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Lie algebra methods for the applications to the statistical theory of turbulence [1]

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Abstract

Approximate Lie symmetries of the Navier-Stokes equations are used for the applications to scaling phenomenon arising in turbulence. In particular, we show that the Lie symmetries of the Euler equations are inherited by the Navier-Stokes equations in the form of approximate symmetries that allows to involve the Reynolds number dependence into scaling laws. Moreover, the optimal systems of all finite-dimensional Lie subalgebras of the approximate symmetry transformations of the Navier-Stokes are constructed. We show how the scaling groups obtained can be used to introduce the Reynolds number dependence into scaling laws explicitly for stationary parallel turbulent shear flows. This is demonstrated in the framework of a new approach to derive scaling laws based on symmetry analysis [11]-[13].

1 Introduction

In this paper we develop and generalize the results devoted to approximate symmetry analysis of the Navier-Stokes equation obtained in [7]. The aim is, first, to transfer the results of calculations of approximate Lie symmetries of the Navier-Stokes equation to the case of arbitrary-order approximate symmetries and, then, to give their application to scaling phenomenon arising in the statistical theory of turbulence. We also present the procedure of construction of the so-called optimal systems of all finite-dimensional Lie subalgebras corresponding to the approximate group transformations obtained.

The basic idea of approximate symmetries, and of approximate transformation groups may be found in Fushchich and Shtelen [5], Euler et al [3], [4], Ibragimov, Baikov and

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Gazizov [1], [2]. We do not review in details the papers devoted to the concept of approximate symmetries which is a combination of the Symmetry Group Analysis of differential equations on the basis of Lie (Lie-Bäcklund) groups and the Theory of Perturbation of differential equations. The notion of approximate symmetry is used when the symmetry properties of equations involving a small parameters are studied. In the main two approaches reasonably well known are due to Fushchich, Shtelen and Euler on one hand and due to Baikov *et al* on the other and they both employ the perturbation techniques. In the first approach the dependent variables are expanded in a perturbation series and (a system of) equations under consideration are separated at each order of approximation. The exact symmetries of a coupled system obtained in the framework of the theory of perturbation are defined to be the approximate symmetries of the original (system of) equations with the small parameter. The second approach is quite different, the Lie operator is expanded only in a perturbation series so that an approximate infinitesimal operator can be constructed. Gazizov in [6] presented a criterion which enables us to connect these approaches. In general, symmetry operators obtained in the framework of developed by Fushchich, Shtelen and Euler and approximate symmetry operators in the sense by Baikov *et al* (based on the theory of approximate transformation groups) are not equivalent to each other. The corresponding examples are given by Gazizov [6]. The algorithm for a direct calculation of approximate symmetry operators is more difficult for a realization than in standard Lie method and it can be taken from [6].

We also mention the paper [17] wherein the comparison of the above-mentioned approaches are given in details. Moreover, the authors give another method for construction of approximate symmetries which is in consistence with the perturbation theory. Higher symmetries including a small parameters are studied in [18] to construct approximate solutions of perturbed equations. The reccurent relations for an expansion of symmetry by a small parameter are found for evolution equations with one spatial variable [18]).

Using the criterion obtained by Gazizov, we showed in [7] that the first-order approximate symmetries in the sense by Baikov *et al* of the Navier-Stokes equations can be derived by the exact symmetries of the coupled system coming out from the Navier-Stokes equations. The nice form of the coefficient functions to the infinitesimal operator of the first-order approximate system (coupled system) associated with the Navier-Stokes equations enabled us to obtain the above-mentioned result. Moreover, we proved that the Lie symmetries of the Euler equations are inherited by the Navier-Stokes equations in the form of approximate symmetries. This factum is especially important in the line of their application to the theory of turbulence. From the knowledge of the approximate symmetries of the Navier-Stokes equations which include two scaling symmetry operators we can obtain a broad variety of results for the turbulent flows since the symmetries of fluid motion are admitted by all statistical quantities of turbulent flow. In the context of Symmetry Group Methods the approach to derive certain turbulent scaling laws arising in the statistical theory of turbulence was given in [11]. In particular, it unifies a large set of scaling laws for the mean velocity of stationary parallel turbulent shear flows. The approach is derived from the Reynolds averaged Navier-Stokes equations, the fluctuations equations, and the velocity product equations, which are the dyad product of the velocity fluctuations with the equations for the velocity fluctuations. Therefore, it was shown that the knowledge of symmetries make it possibly to derive a family of scaling laws but these scaling laws are fixed by using the symmetries of the Euler equations. Reconsidering the derivation of the different scaling laws in [12] we note that the use of symmetries of the Navier-Stokes equations do not enable us to introduce the Reynolds number dependence into scaling laws explicitly. In fact, viscosity is symmetry breaking one scaling symmetry and as a consequence the entire scaling law theory will broke down. That is why it is seen to be important to pursue research on developing the theory of approximate transformation groups for the applications to the statistical theory of turbulence.

We do not review the results by Barenblatt and Chorin about investigation of the influence of the intermittency phenomenon on certain scaling laws presented by the von Kármán-Prandtl universal logarithmic law of the wall (in the intermediate region of wall-bounded turbulence), and the Kolmogorov-Obukhov scaling for the local structure of turbulence. We only mention that the concept of the so-called incomplete similarity and intermediate asymptotics was used to make a correction of the classical scaling laws when the Reynolds number is finite but large. The analysis extended the classical form of dependency between the velocity gradient and the spatial coordinate y, the shear stress at the wall τ, the pipe diameter d, the kinematic viscosity ν and density ρ without using the Navier-Stokes equations directly. For details see [14]–[16].

Our aim is to find approximate symmetries which to leading order correspond to the Euler equations but to higher order allows for the Reynolds number dependence of a turbulent motion. In Section 2, we calculate arbitrary-order approximate symmetries of the Navier-Stokes equations applying the theory of approximate symmetries, and the Navier-Stokes equations are considered as a perturbation of the Euler equations. We construct the so-called approximate Lie symmetry tangent vector field to the manifold defined by the Navier-Stokes equations which is motivated by their application to the theory of turbulence. In particular, we show that the Lie symmetries of the Euler equations are inherited by the Navier-Stokes equations in the form of approximate symmetries (arbitrary-order approximation). Moreover, the optimal systems of all finite-dimensional Lie subalgebras of the approximate symmetry transformation of the Navier-Stokes are presented in Section 3. In Section 4, we show how the scaling groups obtained can be used to introduce the Reynolds number dependence into scaling laws explicitly for stationary parallel turbulent shear flows.

2 Lie symmetries of the S-order approximate system for the Navier-Stokes equations

Let us consider the Navier- Stokes equation

$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} + \nabla p = \nu\Delta\vec{u}, \qquad \mathrm{div}\ \vec{u} = 0 \tag{2.1}$$

and the perturbation series for $\vec{u}$ and p in this null viscosity ν

$$\begin{aligned} u^\alpha(\vec{x},t) &= u_0^\alpha(\vec{x},t) + \sum_{s=1}^{S}\nu^s u_s^\alpha(\vec{x},t) + o(\nu^S), \quad s=1,\dots,S \\ p(\vec{x},t) &= p_0(\vec{x},t) + \sum_{s=1}^{S}\nu^s p_s(\vec{x},t) + o(\nu^S). \end{aligned} \tag{2.2}$$

Inserting these series into the Navier-Stokes equations, we obtain the S-order approximate system for the Navier-Stokes equations in the following denoted by *coupled system.* In a first step we present the exact Lie symmetry for this coupled system obtained. This symmetry is called S-order approximate symmetries of the Navier-Stokes equations. The infinitesimal operator for this system of any order approximation can be written in the following form

$$X_S = \xi^0 \frac{\partial}{\partial t} + \xi^i \frac{\partial}{\partial x^i} + \eta^{0,\alpha} \frac{\partial}{\partial u_0^\alpha} + \eta^{s,\alpha} \frac{\partial}{\partial u_s^\alpha} + \zeta^0 \frac{\partial}{\partial p_0} + \zeta^s \frac{\partial}{\partial p_s}, \quad s = 1, \ldots S. \tag{2.3}$$

Remark 1. We note that symmetries of a coupled system obtained in the framework of the reduction of differential equations with a small parameter using the expansion of the depending variables asymptotically in terms of a small parameter have been considered by Fushchich, Shtelen and Euler *et al.* in [3] – [5] as approximate symmetries of the corresponding differential equations with the small parameter.

