 + a_1)s_2, (a_1 + a_2)s_3, (b_2 + b_3)t_1, (b_1 + b_3)t_2, (b_1 + b_2)t_3)$$

$$X_5 = (-\tfrac{a}{2}s_1 + b_1t_1, -\tfrac{a}{2}s_2 + b_2t_2, b_3t_3, b_1s_1 - \tfrac{a}{2}t_1, b_2s_2 - \tfrac{a}{2}t_2, (b_3 - a)s_3 - at_3)$$

$$X_6 = (a_2s_1, \quad a_1s_2, \quad (a_1 + a_2)s_3 - bt_3, \quad ct_1, \quad ct_2, \quad 0)$$

$$X_7 = (as_1, \quad as_2, \quad 0, \quad c_1s_3 + bt_1, \quad c_2s_3 + bt_2, \quad 0),$$

where $b_1 = \pm b_2$ for the vector field X_5. The corresponding bi-integrable systems with quadratic integrals of motion have been completely described in [11].

We have to keep in the mind that the vector fields X_k and the corresponding bivectors $P_1 = \mathcal{L}_{X_k}P_0$ are defined up to canonical transformations of the vectors s and t, which preserve the form of the canonical bivector P_0 (2.2) [6].

3 Quadratic Poisson bivectors on $so^*(4)$ and $e^*(3)$

After the linear Poisson structures, it is natural to look at quadratic structures. Substituting the following anzats for the components of the Liouville vector fields $X_m = \sum_{i\geq j}^6 a_m^{ij} z_i z_j$ into (1.3) one gets a highly overdetermined system of quadratic equations on the 126 complex coefficients a_m^{ij}. Unfortunately, we cannot get and classify all the solutions of this system even by using modern computers and modern software.

So, in this section we will suppose that

$$P_1 dC_{1,2} = 0. \tag{3.1}$$

It means that symplectic leaf of P_1 are contained in those of P_0. In general, the bivector P_1 could have some more Casimirs, so that their symplectic leaf could be smaller. For example, if $X = 0$, then equation (3.1) is true, but the symplectic foliations of P_0 and P_1 are different. However in this case $P_1 = 0$ is not a linear or a quadratic bivector on $so^*(4)$.

Proposition 1. *For all the quadratic bivectors $P_1 = \mathcal{L}_X(P_0)$ on $so^*(4)$ the restriction (3.1) leads to quadratic Poisson structures having the same foliation by symplectic leaves as P_0.*

Below we will prove this Proposition by using so-called Darboux-Nijenhuis variables on $so^*(4)$.

The additional restriction on $P_1 = \mathcal{L}_X(P_0)$ is a linear equation with respect to X, which allows us to get solutions of the following system of equations

$$\mathcal{L}_X(P_0)dC_{1,2} = -P_0 dX(C_{1,2}) = 0 \qquad \text{and} \qquad [\mathcal{L}_X(P_0), \mathcal{L}_X(P_0)] = 0 \tag{3.2}$$

Any solution X of this system (3.2) gives rise to a Poisson bivector $P_1 = \mathcal{L}_X(P_0)$ fulfilling equations (1.1) and (3.1).

We solved equations (3.2) in the class of quadratic vector fields $X = \sum X_m \partial_m$ with the components $X_m = \sum_{i\geq j} a_m^{ij} z_i z_j$ by using one of the modern computer algebra system and got the following three solutions.

Proposition 2. *Let a and b are numerical vectors and α, β are arbitrary parameters. The vector field $X = (\mathcal{X}_1, \mathcal{X}_2)$ with components*

$$\mathcal{X}_1 = \alpha\big(s \wedge (a \wedge s)\big) + \beta\langle b, t\rangle (s \wedge a), \qquad \mathcal{X}_2 = \alpha\big(t \wedge (b \wedge t)\big)$$

generates a first bivector fulfilling equations (1.1) and (3.1)

$$P_1^{(1)} = \begin{pmatrix} 2\alpha\langle a, s\rangle s_M & \beta\big[(a \wedge s) \otimes (b \wedge t)\big] \\ -\beta\big[(a \wedge s) \otimes (b \wedge t)\big]^T & 2\alpha\langle b, t\rangle t_M \end{pmatrix}. \tag{3.3}$$

Using orthogonal transformations of the vectors

$$s \to s' = U_1 s, \qquad \text{and} \qquad t \to t' = U_2 t \tag{3.4}$$

where $U_{1,2}$ are orthogonal matrices, we can always put

$$a = (0, 0, a_3), \qquad b = (0, 0, b_3).$$

Using scaling transformation

$$P_1^{(1)} \to \lambda P_1^{(1)}, \qquad \lambda \in \mathbb{R}\,, \tag{3.5}$$

we can put $\alpha = 1$ or $\beta = 1$.

