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Lie symmetries of a Painlevé-type equation without Lie symmetries

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Abstract

We use a method inspired by the Jacobi last multiplier [M.C. Nucci, Jacobi last multiplier and Lie symmetries: a novel application of an old relationship, *J. Nonlinear Math. Phys. 12*, 284-304 (2005)] in order to find Lie symmetries of a Painlevé-type equation without Lie point symmetries.

1 Introduction

Lie group analysis is the most powerful tool to find the general solution of ordinary differential equations (ODEs). However it is useless when applied to systems of n first-order equations because they admit an infinite number of symmetries, and there is no systematic way to find even a one-dimensional Lie symmetry algebra, apart from trivial groups like translations in time admitted by autonomous systems. One may try to derive an admitted n-dimensional solvable Lie symmetry algebra by making an ansatz on the form of its generators but when successful (rarely) it is just a lucky guess. However, in [23] we have remarked that any system of n first-order equations could be transformed into an equivalent system where at least one of the equations is of second-order. Then the admitted Lie symmetry algebra is no longer infinite-dimensional, and nontrivial symmetries of the original system could be retrieved [23]. This idea has been successfully applied in several instances ([23], [30], [24], [25], [15], [26], [4], [7], [8]). Also in [18] we have shown that first integrals can be obtained by Lie group analysis even if the system under study does not come from a variational problem, i.e., without making use of Noether's theorem [21]. If we consider a system of first-order equations and, by eliminating one of the dependent variables, derive an equivalent system which has one equation of second-order, then Lie group analysis applied to that equivalent system yields the first integral(s) of the original system which do(es) not contain the eliminated dependent variable. Of course this requires that such first integrals exist. The procedure should be repeated as many times as there are dependent variables in order to find all such first integrals. The first integrals correspond to the characteristic curves of determining equations of parabolic type which are constructed by the method of Lie group analysis. We have used this method for finding first integrals in [25] and [15]. Unfortunately the reduction method is not the ultimate method for finding symmetries. Therefore in [27] we devised another method which employs the Jacobi last multiplier ([10], [11], [12], [13], [16], [17], [1], [31]) in order to transform any system of n first-order equations into an equivalent system of n equations where one of the equations is of second order, namely the order of the system is raised by one. In [27], among other examples, the method was successfully applied to the following second-order equation [14][Ch. 6, 542*ff*]:

$$y'' = \frac{y'^2}{y} + f'(x)y^{p+1} + pf(x)y'y^p, \tag{1.1}$$

where $p \neq 0$ is a real constant and $f \neq 0$ is an arbitrary function of the independent variable x. This equation does not possess Lie point symmetries for general $f(x)$ and yet is trivially integrable [6]. Our method led to an equivalent system of two equations, one of first order and the other of second order, which admits enough Lie symmetries in order to integrate it by quadrature. Indeed this method finds Lie symmetries of system of first-order equations without any guesswork.

In [19] and [20] Muriel and Romero introduced the so-called λ-symmetries which were included into telescopic symmetries by Pucci and Saccomandi [29]. Again as in the case of Lie point symmetries of first-order equations the real problem is how to find solutions of the determining equations. This has been clearly stated by Pucci and Saccomandi themselves on page 6154 "This, obviously, does not mean that we are always able to solve the determining equations of telescopic vector fields for which an ordinary differential equation is a relative invariant, but this situation occurs also for classical Lie symmetries of differential equations (for example, the simple case of first-order ordinary differential equations)."

In [3] and [5] Cicogna, Gaeta and Morando provided a geometrical characterization of λ-symmetries for both ordinary and partial differential equations. They did not address the problem of solving the determining equation in either paper. In a recent paper [2] Ferraioli showed that λ-symmetries correspond to a special type of nonlocal symmetries and in a final remark observed that also telescopic symmetries can be recovered by nonlocal symmetries. Yet there is a lot of guesswork involved in order to find nonlocal symmetries, and on page 5485, Remark 2, he wrote "However, we note that the problem of determining the general solution of (11)[1], should be at least as difficult as solving the given ODE. Therefore in practice it could not be so easy to determine such correspondence (see example 3)".

In this paper we give another exemplification of our method as described in [27] by applying it to example 3 in [2]. It is a second-order Painlevé-type equation [28], which do not admit any Lie point symmetries and is a particular case of Painlevé XIV equation [9]. Our method transforms this equation into a system of three equations, two of first order and one of second order. This system admits a three-dimensional solvable Lie symmetry algebra and therefore can be reduced to one first-order equation which happens to be a Riccati equation and can be easily integrated in terms of Airy functions. Thus we recover through Lie symmetry a first integral admitted by the original equation, although it is not the same known first integral.