In the present section we are primarily interested in the calculation of exact symmetry of the coupled system for the Navier-Stokes equations. However, that this is in view of finding the S-order approximate infinitesimal operator to be derived in the next section. We expand a solution $(\vec{u}, p)$ in a perturbation series according to (2.2) to obtain the coupled system of equations for finding $(\vec{u}_0, \ldots, \vec{u}_S)$, and $(p_0, \ldots, p_S)$

$$\nu^0 : \quad \vec{u}_{0t} + (\vec{u}_0 \cdot \nabla)\vec{u}_0 + \nabla p_0 = 0, \qquad \operatorname{div} \vec{u}_0 = 0. \tag{2.4}$$

$$\nu^s : \quad \vec{u}_{st} + \sum_{k=0}^{s} (\vec{u}_k \cdot \nabla)\vec{u}_{s-k} + \nabla p_s = \Delta \vec{u}_{s-1}, \qquad \operatorname{div} \vec{u}_s = 0, \quad s = 1, \ldots, S. \tag{2.5}$$

The corresponding infinitesimal operator is due to (2.3) and the corresponding prolongation of this operator can be written in the form

$$\tilde{X}_S = X_1 + \eta_m^{0,\alpha} \frac{\partial}{\partial u_{0,m}^\alpha} + \eta_m^{s,\alpha} \frac{\partial}{\partial u_{s,m}^\alpha} + \zeta_m^0 \frac{\partial}{\partial p_{0,m}} + \zeta_m^s \frac{\partial}{\partial p_{s,m}} + \eta_{mn}^{0,\alpha} \frac{\partial}{\partial u_{0,mn}^\alpha} + \eta_{mn}^{s,\alpha} \frac{\partial}{\partial u_{s,mn}^\alpha}, \tag{2.6}$$

where $\eta_m^{s,\alpha} = D_m(\eta^{s,\alpha}) - u_{s,j}^\alpha D_m(\xi^j)$, $\zeta_m^s = D_m(\zeta^s) - p_{s,j} D_m(\xi^j)$, $\eta_{mn}^{s,\alpha} = D_n(\eta_m^{s,\alpha}) - u_{s,mr}^\alpha D_n(\xi^r)$, $s = 0, 1, \ldots, S$. Here D_m denotes the total derivative operator

$$D_m = \frac{\partial}{\partial x^m} + u_{s,m}^\alpha \frac{\partial}{\partial u_s^\alpha} + u_{s,mn}^\beta \frac{\partial}{\partial u_{s,n}^\beta} + p_{s,m} \frac{\partial}{\partial p_s} + p_{s,mn} \frac{\partial}{\partial p_{s,n}} + \cdots .$$

Applying the operator $\tilde{X}_S$ to the coupled system (2.4), (2.5) we find that the smooth coefficients ξ^0, ξ^1, $\eta^{0,i}$, $\eta^{s,i}$, ζ^0 and ζ^s, $s = 1, \ldots, S$ are given by

$$\begin{aligned}
\xi^0 &= (2a_0 + b_0)t + c_0, \\
\xi^i &= (a_0 + b_0)x^i + a_{ij}x^j + h_j(t), \\
\eta^{0,i} &= -a_0 u_0^i + a_{ij} u_0^j + h_j'(t), \\
\eta^{s,i} &= -a_0 u_s^i + a_{ij} u_s^j - s b_0 u_1^i, \\
\zeta^0 &= -2a_0 p_0 - x^i h_i''(t) + g^0(t), \\
\zeta^s &= -2a_0 p_s + g^s(t) - s b_0 p_s,
\end{aligned} \tag{2.7}$$

where a_0, b_0, c_0 are arbitrary constants, the numbers a_{ij} are connected by the relationships $a_{ii} = 0$, $a_{ij} + a_{ji} = 0$ for $i \neq j$ and h_j, g^0, g^s are arbitrary smooth functions of the variable t. We note that the functions ξ^0, ξ^i, $\eta^{0,i}$, ζ^0 coincide with the coefficient functions of the infinitesimal operator for the Euler equations.

To adopt the symmetry operator (2.3), (2.7) for their application in turbulence, we need to rewrite (approximately) this operator (or the Lie symmetry vector field) in the original variables $(\partial/\partial t, \partial/\partial x^i, \partial/\partial u^\alpha, \partial/\partial p)$. For this aim we use the concept of Approximate Group Transformations by Ibragimov, Baikov and Gazizov [1], [2] wherein the infinitesimal operator is expanded in a perturbation series with the small parameter ν.

3 Approximate Lie symmetry of the Navier-Stokes equations

Following the paper [6], we consider a family G of invertible transformations

$$\begin{aligned}
&\bar{x}^i \approx \omega^i(t,\vec{x},\vec{u},p,a;\nu) \equiv \omega_0^i(t,\vec{x},\vec{u},p,a) + \nu\omega_1^i(t,\vec{x},\vec{u},p,a) + \cdots + \nu^S\omega_S^i(t,\vec{x},\vec{u},p,a) + o(\nu^S),\\
&\bar{t} \approx \lambda(t,\vec{x},\vec{u},p,a;\nu) \equiv \lambda_0(t,\vec{x},\vec{u},p,a) + \nu\lambda_1(t,\vec{x},\vec{u},p,a) + \cdots + \nu^S\lambda_S(t,\vec{x},\vec{u},p,a) + o(\nu^S),\\
&\bar{u}^\alpha \approx \tau^\alpha(t,\vec{x},\vec{u},p,a;\nu) \equiv \tau_0^\alpha(t,\vec{x},\vec{u},p,a) + \nu\tau_1^\alpha(t,\vec{x},\vec{u},p,a) + \cdots + \nu^S\tau_S^\alpha(t,\vec{x},\vec{u},p,a) + o(\nu^S),\\
&\bar{p} \approx \mu(t,\vec{x},\vec{u},p,a;\nu) \equiv \mu_0(t,\vec{x},\vec{u},p,a) + \nu\mu_1(t,\vec{x},\vec{u},p,a) + \cdots + \nu\mu_S(t,\vec{x},\vec{u},p,a) + o(\nu^S),\\
&\bar{\nu} \approx \nu\theta(a;\nu) \equiv \nu\theta_1(a) + \cdots + \nu^S\theta_S(a) + o(\nu^S),
\end{aligned}$$

where $f \approx g$ means that $f - g = o(\nu^S)$, $|o(\nu^S)| \leq C\nu^{S+1}$. According to the theory of approximate Lie symmetries (see, for example [1]), the S-order approximate infinitesimal operator can be written in the form

$$\begin{aligned}
X_S^{appr} &= [\xi_{(0)}^0(t,\vec{x},\vec{u},p) + \nu\xi_{(1)}^0(t,\vec{x},\vec{u},p) + \cdots + \nu^S\xi_{(S)}^0(t,\vec{x},\vec{u},p)]\frac{\partial}{\partial t}\\
&+ [\xi_{(0)}^i(t,\vec{x},\vec{u},p) + \nu\xi_{(1)}^i(t,\vec{x},\vec{u},p) + \cdots + \nu^S\xi_{(S)}^i(t,\vec{x},\vec{u},p)]\frac{\partial}{\partial x^i}\\
&+ [\eta_{(0)}^\alpha(t,\vec{x},\vec{u},p) + \nu\eta_{(1)}^\alpha(t,\vec{x},\vec{u},p) + \cdots + \nu^S\eta_{(S)}^\alpha(t,\vec{x},\vec{u},p)]\frac{\partial}{\partial u^\alpha}\\
&+ [\zeta_{(0)}(t,\vec{x},\vec{u},p) + \nu\zeta_{(1)}(t,\vec{x},\vec{u},p) + \cdots + \nu^S\zeta_{(S)}(t,\vec{x},\vec{u},p)]\frac{\partial}{\partial p} + [\nu\kappa_1 + \cdots + \nu^S\kappa_S]\frac{\partial}{\partial \nu},
\end{aligned}$$

where

$$\begin{aligned}
\xi_{(s)}^0 &= \left.\frac{\partial\lambda_{(s)}}{\partial a}\right|_{a=0}, \quad \xi_{(s)}^i = \left.\frac{\partial\omega_{(s)}^i}{\partial a}\right|_{a=0},\\
\eta_{(s)}^\alpha &= \left.\frac{\partial\tau_{(s)}^\alpha}{\partial a}\right|_{a=0}, \quad \zeta_{(s)} = \left.\frac{\partial\mu_{(s)}}{\partial a}\right|_{a=0}, \quad \kappa_s = \left.\frac{\partial\theta_s}{\partial a}\right|_{a=0}, \quad s = 1,\ldots,S.
\end{aligned}$$

An algorithm for the direct calculation of the coefficients of X_S^{appr} can be taken from [1].