Proposition 3. *Let a and b are two complex vectors, such that*

$$\langle a, b\rangle = \langle b, b\rangle = 0,$$

where $\langle x, y\rangle = \sum_{i=1}^3 x_i y_i$ is the inner product of two vectors. The vector field $X = (\mathcal{X}_1, \mathcal{X}_2)$ with components

$$\mathcal{X}_1 = \tfrac{1}{2}\, s \wedge (a \wedge s),$$

$$\mathcal{X}_2 = -\tfrac{1}{2}\, t \wedge (a \wedge t) + i|a|^{-1}\langle a, s\rangle (a \wedge t) - i\langle b, s\rangle (b \wedge t),$$

generates a second bivector fulfilling equations (1.1) and (3.1)

$$P_1^{(2)} = \begin{pmatrix} \langle a,s\rangle s_M & -i\Big[|a|^{-1}(a\wedge s)\otimes(a\wedge t) - (b\wedge s)\otimes(b\wedge t)\Big] \\ i\Big[|a|^{-1}(a\wedge s)\otimes(a\wedge t) - (b\wedge s)\otimes(b\wedge t)\Big]^T & -\langle a,t\rangle t_M \end{pmatrix}, \tag{3.6}$$

here $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ and $(x \otimes y)_{ij} = x_i y_j$.

Using orthogonal transformations (3.4) and scaling (3.5) we can always put

$$a = (0, 0, 1), \qquad b = (b_1, ib_1, 0)\,.$$

Proposition 4. *Let a, b and c are three complex vectors, such that*

$$\langle a, b\rangle = \langle a, c\rangle = \langle b, b\rangle = \langle c, c\rangle = 0, \qquad b \wedge c \neq 0.$$

The vector field $X = (\mathcal{X}_1, \mathcal{X}_2)$ with components

$$\mathcal{X}_1 = \tfrac{1}{2}\, s \wedge (a \wedge s),$$

$$\mathcal{X}_2 = \tfrac{1}{2}\, t \wedge (a \wedge t) - i|a|^{-1}\langle a, s\rangle (a \wedge t) - i\langle b, s\rangle (c \wedge t),$$

generates a third bivector fulfilling equations (1.1) and (3.1)

$$P_1^{(3)} = \begin{pmatrix} \langle a,s\rangle s_M & i\Big[|a|^{-1}(a\wedge s)\otimes(a\wedge t)+(b\wedge s)\otimes(c\wedge t)\Big] \\ -i\Big[|a|^{-1}(a\wedge s)\otimes(a\wedge t)+(b\wedge s)\otimes(c\wedge t)\Big]^T & \langle a,t\rangle t_M \end{pmatrix}. \tag{3.7}$$

Using orthogonal transformations (3.4) and scaling (3.5) we can always put

$$a = (0, 0, 1), \qquad b = (b_1, ib_1, 0), \qquad c = (c_1, -ic_1, 0)\,.$$

Let us consider the canonical bivector on $M = e^*(3)$

$$P_0 = \begin{pmatrix} 0 & x_M \\ x_M & J_M \end{pmatrix} \tag{3.8}$$

and the corresponding Casimir functions

$$C_1 = |x|^2 = x_1^2 + x_2^2 + x_3^2, \qquad C_2 = \langle x, J\rangle = x_1J_1 + x_2J_2 + x_3J_3.$$

On $M = e^*(3)$ the system of equations (3.2) has only one nontrivial solution in the class of quadratic vector fields $X = \sum X_m\partial_m$ with the components $X_m = \sum_{i\geq j} a_m^{ij} z_iz_j$, where $z = (x_1, x_2, x_3, J_1, J_2, J_3)$.

Proposition 5. *If a and b are two vectors, such that $|a| = 0$, and α is arbitrary parameter, then the following bivector on $M = e^*(3)$*

$$P_1^{(4)} = \begin{pmatrix} \langle a,x\rangle, & (x\wedge J)\otimes a+\langle a,J\rangle x_M+\frac{1}{2}\Big(\frac{1}{\alpha}+\alpha\langle a,b\rangle\Big)\Big(x\otimes x-|x|^2\Big)+\alpha(b\wedge x)\otimes(a\wedge x) \\ *, & \langle a,J\rangle J_M+\langle b,x\rangle x_M+\frac{1}{2}\Big(\frac{1}{\alpha}+\alpha\langle a,b\rangle\Big)(x\wedge J)_M-\alpha\Big((a\wedge x)\wedge(b\wedge J)\Big)_M \end{pmatrix} \tag{3.9}$$

satisfies equations (1.1) and (3.1).

This bivector $P_1^{(4)}$ is a Lie derivative of P_0. For the brevity we omit the explicit expression for the corresponding Liouville vector field X.

In the next Section we prove that the Poisson bivectors $P_1^{(m)}$ have a common symplectic foliation with P_0 and that the Poisson bivectors $P_1^{(2)}$ and $P_1^{(3)}$ are equivalent.