[1]It is a linear first-order partial differential equation.

2 A Painlevé-type equation

Let us consider the following Painlevé-type second-order ordinary differential equation [28], [9], [2]:

$$w'' = \frac{w'^2}{w} + \left(w + \frac{t}{w}\right) w' - 1, \tag{2.1}$$

which is a particular case[2] of the Painlevé XIV equation [9]:

$$w'' = \frac{w'^2}{w} + \left(Q(t)w + \frac{S(t)}{w}\right) w' + Q'(t)w^2 - S'(t), \tag{2.2}$$

which it is known to possess a first integral of the Riccati type, i.e.:

$$w' = Q(t)w^2 + kw - S(t), \tag{2.3}$$

where k is an arbitrary constant. Equation (2.1) does not possess any Lie point symmetries[3]. We can trivially put equation (2.1) into the form of an autonomous system of three first-order ODEs, i.e.:

$$\begin{aligned} w_1' &= w_2 && \equiv W_1, \\ w_2' &= \frac{w_2^2}{w_1} + \left(w_1 + \frac{w_3}{w_1}\right) w_2 - 1 && \equiv W_2, \\ w_3' &= 1 && \equiv W_3, \end{aligned} \tag{2.4}$$

where $w_1 = w, w_2 = w', w_3 = t$. If we use the method introduced in [23] to raise the order of at least one of the equations of this system, no nontrivial Lie symmetry can be found. We then consider the equation for the Jacobi last multiplier [11, 12, 13, 31], i.e.:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + \sum_{i=1}^{3} \frac{\partial(W_i)}{\partial w_i} = 0, \tag{2.5}$$

and raise the order accordingly[4], namely by introducing a new dependent variable $r_3 = r_3(t)$ such that

$$a\frac{\mathrm{d}}{\mathrm{d}t}(\log r_3) + \frac{2w_2}{w_1} + \left(w_1 + \frac{w_3}{w_1}\right) = 0, \tag{2.6}$$

with a an arbitrary constant. This yields the substitution:

$$w_3 = -a\frac{r_3'}{r_3}w_1 - w_1^2 - 2w_2, \tag{2.7}$$

and system (2.4) is transformed into the following system of two first-order ODEs and one second-order ODE:

$$w_1' = w_2$$

[2]It corresponds to assume $Q(t) = 1$ and $S(t) = t$.
[3]Neither equation (2.2) does for arbitrary $Q(t), S(t)$.
[4]This is precisely the method introduced in [27].

$$
\begin{aligned}
w_2' &= -\frac{a w_2 r_3'}{r_3} - 1 - \frac{w_2^2}{w_1} \\
r_3'' &= \frac{r_3'^2}{r_3} + \frac{w_2 r_3'}{w_1} - \frac{2 r_3 w_2}{a} + \frac{r_3}{a w_1} + \frac{2 r_3 w_2^2}{a w_1^2}.
\end{aligned} \tag{2.8}
$$

When Lie group analysis of this system is performed[5], a linear partial differential equation of parabolic structure is obtained, as expected [18]. Its characteristic curve is given by $w_2 r_3^a$. Consequently we introduce a new dependent variable $r_2 = r_2(t)$ such that $w_2 = r_2/r_3^a$. Then system (2.8) is transformed into the following system[6]:

$$
\begin{aligned}
w_1' &= r_3^2 r_2, \\
r_2' &= -\frac{r_3^4 r_2^2 + w_1}{r_3^2 w_1}, \\
r_3'' &= \frac{-2r_3^6 r_2^2 + 2r_3^4 r_2 w_1^2 + 2r_3^3 r_3' r_2 w_1 - r_3^2 w_1 + 2r_3'^2 w_1^2}{2 r_3 w_1^2},
\end{aligned} \tag{2.9}
$$

which admits a three-dimensional solvable Lie point symmetry algebra L_3 generated by the following operators:

$$
\Gamma_1 = \partial_t, \qquad \Gamma_2 = \frac{1}{2w_1}\left(r_3\partial_{r_3} - 2r_2\partial_{r_2} + 2w_1\partial_{w_1}\right), \qquad \Gamma_3 = -\frac{1}{2} r_3\partial_{r_3} + r_2\partial_{r_2}. \tag{2.10}
$$