In the case of $S = 1$, the following assertion establishes a relationship between the operators X_1 and X_1^{appr}:

The operator X_1 of an exact symmetry of the coupled system (2.4) can be rewritten in the form of an approximate infinitesimal operator X_1^{appr} if and only if it has the form

$$\begin{aligned} X_1 &= \xi^0_{(0)}\frac{\partial}{\partial t} + \xi^i_{(0)}\frac{\partial}{\partial x^i} + \eta^\alpha_{(0)}\frac{\partial}{\partial u^\alpha_0} + \zeta_{(0)}\frac{\partial}{\partial p_0} \\ &+ [\eta^\alpha_{(1)} - \xi^j_{(1)}u^\alpha_{0,j} + \frac{\partial \eta^\alpha_{(0)}}{\partial u^\beta_0}u^\beta_1 - \frac{\partial \xi^j_{(0)}}{\partial u^\beta_0}u^\alpha_{0,j}u^\beta_1 - \kappa_1 u^\alpha_1]\frac{\partial}{\partial u^\alpha_1} \\ &+ [\zeta_{(1)} - \xi^j_{(1)}p_{0,j} + \frac{\partial \zeta_0}{\partial p_0}p_1 - \frac{\partial \xi^j_{(0)}}{\partial p_0}p_{0,j}p_1 - \kappa_1 p_1]\frac{\partial}{\partial p_1} \end{aligned} \tag{3.1}$$

which is obtained by substituting (2.2) for $S = 1$ (the first-order approximation) into the operator X_1^{appr} and expanding the coefficient functions into Taylor series (for details see Theorem 1 [6]). It is worthwhile noticing that, thanks to the nice form of the coefficient functions obtained in (2.7) of the operator X_1 we are able to apply (3.1) for calculating X_1^{appr}. Indeed, comparing (2.7) and (3.1) we obtain that

$$\xi^i_{(1)} \equiv 0, \quad \eta^\alpha_{(1)} \equiv 0, \quad \zeta_{(1)} = g^1(t), \quad \kappa_1 = b_0. \tag{3.2}$$

taking into account that (see (2.7))

$$\frac{\partial \xi^j_{(0)}}{\partial u^\beta_0} = 0, \quad \frac{\partial \xi^j_{(0)}}{\partial p_0} = 0, \quad \frac{\partial \eta^\alpha_{(0)}}{\partial u^\beta_0} = a_{\alpha\beta} \quad (\beta \neq \alpha), \quad \frac{\partial \eta^\alpha_{(0)}}{\partial u^\alpha_0} = -a_0, \quad \frac{\partial \zeta_0}{\partial p_0} = -2a_0.$$

As a result, we obtain that the operator X_1 is transformed to

$$X_1^{appr} = \xi^0\frac{\partial}{\partial t} + \xi^i\frac{\partial}{\partial x^i} + \eta^{0,\alpha}\frac{\partial}{\partial u^\alpha} + [\zeta^0 + \nu g^1(t)]\frac{\partial}{\partial p} + \nu b_0\frac{\partial}{\partial \nu}. \tag{3.3}$$

Moreover, this operator is admitted by the Navier-Stokes equations in the sense of the first-order approximation of the theory of approximate transformation groups and the operator X_0 (unperturbed term of X_1^{appr})

$$X_0 = \xi^0\frac{\partial}{\partial t} + \xi^i\frac{\partial}{\partial x^i} + \eta^{0,\alpha}\frac{\partial}{\partial u^\alpha} + \zeta^0\frac{\partial}{\partial p}$$

coincides with the infinitesimal operator for the Euler equations. Therefore we showed that the operator X_0 is inherited [8] by the Navier-Stokes equations in the form of approximate symmetry (3.3) (see, [7]).

Remark 2. We note that an infinitesimal operator admitted by an unperturbed equation cannot be always extended in the form of approximate symmetry operator of the perturbed equation under consideration, see [2], [6].

After observing that the first-order approximate Lie symmetries of the Navier-Stokes equations can be easily calculated by using the exact symmetries obtained of the corresponding coupled system, we repeat this derivation to manage with the general case of arbitrary-order approximation. In fact, we need only to generalize the formula (3.1) for arbitrary integer numbers S. In order to reduce calculation, we present this derivation for a non-point approximate operator [6] of the form

$$\begin{aligned} Y_S^{appr} &= \left[f^\alpha_{(0)}(\vec{x},\vec{w},\vec{w}_x,\dots)+\nu f^\alpha_{(1)}(\vec{x},\vec{w},\vec{w}_x,\dots)+\dots+\nu^S f^\alpha_{(S)}(\vec{x},\vec{w},\vec{w}_x,\dots)\right]\frac{\partial}{\partial w^\alpha} \\ &+ \left[\nu\kappa_1+\dots+\nu^S\kappa_S\right]\frac{\partial}{\partial \nu}, \end{aligned} \tag{3.4}$$

where $\vec{w}=(\vec{u},p)$ and then give "a point symmetry version". With the help of Taylor formula

$$\begin{aligned} f(\vec{y}_0 &+ \nu\vec{y}_1+\dots+\nu^S\vec{y}_S) = f(\vec{y}_0)+\nu\frac{\partial f(\vec{y}_0)}{\partial y_0^{\beta_1}}y_1^{\beta_1}+\nu^2\left[\frac{\partial f(\vec{y}_0)}{\partial y_0^{\beta_1}}y_2^{\beta_1}+\frac{1}{2}\frac{\partial^2 f(\vec{y}_0)}{\partial y_0^{\beta_1}\partial y_0^{\beta_2}}y_1^{\beta_1}y_1^{\beta_2}\right] \\ &+ \nu^3\left[\frac{\partial f(\vec{y}_0)}{\partial y_0^{\beta_1}}y_3^{\beta_1}+\frac{1}{2}\frac{\partial^2 f(\vec{y}_0)}{\partial y_0^{\beta_1}\partial y_0^{\beta_2}}(y_1^{\beta_1}y_2^{\beta_2}+y_1^{\beta_2}y_2^{\beta_1})+\frac{1}{3}\frac{\partial^3 f(\vec{y}_0)}{\partial y_0^{\beta_1}\partial y_0^{\beta_2}\partial y_0^{\beta_3}}y_1^{\beta_1}y_1^{\beta_2}y_1^{\beta_3}\right] \quad (3.5) \\ &\vdots \\ &+ \nu^S\left[\sum_{|\sigma|=1}^{S}\frac{1}{\sigma!}\frac{\partial^{|\sigma|}f(\vec{y}_0)}{(\partial y_0^1)^{\sigma_1}\dots(\partial y_0^N)^{\sigma_N}}\sum_{|\mu|=S}y_{(\mu)}\right]+o(\nu^S), \end{aligned}$$

where $\sigma=(\sigma_1,\dots\sigma_N)$ is a multi-index; $|\sigma|=\sigma_1+\dots+\sigma_N$, $\sigma!=\sigma_1!\dots\sigma_N!$, $\sigma_k=0,\dots,S$; $y_{(\mu)}=y^1_{(\mu_1)}\dots y^N_{(\mu_N)}$ and

$$y^k_{(\mu_k)}=\sum_{i_1+\dots+i_{\sigma_k}=\mu_k}y^k_{i_1}\dots y^k_{i_{\sigma_k}}.$$

Here $\mu=(\mu_1,\dots,\mu_N)$ is a multi-index associated with σ (see [2]) and we can write

$$\begin{aligned} & f^\alpha_{(0)}(\vec{x},\vec{w}_0+\nu\vec{w}_1+\dots+\nu^S\vec{w}_S,\vec{w}_{0x}+\nu\vec{w}_{1x}+\dots+\nu^S\vec{w}_{Sx},\dots) \\ =\; & f^\alpha_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)+\nu\left[\frac{\partial f^\alpha_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_0^{\beta_1}}w_1^{\beta_1}+\frac{\partial f^\alpha_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}}w_{1j_1}^{\beta_1}+\dots\right] \\ +\; & \nu^2\left[\frac{\partial f^\alpha_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_0^{\beta_1}}w_2^{\beta_1}+\frac{1}{2}\frac{\partial^2 f^\alpha_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_0^{\beta_1}\partial w_0^{\beta_2}}w_1^{\beta_1}w_1^{\beta_2}\right. \\ +\; & \left.\frac{\partial f^\alpha_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}}w_{2j_1}^{\beta_1}+\frac{1}{2}\frac{\partial^2 f^\alpha_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}\partial w_{0j_2}^{\beta_2}}w_{1j_1}^{\beta_1}w_{1j_2}^{\beta_2}+\dots\right] \end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&+\ \nu^3\left[\frac{\partial f^{\alpha}_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_0^{\beta_1}}w_3^{\beta_1}+\frac{1}{2}\frac{\partial^2 f^{\alpha}_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_0^{\beta_1}\partial w_0^{\beta_2}}(w_1^{\beta_1}w_2^{\beta_2}+w_1^{\beta_2}w_2^{\beta_1})\right.\\
&+\ \frac{1}{3}\frac{\partial^3 f^{\alpha}_{(0)}(\vec{u}_0,\vec{w}_{0x},\dots)}{\partial w_0^{\beta_1}\partial w_0^{\beta_2}\partial w_0^{\beta_3}}w_1^{\beta_1}w_1^{\beta_2}w_1^{\beta_3}+\frac{\partial f^{\alpha}_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}}w_{3j_1}^{\beta_1}\\
&+\ \left.\frac{1}{2}\frac{\partial^2 f^{\alpha}_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}\partial w_{0j_2}^{\beta_2}}(w_{1j_1}^{\beta_1}w_{2j_2}^{\beta_2}+w_{1j_2}^{\beta_2}w_{2j_1}^{\beta_1})+\frac{1}{3}\frac{\partial^3 f^{\alpha}_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}\partial w_{0j_2}^{\beta_2}\partial w_{0j_3}^{\beta_3}}w_{1j_1}^{\beta_1}w_{1j_2}^{\beta_2}w_{1j_3}^{\beta_3}+\dots\right]\\
&\ \vdots\\
&+\ \nu^S\left[\sum_{|\sigma|=1}^{S}\frac{1}{\sigma!}\frac{\partial^{|\sigma|}f^{\alpha}_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{(\partial w_0^1)^{\sigma_1}\dots(\partial w_0^N)^{\sigma_N}}\sum_{|\mu|=S}w_{(\mu)}+\dots\right]+o(\nu^S)
\end{aligned}
$$