4 The Darboux-Nijenhuis variables

Let us consider a bi-hamiltonian manifold M with the non degenerate Poisson bivectors P_0 and P_1. By definition a set of local coordinates (q_i, p_i) on M is called a set of Darboux-Nijenhuis coordinates if they are canonical with respect to the symplectic form

$$\omega = P_0^{-1} = \sum_{i=1}^{n} dp_i \wedge dq_i$$

and put the recursion operator $N = P_1 P_0^{-1}$ in diagonal form,

$$N = \sum_{i=1}^{n} q_i \left(\frac{\partial}{\partial q_i} \otimes dq_i + \frac{\partial}{\partial p_i} \otimes dp_i \right). \tag{4.1}$$

This means that the only nonzero Poisson brackets are

$$\{q_i, p_j\}_0 = \delta_{ij}, \qquad \{q_i, p_j\}_1 = q_i \delta_{ij}\,. \tag{4.2}$$

According to (4.1) coordinates q_i are eigenvalues of N, i.e they are roots of the minimal polynomial $\mathcal{A}(\lambda) = \big(\det(N - \lambda \mathrm{I})\big)^{1/2}$ of N.

In order to get momenta $p_{1,2}$ we can directly solve equations (4.2) with respect to the functions $p_{1,2}(s,t)$. The standard computation problem is that variables $p_{1,2}$ are defined up to canonical transformation $p_i \to p_i + f_i(q_i)$, where f_i are arbitrary functions on q_i.

In order to construct recursion operator N on the generic symplectic leaves of $so^*(4)$ we will use analog of the Andoyer variables [1, 2].

4.1 Analog of the Andoyer variables on $so^*(4)$

Let us introduce the following analog of the Andoyer variables [1]

$$u_1 = s_3 + t_3, \qquad v_1 = -i \ln \left(\frac{s_2 + is_1 + t_2 + it_1}{\sqrt{(t_1+s_1)^2 + (s_2+t_2)^2}} \right) \tag{4.3}$$

and

$$\begin{aligned} u_2 &= \sqrt{C_1 + C_2 + 2s_1t_1 + 2s_2t_2 + 2s_3t_3}, \\ v_2 &= \arccos\left(\frac{t_3C_1 - s_3C_2 + (t_3 - s_3)(s_1t_1 + s_2t_2 + s_3t_3)}{\sqrt{(t_1+s_1)^2 + (s_2+t_2)^2}\sqrt{C_1C_2 - (s_1t_1 + s_2t_2 + s_3t_3)^2}} \right) \end{aligned} \tag{4.4}$$

The inverse transformation in coordinates

$$J_i = s_i + t_i, \qquad x_i = \varkappa(s_i - t_i), \qquad \varkappa \in \mathbb{C}\,, \tag{4.5}$$

looks like

$$J_1 = \sqrt{u_2^2 - u_1^2}\sin v_1, \qquad J_2 = \sqrt{u_2^2 - u_1^2}\cos v_1, \qquad J_3 = u_1,$$

and

$$x_1 = \frac{\mathcal{C}_1\sqrt{1-\frac{u_1^2}{u_2^2}} + u_1\sqrt{\mathcal{C}_2 - \varkappa^2 u_2^2 - \frac{\mathcal{C}_1^2}{u_2^2}}\cos v_2}{u_2}\sin v_1 + \sqrt{\mathcal{C}_2 - \varkappa^2 u_2^2 - \frac{\mathcal{C}_1^2}{u_2^2}}\sin v_2 \cos v_1\,,$$

$$x_2 = \frac{\mathcal{C}_1\sqrt{1-\frac{u_1^2}{u_2^2}} + u_1\sqrt{\mathcal{C}_2 - \varkappa^2 u_2^2 - \frac{\mathcal{C}_1^2}{u_2^2}}\cos v_2}{u_2}\cos v_1 + \sqrt{\mathcal{C}_2 - \varkappa^2 u_2^2 - \frac{\mathcal{C}_1^2}{u_2^2}}\sin v_2 \sin v_1\,,$$

$$x_3 = \mathcal{C}_1 \frac{u_1}{u_2^2} - \sqrt{1 - \frac{u_1^2}{u_2^2}}\sqrt{\mathcal{C}_2 - \varkappa^2 u_2^2 - \frac{\mathcal{C}_1^2}{u_2^2}}\cos v_2$$

Here

$$\mathcal{C}_1 \;=\; x_1J_1+x_2J_2+x_3J_3=\varkappa(C_1-C_2),$$

$$\mathcal{C}_2 \;=\; x_1^2+x_2^2+x_2^2+\varkappa^2(J_1^2+J_2^2+J_3^2)=2\varkappa^2(C_1+C_2).$$

In (x,J)-variables (4.5) canonical bivector P_0 (2.2) on $so^*(4)$ reads as

$$P_0=\begin{pmatrix} \varkappa^2 J_M & x_M \\ x_M & J_M \end{pmatrix}.$$

At $\varkappa\to 0$ this bivector is reduced to the canonical bivector P_0 on $e^*(3)$, which is the dual space to the algebra $e(3)$ of Euclidean group $E(3)$. After contraction $\varkappa\to 0$ our variables u,v (4.3-4.4) coincide with the Andoyer variables on $e^*(3)$ [1].

The projection of the canonical bivector P_0 on the generic symplectic leaves of $M=so^*(4)$ or $M=e^*(3)$ in (u,v)-variables (4.3-4.4) looks like

$$\widehat{P}_0=\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Using this analog of the Andoyer variables we can easy obtain the projection $\widehat{P}_1$ of any bivector P_1 fulfilling (3.1).