This implies that we can reduce system (2.9) to a single first-order ODE. Indeed the differential invariants of order ≤ 1 of the algebra L_3, i.e.[7]:

$$
\tilde{u} = -\frac{-2r_3' w_1 + r_3^3 r_2}{2 r_3 w_1}, \qquad \tilde{y} = r_2 r_3^2 \tag{2.11}
$$

yield the following equation of first order:

$$
\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\tilde{y}} = \frac{\tilde{y}}{2\tilde{y}\tilde{u} - 1} \tag{2.12}
$$

which can be easily integrated in terms of Airy functions[8], i.e.:

$$
\frac{\tilde{u}\mathrm{AiryAi}(\tilde{u}^2 - \tilde{y}) + \mathrm{AiryAi}(1, \tilde{u}^2 - \tilde{y})}{\tilde{u}\mathrm{AiryBi}(\tilde{u}^2 - \tilde{y}) + \mathrm{AiryBi}(1, \tilde{u}^2 - \tilde{y})} = a_1, \tag{2.13}
$$

with a_1 an arbitrary constant. We notice that this equation suggests the change of independent variable $\tilde{Y} = \tilde{u}^2 - \tilde{y}$ into equation (2.12) which thus transforms into the following Riccati equation:

$$
\frac{\mathrm{d}\tilde{u}}{\mathrm{d}\tilde{Y}} = \tilde{u}^2 - \tilde{Y}, \tag{2.14}
$$

[5] Here and throughout this paper, we use our own interactive REDUCE program [22].

[6] At this stage, we have posed $a = -2$, but our choice is absolutely arbitrary [27], and, of course, does not affect the Lie symmetries admitted by the system.

[7] In the original variables w_i, $(i = 1, 2, 3)$:

$$
\tilde{u} = \frac{w_1^2 + w_2 + w_3}{2w_1}, \qquad \tilde{y} = w_2
$$

[8] We use MAPLE 9.

the general solution of which is indeed given by

$$\tilde{u} = -\frac{a_1 \mathrm{AiryAi}(1, \tilde{Y}) + \mathrm{AiryBi}(1, \tilde{Y})}{a_1 \mathrm{AiryAi}(\tilde{Y}) + \mathrm{AiryBi}(\tilde{Y})}. \tag{2.15}$$

It is remarkable that the Riccati equation (2.14) is quite similar to the known first integral of Riccati type (2.3), i.e.

$$w' = w^2 + kw - t, \tag{2.16}$$

although the dependent and independent variables are not the same.
Finally, in the original variables w_i, $(i = 1, 2, 3)$, formula (2.13) becomes:

$$\frac{(w_1^2 + w_2 + w_3)\mathrm{AiryAi}\,(\xi) + 2w_1 \mathrm{AiryAi}\,(1, \xi)}{(w_1^2 + w_2 + w_3)\mathrm{AiryBi}\,(\xi) + 2w_1 \mathrm{AiryBi}\,(1, \xi)} = a_1, \tag{2.17}$$

where

$$\xi = \frac{w_1^4 + 2(w_3 - w_2)w_1^2 + (w_3 + w_2)^2}{4w_1^2}, \tag{2.18}$$

namely:

$$\frac{(w^2 + w' + t)\mathrm{AiryAi}\,(\xi) + 2w\mathrm{AiryAi}\,(1, \xi)}{(w^2 + w' + t)\mathrm{AiryBi}\,(\xi) + 2w\mathrm{AiryBi}\,(1, \xi)} = a_1, \tag{2.19}$$

where

$$\xi = \frac{w^4 + 2(t - w')w^2 + (t + w')^2}{4w^2}. \tag{2.20}$$

3 Final remarks

There are many equations which apparently do not possess any Lie point symmetry. Yet they can be solved in terms of known functions, or a first integral is known. Equations (1.1) and (2.1) are such equations. One could write them as systems of first-order equations, which are known to possess an infinite number of elusive Lie point symmetries. The Jacobi last multiplier suggests a transformation which transforms a system of first-order equations into another system with at least one equation of second order. Thus, the admitted Lie symmetry algebra is finite, and hopefully, nontrivial.
The successful application of our method to equation (2.1) does not imply that we have found the ultimate method. Indeed more work shall be dedicated to this subject and we welcome any method which can shed light on the apparent "lack" of symmetries in so many equations of relevance in Physics.

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