where the lower index $j_k=1,\dots,n$ denotes the corresponding partial derivative of the functions w_s^{γ}, $\gamma=1,\dots N$ and $s=0,\dots S$. The symbol $\dots$ denotes the omitted terms arising due to the Taylor formula and can be easily reconstructed. For the applications to the coupled system (2.4),(2.5) these terms are not used in further calculations due to the linear functional dependence of the coefficients $\eta^{0,i}$, $\eta^{s,i}$, ζ^0 and ζ^s on u_s and p_s in the formula (2.7). Repeating the formula (3.6) for each term $\nu^s f_s^{\alpha}$, $s=1,\dots S$ and using the perturbation series for $\vec{u}$, p, we can rewrite the terms in square brackets of the operator

$$
Y_{1,S}^{appr}=\left[f^{\alpha}_{(0)}(\vec{x},\vec{w},\vec{w}_x,\dots)+\nu f^{\alpha}_{(1)}(\vec{x},\vec{w},\vec{w}_x,\dots)+\dots+\nu^S f^{\alpha}_{(S)}(\vec{x},\vec{w},\vec{w}_x,\dots)\right]\frac{\partial}{\partial w^{\alpha}},
$$

in the form

$$
\begin{aligned}
&f^{\alpha}_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)+\nu\left[f^{\alpha}_{(1)}(\vec{w}_0,\vec{w}_{0x},\dots)\right.\\
&+\ \left.\frac{\partial f^{\alpha}_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_0^{\beta_1}}w_1^{\beta_1}+\frac{\partial f^{\alpha}_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}}w_{1j_1}^{\beta_1}+\dots\right]\\
&+\ \nu^2\left[f^{\alpha}_{(2)}(\vec{w}_0,\vec{w}_{0x},\dots)+\frac{\partial f^{\alpha}_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_0^{\beta_1}}w_2^{\beta_1}+\frac{1}{2}\frac{\partial^2 f^{\alpha}_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_0^{\beta_1}\partial w_0^{\beta_2}}w_1^{\beta_1}w_1^{\beta_2}\right.\\
&+\ \frac{\partial f^{\alpha}_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}}w_{2j_1}^{\beta_1}+\frac{1}{2}\frac{\partial^2 f^{\alpha}_{(0)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}\partial w_{0j_2}^{\beta_2}}w_{1j_1}^{\beta_1}w_{1j_2}^{\beta_2}+\dots+\frac{\partial f^{\alpha}_{(1)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_0^{\beta_1}}w_1^{\beta_1}\\
&+\ \left.\frac{\partial f^{\alpha}_{(1)}(\vec{w}_0,\vec{w}_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}}w_{2j_1}^{\beta_1}+\dots\right] \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
&+\ \nu^3\Bigg[f^\alpha_{(3)}(\vec w_0,\vec w_{0x},\dots)+\frac{\partial f^\alpha_{(0)}(\vec w_0,\vec w_{0x},\dots)}{\partial w_0^{\beta_1}}w_3^{\beta_1}+\frac{1}{2}\frac{\partial^2 f^\alpha_{(0)}(\vec w_0,\vec w_{0x},\dots)}{\partial w_0^{\beta_1}\partial w_0^{\beta_2}}(w_1^{\beta_1}w_2^{\beta_2}+w_1^{\beta_2}w_2^{\beta_1})\\
&+\ \frac{1}{3}\frac{\partial^3 f^\alpha_{(0)}(\vec u_0,\vec w_{0x},\dots)}{\partial w_0^{\beta_1}\partial w_0^{\beta_2}\partial w_0^{\beta_3}}w_1^{\beta_1}w_1^{\beta_2}w_1^{\beta_3}+\frac{\partial f^\alpha_{(0)}(\vec w_0,\vec w_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}}w_{3j_1}^{\beta_1}\\
&+\ \frac{1}{2}\frac{\partial^2 f^\alpha_{(0)}(\vec w_0,\vec w_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}\partial w_{0j_2}^{\beta_2}}(w_{1j_1}^{\beta_1}w_{2j_2}^{\beta_2}+w_{1j_2}^{\beta_2}w_{2j_1}^{\beta_1})+\frac{1}{3}\frac{\partial^3 f^\alpha_{(0)}(\vec w_0,\vec w_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}\partial w_{0j_2}^{\beta_2}\partial w_{0j_3}^{\beta_3}}w_{1j_1}^{\beta_1}w_{1j_2}^{\beta_2}w_{1j_3}^{\beta_3}+\dots\\
&+\ \frac{\partial f^\alpha_{(2)}(\vec w_0,\vec w_{0x},\dots)}{\partial w_0^{\beta_1}}w_1^{\beta_1}+\frac{\partial f^\alpha_{(2)}(\vec w_0,\vec w_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}}w_{1j_1}^{\beta_1}+\dots+\frac{\partial f^\alpha_{(1)}(\vec w_0,\vec w_{0x},\dots)}{\partial w_0^{\beta_1}}w_2^{\beta_1}\\
&+\ \frac{1}{2}\frac{\partial^2 f^\alpha_{(1)}(\vec w_0,\vec w_{0x},\dots)}{\partial w_0^{\beta_1}\partial w_0^{\beta_2}}w_1^{\beta_1}w_1^{\beta_2}+\frac{\partial f^\alpha_{(1)}(\vec w_0,\vec w_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}}w_{2j_1}^{\beta_1}+\frac{1}{2}\frac{\partial^2 f^\alpha_{(1)}(\vec w_0,\vec w_{0x},\dots)}{\partial w_{0j_1}^{\beta_1}\partial w_{0j_2}^{\beta_2}}w_{1j_1}^{\beta_1}w_{1j_2}^{\beta_2}\dots\Bigg]\\
&\ \ \vdots\\
&+\ \nu^S\left[f^\alpha_{(S)}(\vec x,\vec w,\vec w_x,\dots)+\sum_{s=1}^{S}\sum_{|\sigma|=1}^{s}\frac{1}{\sigma!}\frac{\partial^{|\sigma|}f^\alpha_{(S-s)}(\vec w_0,\vec w_{0x},\dots)}{(\partial w_0^1)^{\sigma_1}\dots(\partial w_0^N)^{\sigma_N}}\sum_{|\mu|=s}w_{(\mu)}+\dots\right]+o(\nu^S).
\end{aligned}
$$

Denote by

$$
Y_{2,S}^{appr}=\left[\nu\kappa_1+\dots+\nu^S\kappa_S\right]\frac{\partial}{\partial\nu} \tag{3.8}
$$

the second operator in the formula (3.4). Direct calculations show that

$$
\frac{\partial}{\partial w^\alpha}=\nu^{-s}\frac{\partial}{\partial w_s^\alpha},\qquad s=0,\dots S,
$$

and

$$
\begin{aligned}
\nu\kappa_1\frac{\partial}{\partial\nu}&=-\kappa_1\left(w_1^\alpha\frac{\partial}{\partial w_1^\alpha}+2w_2^\alpha\frac{\partial}{\partial w_2^\alpha}+\dots+Sw_S^\alpha\frac{\partial}{\partial w_S^\alpha}\right)\\
\nu^2\kappa_2\frac{\partial}{\partial\nu}&=-\kappa_2\left(w_1^\alpha\frac{\partial}{\partial w_2^\alpha}+2w_2^\alpha\frac{\partial}{\partial w_3^\alpha}+\dots+Sw_{S+1}^\alpha\frac{\partial}{\partial w_{S+1}^\alpha}\right)\\
&\ \ \vdots\\
\nu^S\kappa_S\frac{\partial}{\partial\nu}&=-\kappa_S\left(w_1^\alpha\frac{\partial}{\partial w_S^\alpha}+2w_2^\alpha\frac{\partial}{\partial w_{S+1}^\alpha}+\dots+Sw_{2S-1}^\alpha\frac{\partial}{\partial w_{2S-1}^\alpha}\right).
\end{aligned} \tag{3.9}
$$