So, we can introduce symplectic form $\omega=\widehat{P}_0^{-1}$, recursion operator $N=\widehat{P}_1\widehat{P}_0^{-1}$ and Darboux-Nijenhuis variables on the generic symplectic leaves of $M=so^*(4)$ for all the bivectors $P_1^{(m)}$ from the Section 3.

4.2 First bivector

Let us consider bivector $P_1^{(1)}$ (3.3) on $M=so^*(4)$. Using orthogonal transformations (3.4) we can always put

$$a=(0,0,a_3),\qquad b=(0,0,b_3).$$

In this case bivector $P_1^{(1)}$ gives rise to the following Darboux-Nijenhuis coordinates

$$q_1=2\alpha a_3 s_3,\qquad q_2=2\alpha b_3 t_3\,, \tag{4.6}$$

and momenta

$$\begin{aligned} p_1&=\frac{1}{2\alpha\, a_3}\arctan\left(\frac{s_1}{s_2}\right)-\frac{\beta a_3}{4\alpha^2\, b_3}\ln(a_3s_3-b_3t_3),\\ p_2&=\frac{1}{2\alpha\, b_3}\arctan\left(\frac{t_1}{t_2}\right)+\frac{\beta\, a_3}{4\alpha^2\, b_3}\ln(a_3s_3-b_3t_3), \end{aligned} \tag{4.7}$$

which satisfy relations (4.2). It means that projections of P_0 and P_1 on the generic symplectic leaves of P_0 are equal to

$$\widehat{P}_0=\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix},\qquad \widehat{P}_1^{(1)}=\begin{pmatrix} 0 & 0 & q_1 & 0\\ 0 & 0 & 0 & q_2\\ -q_1 & 0 & 0 & 0\\ 0 & -q_2 & 0 & 0 \end{pmatrix} \tag{4.8}$$

and, therefore, symplectic foliations of P_0 and P_1 coincide. On the other hand it means that generic symplectic leaves are regular semisimple ωN manifolds [3].

4.3 Second bivector

Let us consider bivector $P_1^{(2)}$ (3.6) on $M = so^*(4)$. Using orthogonal transformations (3.4) and scaling (3.5) we can always put

$$a = (0,0,1), \qquad b = (b_1, ib_1, 0)\,. \tag{4.9}$$

In this case the Darboux-Nijenhuis coordinates are roots of the minimal polynomial of the recursion operator $N^{(2)}$

$$\mathcal{A}(\lambda) = (\lambda - q_1)(\lambda - q_2) = \lambda^2 + (t_3 - s_3)\lambda + (t_1 + it_2)(s_1 + is_2)b_1^2 - s_3 t_3 \tag{4.10}$$

whereas the corresponding momenta are equal to

$$p_{1,2} = -i \ln \mathcal{B}(\lambda = q_{1,2}), \tag{4.11}$$

where

$$\mathcal{B}(\lambda) = \big(s_1 - is_2 + b_1^2(t_1 + it_2)\big)\lambda + t_3(s_1 - is_2) + b_1^2 s_3(t_1 + it_2)\,.$$

As above these variables satisfy relations (4.2) and, therefore, symplectic foliations of P_0 and $P_1^{(2)}$ coincide.

According [5], we can easily derive relations (4.2) from the following relations

$$\begin{aligned} \{\mathcal{A}(\lambda), \mathcal{B}(\mu)\}_k &= \frac{i}{\lambda - \mu}\Big(\lambda^k \mathcal{B}(\lambda)\mathcal{A}(\mu) - \mu^k \mathcal{A}(\lambda)\mathcal{B}(\mu)\Big), \\ \{\mathcal{A}(\lambda)\mathcal{A}(\mu)\}_k &= \{\mathcal{B}(\lambda)\mathcal{B}(\mu)\}_k = 0, \qquad k = 0,1. \end{aligned} \tag{4.12}$$

4.4 Third bivector

Let us consider bivector $P_1^{(3)}$ (3.7) on $M = so^*(4)$. Using orthogonal transformations (3.4) and scaling (3.5) we can always put

$$a = (0,0,1), \qquad b = (b_1, ib_1, 0), \qquad c = (c_1, -ic_1, 0)\,. \tag{4.13}$$

In this case the Darboux-Nijenhuis coordinates are roots of the minimal polynomial of the corresponding recursion operators $N^{(3)}$

$$\mathcal{A}(\lambda) = (\lambda - q_1)(\lambda - q_2) = \lambda^2 - (s_3 + t_3)\lambda + c_1 b_1 (t_1 - it_2)(s_1 + is_2) + s_3 t_3\,. \tag{4.14}$$

The corresponding momenta are defined by

$$p_{1,2} = -i \ln \mathcal{B}(\lambda = q_{1,2}), \tag{4.15}$$

where

$$\mathcal{B}(\lambda) = \big((s_1 - is_2) + c_1 b_1 (t_1 - it_2)\big)\lambda + c_1 b_1 (t_1 - it_2) s_3 - (s_1 - is_2) t_3\,.$$

As above these variables satisfy relations (4.2) and, therefore, symplectic foliations of P_0 and $P_1^{(3)}$ coincide.

It is easy to see that polynomial $\mathcal{A}(\lambda)$ (4.10) and polynomial $\mathcal{A}(\lambda)$ (4.14) coincide after the following canonical transformation

$$t_2 \to -t_2, \qquad t_3 \to -t_3\,, \qquad \text{at} \quad \hat{b}_1^2 = c_1 b_1. \tag{4.16}$$

Here $\hat{b}_1$ is the entry of the vector b (4.9), whereas b_1, c_1 are the entries of the vectors b and c (4.13). So, the corresponding Darboux-Nijenhuis variables and, therefore, bivectors $P_1^{(2)}$ (3.6) and $P_1^{(3)}$ (3.7) are equivalent up to canonical transformations.

4.5 Fourth bivector

Let us consider bivector $P_1^{(4)}$ (3.9) on $M = e^*(3)$. The Darboux-Nijenhuis coordinates are roots of the following minimal polynomial of the recursion operator N

$$\begin{aligned}\mathcal{A}(\lambda) &= \lambda^2 + \Big(\langle a, J\rangle - \langle a\wedge b, x\rangle\alpha\Big)\lambda + \frac{\alpha^2\langle a,b\rangle}{4}\Big(\langle a,x\rangle\langle b,x\rangle - \langle a\wedge xv, b\wedge x\rangle\Big)\\ &+ \frac{\alpha}{2}\langle a,b\rangle\langle a\wedge x, J\rangle - \frac{1}{2\alpha}\langle a\wedge x, J\rangle + \frac{1}{4\alpha^2}\langle x,x\rangle - \frac{1}{2}\langle a,x\rangle\langle b,x\rangle\end{aligned} \tag{4.17}$$

Linear canonical transformations of $e^*(3)$ consist of rotations

$$x \to \alpha\,\mathrm{U}\,x\,, \qquad J \to \mathrm{U}\,J\,, \tag{4.18}$$

where α is an arbitrary parameter and U is an orthogonal matrix, and shifts

$$x \to x\,, \qquad J \to J + \mathrm{S}\,x\,, \tag{4.19}$$

where S is an arbitrary 3×3 skew-symmetric constant matrix [6]. Using these canonical transformations of the vectors x and J we can always put

$$\mathcal{A}(\lambda) = \lambda^2 + J_3\lambda + \frac{1}{2\alpha}(x_2J_1 - x_1J_2) + \frac{1}{4\alpha^2}(x_1^2 + x_2^2 + x_3^2).$$

However, in this case we could not directly solve equations (4.2) with respect to the functions $p_{1,2}(x, J)$.

Nevertheless, projections of P_0 and $P_1^{(4)}$ on the generic symplectic leaves of P_0 are given by (4.8) and, therefore, symplectic foliations of P_0 and $P_1^{(4)}$ coincide.

4.6 Higher order Poisson bivectors

For all the considered above Poisson bivectors $P_1^{(m)}$ we can build the higher order Poisson structures by using the corresponding Darboux-Nijenhuis variables. Namely, if we restrict our bivectors P_0 and $P_1^{(m)}$ on their common symplectic leaves and postulate the following Poisson brackets between the Darboux-Nijenhuis variables

$$\{q_i, q_j\}_k = \{p_i, p_j\}_k = 0, \qquad \{q_i, p_j\} = q_i^k\delta_{ij},$$

then after transformations $(q, p) \to (u, v) \to (s, t)$ one gets $k+1$ order Poisson bivectors on $so^*(4)$.

As an example we present the cubic Poisson bivector

$$P_2^{(3)} = \begin{pmatrix} i\Big(a^{-1}\langle a,s\rangle^2-\langle b,s\rangle\langle c,t\rangle\Big)s_M, & -\langle a,s+t\rangle\Big[a^{-1}(a\wedge s)\otimes(a\wedge t)+(b\wedge s)\otimes(c\wedge t)\Big] \\ * & i\Big(a^{-1}\langle a,t\rangle^2-\langle b,s\rangle\langle c,t\rangle\Big)t_M \end{pmatrix}, \tag{4.20}$$

associated with the variables (4.14)- (4.15) and, therefore, compatible with the quadratic tensor $P_1^{(3)}$ (3.7).

5 Bi-integrable systems

5.1 The Jacobi method

In order to get bi-integrable systems on M we can identify Darboux-Nijenhuis variables with the separated variables and substitute all the pairs of variables q_j, p_j into the separated equations

$$\Phi_j(q_j, p_j, \alpha_1, \alpha_2) = 0, \qquad j = 1, 2, \tag{5.1}$$

where Φ_j are functions on p_j, q_j and two parameters $\alpha_{1,2}$ only.

According to the Jacobi theorem if we solve the separated equations (5.1) with respect to parameters $\alpha_{1,2}$ one gets a pair of independent integrals of motion

$$\alpha_{1,2} = H_{1,2}(p, q), \tag{5.2}$$

as functions on the phase space $M = so^*(4)$, which are in the bi-volution

$$\{H_1, H_2\}_0 = \{H_1, H_2\}_1 = 0 \tag{5.3}$$

with respect to the Poisson brackets associated with bivectors P_0 and P_1 (see Propositions 1-3 in [12]).