Considering only a finite number of terms of a given order of approximation in the non-point symmetry operator (3.4), we obtain the following constraints on the function θ:

$$
\kappa_s=\left.\frac{\partial\theta_s}{\partial a}\right|_{a=0}\equiv 0,\quad s=2,\dots,S. \tag{3.10}
$$

Then the operator Y_S^{appr} in the variables w_s^α is transformed to

$$\begin{aligned}
X_S = {} & f^\alpha_{(0)}(\vec{w}_0, \vec{w}_{0x}, \dots)\frac{\partial}{\partial w_0^\alpha} + \Big[f^\alpha_{(1)}(\vec{w}_0, \vec{w}_{0x}, \dots) \\
+ {} & \frac{\partial f^\alpha_{(0)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_0^{\beta_1}} w_1^{\beta_1} + \frac{\partial f^\alpha_{(0)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_{0j_1}^{\beta_1}} w_{1j_1}^{\beta_1} + \dots - \kappa_1 w_1^\alpha \Big] \frac{\partial}{\partial w_1^\alpha} \\
+ {} & \Big[f^\alpha_{(2)}(\vec{w}_0, \vec{w}_{0x}, \dots) + \frac{\partial f^\alpha_{(0)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_0^{\beta_1}} w_2^{\beta_1} + \frac{1}{2}\frac{\partial^2 f^\alpha_{(0)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_0^{\beta_1}\partial w_0^{\beta_2}} w_1^{\beta_1} w_1^{\beta_2} \\
+ {} & \frac{\partial f^\alpha_{(0)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_{0j_1}^{\beta_1}} w_{2j_1}^{\beta_1} + \frac{1}{2}\frac{\partial^2 f^\alpha_{(0)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_{0j_1}^{\beta_1}\partial w_{0j_2}^{\beta_2}} w_{1j_1}^{\beta_1} w_{1j_2}^{\beta_2} + \dots + \frac{\partial f^\alpha_{(1)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_0^{\beta_1}} w_1^{\beta_1} \\
+ {} & \frac{\partial f^\alpha_{(1)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_{0j_1}^{\beta_1}} w_{2j_1}^{\beta_1} + \dots - 2\kappa_2 w_2^\alpha \Big] \frac{\partial}{\partial w_2^\alpha} \qquad (3.11) \\
+ {} & \Big[f^\alpha_{(3)}(\vec{w}_0, \vec{w}_{0x}, \dots) + \frac{\partial f^\alpha_{(0)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_0^{\beta_1}} w_3^{\beta_1} + \frac{1}{2}\frac{\partial^2 f^\alpha_{(0)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_0^{\beta_1}\partial w_0^{\beta_2}} (w_1^{\beta_1} w_2^{\beta_2} + w_1^{\beta_2} w_2^{\beta_1}) \\
+ {} & \frac{1}{3}\frac{\partial^3 f^\alpha_{(0)}(\vec{u}_0, \vec{w}_{0x}, \dots)}{\partial w_0^{\beta_1}\partial w_0^{\beta_2}\partial w_0^{\beta_3}} w_1^{\beta_1} w_1^{\beta_2} w_1^{\beta_3} + \frac{\partial f^\alpha_{(0)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_{0j_1}^{\beta_1}} w_{3j_1}^{\beta_1} \\
+ {} & \frac{1}{2}\frac{\partial^2 f^\alpha_{(0)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_{0j_1}^{\beta_1}\partial w_{0j_2}^{\beta_2}} (w_{1j_1}^{\beta_1} w_{2j_2}^{\beta_2} + w_{1j_2}^{\beta_2} w_{2j_1}^{\beta_1}) + \frac{1}{3}\frac{\partial^3 f^\alpha_{(0)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_{0j_1}^{\beta_1}\partial w_{0j_2}^{\beta_2}\partial w_{0j_3}^{\beta_3}} w_{1j_1}^{\beta_1} w_{1j_2}^{\beta_2} w_{1j_3}^{\beta_3} + \dots \\
+ {} & \frac{\partial f^\alpha_{(2)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_0^{\beta_1}} w_1^{\beta_1} + \frac{\partial f^\alpha_{(2)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_{0j_1}^{\beta_1}} w_{1j_1}^{\beta_1} + \dots + \frac{\partial f^\alpha_{(1)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_0^{\beta_1}} w_2^{\beta_1} \\
+ {} & \frac{1}{2}\frac{\partial^2 f^\alpha_{(1)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_0^{\beta_1}\partial w_0^{\beta_2}} w_1^{\beta_1} w_1^{\beta_2} + \frac{\partial f^\alpha_{(1)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_{0j_1}^{\beta_1}} w_{2j_1}^{\beta_1} \\
+ {} & \frac{1}{2}\frac{\partial^2 f^\alpha_{(1)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{\partial w_{0j_1}^{\beta_1}\partial w_{0j_2}^{\beta_2}} w_{1j_1}^{\beta_1} w_{1j_2}^{\beta_2} + \dots - 3\kappa_3 w_3^\alpha \Big] \frac{\partial}{\partial w_3^\alpha} \\
\vdots \\
+ {} & \Big[f^\alpha_{(S)}(\vec{x}, \vec{w}, \vec{w}_x, \dots) + \sum_{s=1}^{S}\sum_{|\sigma|=1}^{s} \frac{1}{\sigma!}\frac{\partial^{|\sigma|} f^\alpha_{(S-s)}(\vec{w}_0, \vec{w}_{0x}, \dots)}{(\partial w_0^1)^{\sigma_1}\dots(\partial w_0^N)^{\sigma_N}} \sum_{|\mu|=s} w_{(\mu)} + \dots - S\kappa_S w_S^\alpha \Big] \frac{\partial}{\partial w_S^\alpha}.
\end{aligned}$$

Therefore we are able to prove the following extension of Gazizov's result (see Theorem 1 [6]).

Let a non-point approximate symmetry operator be of the form

$$\begin{aligned}
Y_S^{appr} &= \Big[f^\alpha_{(0)}(\vec{x}, \vec{w}, \vec{w}_x, \dots) + \nu f^\alpha_{(1)}(\vec{x}, \vec{w}, \vec{w}_x, \dots) + \dots + \nu^S f^\alpha_{(S)}(\vec{x}, \vec{w}, \vec{w}_x, \dots) \Big] \frac{\partial}{\partial w^\alpha} \\
&+ \nu\kappa_1 \frac{\partial}{\partial \nu}, \qquad (3.12)
\end{aligned}$$

then the operator (3.12) takes the form (3.11) in the variables w_s^α. Moreover, if the equation under consideration approximately admits the operator Y_S^{appr}, then the corresponding coupled system admits the operator X_S. The proof directly follows from the definition of approximate infinitesimal operator [1], [2], substitution of the perturbation series for $\vec{w}$ into the original equation and separation with respect to ν exploiting the results of calculations done above.

If (3.12) is an approximate Lie symmetry operator i.e. it has the form

$$\begin{aligned} X_S^{appr} &= [\xi^0_{(0)}(t,\vec{x},\vec{w}) + \nu\xi^0_{(1)}(t,\vec{x},\vec{w}) + \cdots + \nu^S\xi^0_{(S)}(t,\vec{x},\vec{w})]\frac{\partial}{\partial t} \\ &+ [\xi^i_{(0)}(t,\vec{x},\vec{w}) + \nu\xi^i_{(1)}(t,\vec{x},\vec{w}) + \cdots + \nu^S\xi^i_{(S)}(t,\vec{x},\vec{w})]\frac{\partial}{\partial x^i} \\ &+ [\eta^\alpha_{(0)}(t,\vec{x},\vec{w}) + \nu\eta^\alpha_{(1)}(t,\vec{x},\vec{w}) + \cdots + \nu^S\eta^\alpha_{(S)}(t,\vec{x},\vec{w})]\frac{\partial}{\partial w^\alpha} + \nu\kappa_1\frac{\partial}{\partial \nu}, \end{aligned}$$