For instance, substituting the Darboux-Nijenhuis variables p_j, q_j (4.6-4.7) associated with the first bivector $P_1^{(1)}$ into the relations

$$\Phi_1 = 2q_1^2 + 2V(p_1) - H_1 - H_2, \qquad \Phi_2 = 2q_2^2 - H_1 + H_2,$$

where $V = e^{4i\alpha a_3 p_1} + e^{-4i\alpha a_3 p_1}$, then at $\beta = -i\alpha$, $b_3 = a_3 = 1$ one gets

$$H_{1,2} = (q_1^2 \pm q_2^2) + V = 4\alpha^2(s_3^2 \pm t_3^2) + (t_3 - s_3)\frac{s_1 + is_2}{s_1 - is_2} + (t_3 - s_3)^{-1}\frac{s_1 - is_2}{s_1 + is_2}\,.$$

Using other separated equations $\Phi_{1,2} = 0$ we can get many other more complicated bi-integrable systems on $so^*(4)$.

5.2 Quadratic integrals of motion

In this Section we will substitute all the known pairs of integrals of motion $H_{1,2}$ on $so^*(4)$ into the equations (5.3) and try to found bi-integrable systems associated with one of the Poisson bivectors from Section 3. So, in this Section we will forget about the Darboux-Nijenhuis coordinates and will start with integrals of motion listed in [2] and in [9], which have different physical applications.

In order to describe these bi-integrable systems we prefer to use (x, J) coordinates (4.5) as in [2, 9].

Proposition 6. *If $A = -i\varkappa a$, $|a| = 1$ and B is an arbitrary vector, then at*

$$b \wedge c + 2ia = 0$$

the following integrals of motion

$$\begin{aligned} H_1 &= \langle A, B\rangle |J|^2 - 2\langle A, J\rangle\langle B, J\rangle + \langle B, J\wedge x\rangle, \\ H_2 &= \langle B, J\rangle\Big(2\langle A, J\wedge x\rangle - \varkappa^2\langle J, J\rangle + \langle x, x\rangle\Big) \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \widetilde{H}_1 &= \langle A, J\wedge (B\wedge J)\rangle + \langle B, J\wedge x\rangle \\ \widetilde{H}_2 &= \langle J, B\rangle^2\Big(\langle J\wedge A, J\wedge A\rangle + 2\langle A, J\wedge x\rangle + \langle x, x\rangle\Big) \end{aligned} \tag{5.5}$$

are in bi-involution

$$\{H_1, H_2\}_0 = \{H_1, H_2\}_1 = \{\widetilde{H}_1, \widetilde{H}_2\}_0 = \{\widetilde{H}_1, \widetilde{H}_2\}_0 = 0\,, \tag{5.6}$$

with respect to the Poisson brackets associated with P_0 (2.2) and P_1 (3.7).

Integrable system with the cubic integrals of motion H_2 (5.4) has been proposed in [4]. For this system we know the Lax matrices and the separated variables [4]. The second integrable systems with fourth order integral of motion $\widetilde{H}_2$ (5.5) has been considered in [8].

According to [3] the bi-involutivity of integrals of motion (5.6) is equivalent to the existence of non-degenerate control matrices F and $\widetilde{F}$, such that

$$P_1 dH_i = P_0 \sum_{j=1}^{2} F_{ij}\, dH_j, \qquad P_1 d\widetilde{H}_i = P_0 \sum_{j=1}^{2} \widetilde{F}_{ij}\, d\widetilde{H}_j, \qquad i = 1, 2. \tag{5.7}$$

In our case they look like

$$F = \begin{pmatrix} \langle a, J\rangle & \dfrac{i}{2\varkappa} \\ \dfrac{-iH_2}{2\varkappa\langle B, J\rangle} & 0 \end{pmatrix}, \qquad \widetilde{F} = \begin{pmatrix} \dfrac{\langle a, J\rangle}{2} & \dfrac{i}{4\varkappa\langle B, J\rangle} \\ \dfrac{-i\widetilde{H}_2}{\varkappa\langle B, J\rangle} & \dfrac{\langle a, J\rangle}{2} \end{pmatrix} \tag{5.8}$$

Entries of F are polynomials, whereas one of the entries of $\widetilde{F}$ is a rational function on $so^*(4)$.

The eigenvalues of the matrices F and $\widetilde{F}$ coincide to each other. They are roots of the common characteristic polynomial

$$\mathcal{A}(\lambda) = (\lambda - q_1)(\lambda - q_2) = \lambda^2 - \langle a, J\rangle\lambda + \frac{\langle J, J\rangle}{4} - \frac{i\langle a, x\wedge J\rangle}{2\varkappa} - \frac{\langle x, x\rangle}{4\varkappa^2}\,. \tag{5.9}$$

Of course, this polynomial coincides with the minimal polynomial $\mathcal{A}^{(3)}$ of the recursion operator $N^{(3)}$ after suitable canonical transformation.