then the corresponding operator (3.11) are presented in the form

$$\begin{aligned} X_S &= \xi^0_{(0)}(t,\vec{x},\vec{w}_0)\frac{\partial}{\partial t} + \xi^i_{(0)}(t,\vec{x},\vec{w}_0)\frac{\partial}{\partial x^i} + \eta^\alpha_{(0)}(t,\vec{x},\vec{w}_0)\frac{\partial}{\partial w^\alpha_0} \\ &+ \Bigg[\eta^\alpha_{(1)}(t,\vec{x},\vec{w}_0) - \xi^j_{(1)}(t,\vec{x},\vec{w}_0)w^\alpha_{0j} + \frac{\partial\eta^\alpha_{(0)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0}w^{\beta_1}_1 \\ &- \frac{\partial\xi^j_{(0)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0}w^{\beta_1}_1 w^\alpha_{0j} - \kappa_1 w^\alpha_1\Bigg]\frac{\partial}{\partial w^\alpha_1} \end{aligned} \tag{3.13}$$

$$\begin{aligned} &+ \Bigg[\eta^\alpha_{(2)}(t,\vec{x},\vec{w}_0) - \xi^j_{(2)}(t,\vec{x},\vec{w}_0)w^\alpha_{0j} + \frac{\partial\eta^\alpha_{(0)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0}w^{\beta_1}_2 + \frac{\partial\xi^j_{(0)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0}w^{\beta_1}_2 w^\alpha_{0j} \\ &+ \frac{1}{2}\frac{\partial^2\eta^\alpha_{(0)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0\partial w^{\beta_2}_0}w^{\beta_1}_1 w^{\beta_2}_1 - \frac{1}{2}\frac{\partial^2\xi^j_{(0)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0\partial w^{\beta_2}_0}w^{\beta_1}_1 w^{\beta_2}_1 w^\alpha_{0j} + \frac{\partial\eta^\alpha_{(1)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0}w^{\beta_1}_1 \\ &- \frac{\partial\xi^j_{(1)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0}w^{\beta_1}_1 w^\alpha_{0j} - 2\kappa_2 w^\alpha_2\Bigg]\frac{\partial}{\partial w^\alpha_2} \end{aligned}$$

$$\begin{aligned} &+ \Bigg[\eta^\alpha_{(3)}(t,\vec{x},\vec{w}_0) - \xi^j_{(3)}(t,\vec{x},\vec{w}_0)w^\alpha_{0j} + \frac{\partial\eta^\alpha_{(0)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0}w^{\beta_1}_3 - \frac{\partial\xi^j_{(0)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0}w^{\beta_1}_3 w^\alpha_{0j} \\ &+ \frac{1}{2}\frac{\partial^2\eta^\alpha_{(0)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0\partial w^{\beta_2}_0}(w^{\beta_1}_1 w^{\beta_2}_2 + w^{\beta_2}_1 w^{\beta_1}_2) - \frac{1}{2}\frac{\partial^2\xi^j_{(0)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0\partial w^{\beta_2}_0}(w^{\beta_1}_1 w^{\beta_2}_2 + w^{\beta_2}_1 w^{\beta_1}_2)w^\alpha_{0j} \\ &+ \frac{1}{3}\frac{\partial^3\eta^\alpha_{(0)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0\partial w^{\beta_2}_0\partial w^{\beta_3}_0}w^{\beta_1}_1 w^{\beta_2}_1 w^{\beta_3}_1 - \frac{1}{3}\frac{\partial^3\xi^j_{(0)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0\partial w^{\beta_2}_0\partial w^{\beta_3}_0}w^{\beta_1}_1 w^{\beta_2}_1 w^{\beta_3}_1 w^\alpha_{0j} \\ &+ \frac{\partial\eta^\alpha_{(1)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0}w^{\beta_1}_2 - \frac{\partial\xi^j_{(1)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0}w^{\beta_1}_2 w^\alpha_{0j} + \frac{1}{2}\frac{\partial^2\eta^\alpha_{(1)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0\partial w^{\beta_2}_0}w^{\beta_1}_1 w^{\beta_2}_1 \\ &- \frac{1}{2}\frac{\partial^2\xi^j_{(1)}(t,\vec{x},\vec{w}_0)}{\partial w^{\beta_1}_0\partial w^{\beta_2}_0}w^{\beta_1}_1 w^{\beta_2}_1 w^\alpha_{0j} - 3\kappa_3 w^\alpha_3\Bigg]\frac{\partial}{\partial w^\alpha_3} \end{aligned}$$

$$\vdots$$

$$+\left[\eta^{\alpha}_{(S)}(t,\vec{x},\vec{w}_0)-\xi^{j}_{(S)}(t,\vec{x},\vec{w}_0)w^{\alpha}_{0j}+\sum_{s=1}^{S}\sum_{|\sigma|=1}^{s}\frac{1}{\sigma!}\frac{\partial^{|\sigma|}\eta^{\alpha}_{(S-s)}(t,\vec{x},\vec{w}_0)}{(\partial w^1_0)^{\sigma_1}\dots(\partial w^N_0)^{\sigma_N}}\sum_{|\mu|=s}w_{(\mu)}\right.$$

$$\left.-\sum_{s=1}^{S}\sum_{|\sigma|=1}^{s}\frac{1}{\sigma!}\frac{\partial^{|\sigma|}\xi^{j}_{(S-s)}(t,\vec{x},\vec{w}_0)}{(\partial w^1_0)^{\sigma_1}\dots(\partial w^N_0)^{\sigma_N}}\left(\sum_{|\mu|=s}w_{(\mu)}\right)w^{\alpha}_{0j}-S\kappa_S w^{\alpha}_S\right]\frac{\partial}{\partial w^{\alpha}_S}.$$

We apply the formula (3.13) to calculate the approximate Lie symmetry operator of the Navier-Stokes equations. Inspired by the formula (3.13) and the calculation of the first-order approximate infinitesimal operator X_1^{appr} for the Navier-Stokes equations, we obtain, comparing (2.7) and (3.13), that

$$\begin{aligned}
&\xi^j_{(1)}\equiv 0,\quad \eta^{\alpha}_{(1)}\equiv 0,\quad \frac{\partial\eta^{\alpha}_{(0)}}{\partial u^{\beta_1}_0}=a_{\alpha\beta_1}\quad(\alpha\neq\beta_1),\quad \frac{\partial\eta^{\alpha}_{(0)}}{\partial u^{\alpha}_0}=-a_0,\quad \alpha=\overline{1,3}\\
&\eta^4_{(1)}\equiv\zeta_{(1)}=g^1(t),\quad \frac{\partial\eta^4_{(0)}}{\partial p_0}=-2a_0,\quad \eta^4_{(0)}\equiv\zeta_0,\quad \kappa_1=b_0, \qquad (3.14)\\
&\xi^j_{(2)}\equiv 0,\quad \eta^{\alpha}_{(2)}\equiv 0,\quad \alpha=\overline{1,3},\quad \eta^4_{(2)}\equiv\zeta_{(2)}=g^2(t),\quad \frac{\partial^2\xi^j_{(0)}}{\partial w^{\beta_1}_0\partial w^{\beta_2}_0}\equiv 0,\\
&\frac{\partial^2\eta^{\alpha}_{(0)}}{\partial w^{\beta_1}_0\partial w^{\beta_2}_0}\equiv 0,\quad \frac{\partial\xi^j_{(1)}}{\partial w^{\beta_1}_0}\equiv 0,\quad \frac{\partial\eta^{\alpha}_{(1)}}{\partial w^{\beta_1}_0}\equiv 0,\quad \alpha=\overline{1,4}\\
&\vdots\\
&\xi^j_{(S)}\equiv 0,\quad \eta^{\alpha}_S\equiv 0,\quad \alpha=\overline{1,3},\quad \eta^4_{(S)}\equiv\zeta^S=g^S(t),
\end{aligned}$$

higher-order derivatives $\dfrac{\partial^{|\sigma|}\xi^j_{(S-s)}}{(\partial w^1_0)^{\sigma_1}\dots(\partial w^N_0)^{\sigma_N}},\quad \dfrac{\partial^{|\sigma|}\eta^{\alpha}_{(S-s)}}{(\partial w^1_0)^{\sigma_1}\dots(\partial w^N_0)^{\sigma_N}}$ become zero for $|\sigma|\geq 2$ due to (2.7).

Therefore we derive that the S-order approximate symmetry operator X_S for the Navier-Stokes equations is transformed to

$$X^{appr}_S = \xi^0\frac{\partial}{\partial t}+\xi^i\frac{\partial}{\partial x^i}+\eta^{0,\alpha}\frac{\partial}{\partial u^{\alpha}}+[\zeta^0+\nu g^1(t)+\dots+\nu^S g^S(t)]\frac{\partial}{\partial p}+\nu b_0\frac{\partial}{\partial\nu}. \qquad (3.15)$$

This operator is admitted by the Navier-Stokes equations in the sense of S-order approximation of the theory of approximate transformation groups. Moreover, we can see that the infinitesimal operator X_0 of Lie symmetries for the Euler equations is inherited by the Navier-Stokes equations in the form of approximate symmetry (3.15) for arbitrary-order approximation.