Using relations (4.12) we can reconstruct the conjugated momenta $p_{1,2}$. Namely, if numerical vector d satisfies conditions $\langle a,d\rangle=\langle d,d\rangle=0$, the coordinates $q_{1,2}$ (5.9) and the momenta

$$p_{1,2}=-i\ln\mathcal{B}(\lambda=q_{1,2}),\qquad \mathcal{B}(\lambda)=\{\langle d,J\rangle,\mathcal{A}(\lambda)\}_0 \tag{5.10}$$

are the Darboux-Nijenhuis variables fulfilling equations (4.2). The variables $p_{1,2}$ (5.10) are defined up to canonical transformation $p_i\to p_i+f_i(q_i)$, where f_i are arbitrary functions on q_i only.

Proposition 7. *Coordinates $q_{1,2}$ (5.9) and momenta $p_{1,2}$ (5.10) are the separated variables for the integrable systems with integrals of motion $H_{1,2}$ (5.4) and $\widetilde{H}_{1,2}$ (5.5). If*

$$b^*=c,\qquad \textit{and}\qquad d=c,$$

the corresponding separated equations are equal to

$$\begin{aligned}4\varkappa^2\langle a,B\rangle\,q_k^3+q_kH_1-H_2 &= 2\varkappa^2\langle c,a\wedge B\rangle\,(q_k^2-C_1)(q_k^2-C_2)\,q_k\,\mathrm{e}^{-ip_k}\\ &+\ 2\varkappa^2\langle b,a\wedge B\rangle\,q_k\,\mathrm{e}^{ip_k}\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}2\varkappa\langle a,B\rangle\,q_{1,2}^2-\widetilde{H}_1\mp\sqrt{\widetilde{H}_2} &= \varkappa\langle c,a\wedge B\rangle\,(q_{1,2}^2-C_1)(q_{1,2}^2-C_2)\,\mathrm{e}^{-ip_{1,2}}\\ &+\ \varkappa\langle b,a\wedge B\rangle\,\mathrm{e}^{ip_{1,2}}\end{aligned} \tag{5.12}$$

The separated equations (5.11) and (5.12) are related with the parabolic and cartesian Stäckel webs on a plane. The Stäckel matrices S and $\widetilde{S}$ diagonalize the control matrices F and $\widetilde{F}$:

$$F=S\begin{pmatrix}q_1&0\\0&q_2\end{pmatrix}S^{-1},\qquad S=\begin{pmatrix}1&1\\2i\varkappa q_2&2i\varkappa q_1\end{pmatrix},$$

and

$$\widetilde{F}=\widetilde{S}\begin{pmatrix}q_1&0\\0&q_2\end{pmatrix}\widetilde{S}^{-1},\qquad \widetilde{S}=\begin{pmatrix}1&1\\2i\sqrt{\widetilde{H}_2}&-2i\sqrt{\widetilde{H}_2}\end{pmatrix}.$$

The right hand sides of the separated equations (5.11) and (5.12) are the generalized Stäckel potentials.

For special values of A, B and $\varkappa$ the Hamiltonians (5.4) and (5.5) are real functions [4, 8]. There is one integrable system with the complex Hamiltonians

$$\begin{aligned}\widehat{H}_1 &= \alpha J_2^2-\frac{\varkappa^2}{\alpha}J_1^2+x_2J_1-x_1J_2,\qquad \alpha=i\varkappa\,,\\ \widehat{H}_2 &= \alpha(J_1x_2-J_2x_1)(\varkappa^2|J|^2-|x|^2)+\varkappa^2\Big((J_1x_2-J_2x_1)^2+(J_3x_1-J_1x_3)^2\Big)\\ &-\ \alpha^2\Big((J_1x_2-J_2x_1)^2+(J_2x_3-J_3x_2)^2\Big)\end{aligned}$$

which are in bi-involution with respect to the same Poisson brackets at $a_1=a_2=0$ and $a_3=-1$. At the arbitrary value of α the Lax matrices and the separated variables for this system have been constructed in [10, 4].

5.3 Inhomogeneous integrals of motion

There is only one nontrivial linear Poisson bivector, which is compatible with the canonical bivector P_0 and the quadratic bivector $P_1^{(3)}$ (3.7)

$$P_1^{(0)} = \begin{pmatrix} \alpha_1 s_M & 0 \\ 0 & \alpha_2 t_M \end{pmatrix}, \qquad \alpha_{1,2} \in \mathbb{C}.$$

The linear combination

$$P_1^g = P_1^{(3)} + \begin{pmatrix} \alpha_1 s_M & 0 \\ 0 & \alpha_2 t_M \end{pmatrix} \tag{5.13}$$

is an inhomogeneous Poisson bivector on $so^*(4)$ compatible with P_0 and such that $P_1^g dC_{1,2} = 0$.