4 Optimal systems of finite-dimensional Lie subalgebras of the approximate group of transformations

In this section we construct the so-called optimal system of all finite-dimensional Lie subalgebras generated by the approximate symmetry operator X_S^{appr}. According to the theory of approximate Lie symmetries, a vector space L of approximate operators is called an approximate Lie algebra of operators if it is closed (in approximation of some given order) under the approximate commutator

$$[A, B] \in L, \quad [A, B] \approx AB - BA,$$

for any A, $B \in L$. The approximate commutator $[A, B]$ is calculated to the precision indicated. Examples of the approximate symmetries show that such symmetries usually do not form a Lie algebra, but create the the so-called approximate Lie algebra [19].

The approximate Lie algebra L admitted by the Navier-Stokes equations was found in the previous section. Therefore this algebra is decomposed into the semi-direct sum of the finite-dimensional subalgebra L_6 spanned by the generators

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = 2t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i} - u^i\frac{\partial}{\partial u^i} - 2p\frac{\partial}{\partial p}, \\
X_{1,2} &= x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1} + u^1\frac{\partial}{\partial u^2} - u^2\frac{\partial}{\partial u^1} \\
X_{2,3} &= x^2\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^2} + u^2\frac{\partial}{\partial u^3} - u^3\frac{\partial}{\partial u^2} \\
X_{3,1} &= x^3\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^3} + u^3\frac{\partial}{\partial u^1} - u^1\frac{\partial}{\partial u^3} \\
X_6 &= t\frac{\partial}{\partial t} + x^i\frac{\partial}{\partial x^i} + \nu\frac{\partial}{\partial \nu}
\end{aligned} \tag{4.1}$$

and the infinite-dimensional ideal L_∞ with the basis

$$\begin{aligned}
X_{a^i} &= a^i(t)\frac{\partial}{\partial x^i} + a^i_t(t)\frac{\partial}{\partial u^i} - x^i a^i_{tt}(t)\frac{\partial}{\partial p}, \quad i = 1, 2, 3 \\
X_7 &= a^4(t)\frac{\partial}{\partial p}, \quad X_8 = \nu a^5(t)\frac{\partial}{\partial p}, \quad \ldots, \quad X_{7+S} = \nu^S a^{4+S}(t)\frac{\partial}{\partial p}
\end{aligned} \tag{4.2}$$

where $a^k(t)$ are smooth arbitrary functions. The direct calculation of commutators shows us that L_6 forms (exact) Lie algebra of the infinitesimal operators with respect to the (exact) commutator operator.

We construct complete systems of all inequality subalgebras of the Lie algebra L_6 where every subalgebra represents a class of equivalent subalgebras. According to [9], we denote the optimal system by the symbol ΘL_n where n is a dimension of the corresponding system. Recall that inequivalent subalgebras are used to find invariant solutions such that the solutions so obtained cannot be carried over into each other by the admissible transformations of the differential equation under consideration. To construct the optimal system ΘL_n, it is necessary to find the group of inner automorphisms $Aut(L_6)$ of Lie subalgebra L_6.

In order to simplify the calculation of commutators, we consider the algebra L'_6 (which is isomorphic to L_6) spanned by the basis $< Y_1, \dots, Y_6 >$ where

$$Y_1 = X_1,\ Y_2 = 2X_6 - X_2,\ Y_3 = X_2 - X_6,\ Y_4 = X_{2,3},\ Y_5 = X_{3,1},\ Y_6 = X_{1,2}.$$

Table of commutators for the basis operators of this algebra reads

$$\begin{aligned}
&[Y_1, Y_1] = [Y_1, Y_2] = 0,\ [Y_1, Y_3] = Y_1, [Y_1, Y_4] = [Y_1, Y_5] = [Y_1, Y_6] = 0,\\
&[Y_2, Y_1] = [Y_2, Y_2] = [Y_2, Y_3] = [Y_2, Y_4] = [Y_2, Y_5] = [Y_2, Y_6] = 0,\\
&[Y_3, Y_1] = -Y_1,\ [Y_3, Y_2] = [Y_3, Y_3] = [Y_3, Y_4] = [Y_3, Y_5] = [Y_3, Y_6] = 0,\\
&[Y_4, Y_1] = [Y_4, Y_2] = [Y_4, Y_3] = [Y_4, Y_4] = 0,\ [Y_4, Y_5] = -Y_6, [Y_4, Y_6] = -Y_5,\\
&[Y_5, Y_1] = [Y_5, Y_2] = [Y_5, Y_3] = 0,\ [Y_5, Y_4] = Y_6,\ [Y_5, Y_5] = 0,\ [Y_5, Y_6] = Y_6,\\
&[Y_6, Y_1] = [Y_6, Y_2] = [Y_6, Y_3] = 0,\ [Y_6, Y_4] = Y_5,\ [Y_6, Y_5] = -Y_4,\ [Y_6, Y_6] = 0.
\end{aligned}$$

This algebra is decomposed into the direct sum $L'_6 = O_3 \oplus K \oplus T$ of ideals $O_3 =< Y_4, Y_5, Y_6 >$, $K =< Y_1, Y_3 >$, $T =< Y_2 >$ where T is the center of L'_6. $O_3 = RY_4 \oplus RY_5 \oplus RY_6$ is $3D$ simple Lie algebra. Here R denotes the field of real numbers. The group of inner automorphisms $Aut(L'_6) = Aut(O_3) \oplus Aut(K)$ is isomorphic to $G = PO_3 \oplus \mathcal{R}$ where PO_3 is the $3D$ rotational group in R^3 and $\mathcal{R} = Ra \oplus Rt$ is $2D$ non-Abelian subalgebra such that

$$\mathcal{R} = \{\phi_{x,y} | x \in R \setminus 0, y \in R\}$$

is $2D$ non-Abelian subgroup and for $\phi = \phi_{x,y}$ we have $Y_1^\phi = xY_1$, $Y_3^\phi = Y_3 + yY_1$. Here the symbol $(\cdot)^f$ denotes the action of an inner automorphism f on an element $(\cdot)$.

The above decompositions allows us to construct the optimal systems. The transitivity property of $Aut(O_3)$ $(\mathcal{R})$ on the set of vectors from O_3 of an equal length ($\{Y_3+yY_1 | y \in R\}$) is also used. We briefly describe the construction of optimal systems $\Theta L'_i$.

To construct the optimal system $\Theta L'_1$ we consider a vector $v = \alpha v_1 + \beta v_2 + \gamma v_3 \in L'_6$ where $v_1 \in O_3$, $v_2 \in K$, $v_3 \in< Y_2 >$.

At first we suppose that $\alpha \neq 0$. Then $\alpha'^{-1} v = \alpha\alpha'^{-1} v_1 + \beta\alpha'^{-1} v_2 + \gamma\alpha'^{-1} v_3$ where $\alpha'^{-1} \neq 0$ and there exists $\phi \in Aut(O_3)$ such that $v_1^\phi = Y_4 + \beta' v_2 + \gamma' v_3$.

Let $v_2 = \beta_1 Y_3 + \beta_2 Y_1$. Suppose that $\beta_1 \neq 0$ then there exists $\psi \in Aut(K)$ such that $v_2^\psi = \lambda Y_3$ and $Y_4^\psi = Y_4$. Therefore the subalgebra $< v >$ is isomorphic to $< Y_4 + \lambda Y_3 + \mu Y_2 >$ and subalgebras with different λ and μ are not conjugate.

Suppose that $\beta_1 = 0$ and $\beta_2 \neq 0$ then there exists $\theta \in Aut(K)$ such that $(\beta' v_2)^\theta = Y_1$. Therefore the subalgebra $< v >$ is isomorphic to $< Y_1 + Y_4 + \mu Y_2 >$ and subalgebras with different μ are not conjugate.

Now suppose that $\beta_1 = \beta_2 = 0$ i.e. $v_2 = 0$ then the subalgebras $< v >$ is isomorphic to $< Y_4 + \mu Y_2 >$.

In the case of $\alpha = 0$ we can easily find as above that the subalgebra $< v >$ is isomorphic to $< Y_2 + \mu Y_3 >$ or $< Y_1 + Y_2 >$ or $< Y_1 >$ or $< Y_2 >$.

Consider a $2D$ subalgebra P and let the first element of a base of this subalgebra be $v = Y_4 + \lambda Y_3 + \mu Y_2$ then the second element $w = \tau Y_1 + \tau_1 Y_3 + \xi Y_2$ and $[v, w] = \lambda\tau Y_1$. Suppose that $\lambda\tau \neq 0$ then $P =< Y_4 + \lambda Y_3 + \mu Y_2, Y_1 >$.