Let us consider the following inhomogeneous Hamiltonians [4, 9]

$$H_1^g = H_1 \quad + \quad \langle k, J\rangle + \alpha_{12}\langle B, x\rangle, \tag{5.14}$$

$$\begin{aligned} H_2^g = H_2 \quad &+ \quad 2\alpha_{12}\langle B, J\rangle\langle, A, x\rangle - \langle k, A\rangle J^2 - \langle k, J\wedge x\rangle \\ &- \quad \varkappa^2\alpha_{12}^2\langle B, J\rangle - \alpha_{12}\langle k, x\rangle\,, \end{aligned}$$

and

$$\widetilde{H}_1^g \quad = \quad \widetilde{H}_1 + \langle \alpha_3 A + \alpha_4 A\wedge B, J\rangle + \alpha_{12}\langle B, x\rangle, \tag{5.15}$$

$$\begin{aligned} \widetilde{H}_2^g \quad = \quad & \widetilde{H}_2 + 2\langle B, J\rangle\Big[\alpha_3\big(\langle A, J\rangle^2 + \varkappa^2|J|^2 - |x|^2 - 2\langle A, J\wedge x\rangle\big) \\ + \quad & \alpha_4\big(\langle B\wedge A, J\wedge x\rangle + \langle A, J\rangle\langle A, B\wedge J\rangle\big) + \alpha_{12}\langle B, J\rangle\langle A, x\rangle\Big] \\ + \quad & \alpha_3^2\Big(2|x|^2 - \langle A, J\rangle^2 + 2\langle A, J\wedge x\rangle\Big) + 2\alpha_4\alpha_{12}\Big(\langle B, J\rangle\langle A, B\wedge x\rangle\Big) \\ + \quad & 2\alpha_4^2\Big(\langle B, B\rangle\langle A, J\rangle^2 - \varkappa^2\langle B, J\rangle^2 - 2\langle A, B\rangle\langle A, J\rangle\langle B, J\rangle\Big) \\ - \quad & 4\alpha_3\alpha_{12}\langle B, J\rangle\langle A, x\rangle - \alpha_{12}^2\varkappa^2\langle B, J\rangle^2 \\ + \quad & 2\alpha_3\alpha_{12}\Big(\langle A\wedge B, J\wedge x\rangle - \langle A, J\rangle\langle A, B\wedge J\rangle\Big) \\ + \quad & 2\alpha_3\alpha_{12}\Big(\alpha_3\langle A, x\rangle + \alpha_4\langle A, B\wedge x\rangle + \alpha_{12}\varkappa^2\langle B, J\rangle\Big), \end{aligned}$$

where $\alpha_{12} = i(\alpha_1 - \alpha_2)$, $\alpha_1 \ldots, \alpha_4$ are arbitrary parameters and k is an arbitrary numerical vector.

Proposition 8. *The inhomogeneous integrals of motion (5.14) and (5.15) are in bi-involution with respect to the canonical bracket,$\{.,.\}_0$ and the second Poisson bracket $\{.,.\}_1$ associated with the bivector P_1^g (5.13)*

The separated variables for these inhomogeneous bi-integrable systems are different, but they remain the eigenvalues of the corresponding control matrices

$$F^g = F + \frac{1}{2}\begin{pmatrix} \alpha_1+\alpha_2 & 0 \\ \frac{\alpha_1-\alpha_2}{\varkappa}\langle k, x\rangle - \frac{i}{\varkappa}\langle k, J\wedge x\rangle - \langle k, B\rangle|J|^2 & \alpha_1+\alpha_2 \end{pmatrix}$$

and

$$\widetilde{F}^g = \begin{pmatrix} \dfrac{\langle a, J\rangle + \alpha_1 + \alpha_2}{2} & \dfrac{i}{4\varkappa(\langle B, J\rangle - \alpha_3)} \\ -\dfrac{i\widetilde{H}_2^g(\alpha_3 = \alpha_4 = 0)\,(\langle B, J\rangle) - \alpha_3}{\varkappa\langle B, J\rangle^2} & \dfrac{\langle a, J\rangle + \alpha_1 + \alpha_2}{2} \end{pmatrix}.$$

For the inhomogeneous Hamiltonians we have to add to the left hand side of the separated relations (5.11) and (5.12) terms proportional q_k^2 and q_k respectively. Moreover, in the right hand side we have to substitute $((q_k - i\alpha_1)^2 - C_1)((q_k - i\alpha_2)^2 - C_2)$ instead of $(q_k^2 - C_1)(q_k^2 - C_2)$.

6 Conclusion

We classify quadratic Poisson bivectors having the given canonical foliation on $M = so^*(4)$ or $M = e^*(3)$ as their symplectic leaf foliation. The corresponding Darboux-Nijenhuis variables are constructed. A pair of known integrable systems on $so^*(4)$ can be related with one of such quadratic Poisson bivectors. We prove that the corresponding integrals of motion admit separation of variables and the separated coordinates are eigenvalues of the control matrices.

Another approach to the construction of the quadratic and cubic Poisson bivectors P_1 having common symplectic foliations with P_0 has been proposed in [13, 14]. Among the corresponding bi-integrable systems there are generalized periodic Toda lattices, the XXX Heisenberg magnet, the XXX Heisenberg magnet with boundary conditions, the Kowalevski top on $so^*(4)$, the Goryachev-Chaplygin gyrostat on $e^*(3)$, the Kowalevski-Chaplygin-Goryachev gyrostat on $e^*(3)$ and some other well known integrable systems. Now we add to this list two other integrable systems on $so^*(4)$.

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