Now assume that $\tau = 0$, $\tau_1 \neq 0$ then $w = Y_3 + \xi Y_2$ and $P =< Y_4 + \mu_1 Y_2, Y_3 + \mu_2 Y_2 >$.

If $\tau = \tau_1 = 0$ then $w = Y_2$ and $P =< Y_4 + \lambda Y_3, Y_2 >$.

Suppose that $\tau \neq 0$ and $\lambda = 0$ then for $v = Y_4 + \mu_1 Y_2$ we can take $w = Y_3 + \mu_2 Y_2$ for $\tau_1 \neq 0$ or $w = Y_1 + Y_2$ for $\tau_1 = 0$. Therefore $P =< Y_4 + \mu_1 Y_2, Y_3 + \mu_2 Y_2 >$ or $P =< Y_4 + \mu_1 Y_2, Y_1 + Y_2 >$. Finally for $\lambda = \tau = \tau_1 = 0$ we obtain that $P =< Y_4, Y_2 >$.

Let now the first element of a base of P be $v = Y_4 + Y_1 + \mu Y_2$ then the second element $w \in K\oplus < Y_2 >$ and $[v, w] \in < Y_1 >$. Therefore $[v, w] = 0$ (otherwise dim$P > 2$) and $w = Y_1 + \xi Y_2$ or $w = Y_2$. As a result, we obtain $P =< Y_4 + \mu Y_2, Y_1 + \xi Y_2 >$ or $P =< Y_4 + Y_1, Y_2 >$.

The result of classification in the case when $P \subseteq K \oplus T$ is given by the following list: $< Y_1, Y_3 + \mu Y_2 >$, $< Y_1, Y_2 >$, $< Y_2, Y_3 >$ since $< Y_1, Y_3 > \in < Y_1, Y_3 + \mu Y_2 >$ for $\mu = 0$.

Let us consider $3D$ subalgebras $\{S_n\}$. Immediately we can find two inequivalent subalgebras $K \oplus T$ and O_3. Otherwise, we can choose a first element of a base of S_k in the form $v = Y_4 + \lambda Y_3 + \mu Y_1 + \xi Y_2$. Further, since there not exists $2D$ subalgebras in O_3 we can take $S_n =< v > \oplus S_{0n}$ where $2D$ subalgebra $S_{0n} \subseteq K\oplus < Y_2 >$. Therefore $S_{01} =< Y_2, Y_3 >$, $S_{02} =< Y_1, Y_2 >$ and $S_{03} =< Y_3 + cY_2, Y_1 >$. So we found that $S_1 =< Y_4, Y_3, Y_2 >$, $S_2 =< Y_4 + \lambda Y_3, Y_1, Y_2 >$ and $S_3 =< Y_4 + c_1 Y_2, Y_3 + c_2 Y_2, Y_1 >$ since $< Y_1, Y_3 > \subset < Y_3 + cY_2, Y_1 >$ for $c = 0$.

In order to classify $4D$ subalgebras $\{V_n\}$ let us suppose that $O_3 \subset V_l$ then $V_l =< v > \oplus O_3$ where $< v > \in K \oplus T$. Another words $< v > =< Y_3 + \mu Y_2 >$ or $< v > =< Y_1 + Y_2 >$, $< v > =< Y_1 >$ or $< v > =< Y_2 >$ i.e. we have four classes inequivalent subalgebras.

In the case of $O_3 \not\subset V_l$ we obtain that $P =< v > \oplus K \oplus T$ where $v \in O_3$. Therefore $P =< Y_1, Y_2, Y_3, Y_4 >$.

It is clear that for any $5D$ subalgebra W_k to be hold $O_3 \subseteq W_k$. It means that $W_k = O_3 \oplus P_0$ where P_0 is a $2D$ subalgebra in $K \oplus T$.

Therefore we have three classes of inequivalent $5D$ subalgebras.

The results of constructing of the optimal systems $\Theta L_i'$, $i = 1, \ldots, 5$ are presented by the following classes of inequivalent subalgebras:

$$(\Theta L_1') \qquad A0_{c_i} :< Y_4 + c_1 Y_3 + c_2 Y_2 >, \quad A1_{c_1} :< Y_4 + c_1 Y_2 + Y_1 >,$$
$$A2 :< Y_1 + Y_2 >, \quad A3 :< Y_1 >, \quad A4 :< Y_3 >,$$
$$A5_c :< Y_2 + cY_3 >,$$

where c_i is real numbers;

$$(\Theta L_2') \qquad B0_{c_1,c_2} :< Y_4 + c_1 Y_3 + c_2 Y_2, Y_1 >, B1_{c_1} :< Y_4 + c_1 Y_2, Y_1 + Y_2 >,$$
$$B2_{c_2} :< Y_4 + c_1 Y_2, Y_1 >,$$
$$B3_{c_1,d_1} :< Y_3 + c_1 Y_2, Y_4 + d_1 Y_2 >, \quad B4_{c_1} :< Y_3 + c_1 Y_4, Y_2 >,$$
$$B5 :< Y_1, Y_2 >, \quad B6 :< Y_2, Y_4 >,$$
$$B7_{c_1} :< Y_1, Y_3 + c_1 Y_2 >,$$

$$(\Theta L_3') \qquad C0_{c_1,c_2} :< Y_4 + c_1 Y_2, Y_1, Y_3 + c_2 Y_2 >, \quad C1_{c_1} :< Y_4 + c_1 Y_3, Y_1, Y_2 >,$$
$$C2 :< Y_4, Y_3, Y_2 >, \quad C3 : O_3, \quad C4 : K \oplus T,$$

$$
\begin{array}{ll}
(\Theta L_4') & D0_{f_1}\ :< Y_4, Y_5, Y_6, Y_3 + f_1 Y_2 >, \quad D1\ :< Y_4, Y_5, Y_6, Y_2 >, \\
& D2\ :< Y_4, Y_5, Y_6, Y_1 + Y_2 >, \quad D3\ :< Y_4, Y_5, Y_6, Y_1 >, \\
& D4\ :< Y_1, Y_2, Y_3, Y_4 >,
\end{array}
$$

$$
\begin{array}{ll}
(\Theta L_5') & E0\ :< Y_1, Y_3, Y_4, Y_5, Y_6 >, \quad E1\ :< Y_2, Y_3, Y_4, Y_5, Y_6 >, \\
& E2_{c_1}\ :< Y_2 + c_1 Y_3, Y_1, Y_4, Y_5, Y_6 >,
\end{array}
$$

Subalgebra $A5_c$ is a crucial for studying scaling laws in the statistical theory of turbulence. $A5_c$ generates the two-parametric Lie group G^{ac} of symmetry transformations of the Navier-Stokes equations

$$
\bar{x}_i = e^a x_i, \quad \bar{t} = e^{ac} t, \quad \bar{u}_i = e^{a-ac} u_i, \quad \bar{p} = e^{2a(1-c)} p, \quad \bar{\nu} = e^{a(2-c)} \nu. \tag{4.3}
$$

For $c = 1$ this subalgebra coincides with X_6. In the next section we show how this group of scaling symmetries G^{ac} can be applied to derive certain scaling laws. As an example, we rediscovery the so-called universal logarithmic law by von Kármán and Prandtl for distributions of the mean velocity profile for a steady-state planar turbulent flow (or for turbulent flow in a pipe) using the symmetries obtained.

5 Symmetry groups in turbulence

Traditional models of turbulence use the Reynolds decomposition to separate the fluid velocity $\vec{u}$ at a point $\vec{x}$ into its mean and fluctuating components as $\vec{u} = \langle \vec{u} \rangle + \vec{u}'$, where $\langle \vec{u}' \rangle = 0$ and the bracket $\langle \cdot \rangle$ denotes an Eulerian mean which is defined by the formula [20]

$$
\langle \vec{f} \rangle = \int \vec{f} P(\vec{f}) d\vec{f} \equiv M^1 \vec{f},
$$

where $P(\vec{f})$ denotes the so-called density probability distribution function. This function is normalized according to the equality

$$
M^0 \vec{f} = \int P(\vec{f}) d\vec{f} = 1.
$$

We recall that

$$
M^n \vec{f} = \int \vec{f}^n P(\vec{f}) d\vec{f}
$$

is called by the n-order moment of a stochastic quantity $\vec{f}(\vec{x}, t)$. Mathematically, Eulerian averaging commutes with the partial derivatives in space and time, but it does not commute with the time derivative

$$
\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}.
$$

This lack of commutivity between Eulerian averaging and the material time derivative leads to the unknown Reynolds stresses in the motion equations for the Eulerian mean velocity $\langle \vec{u} \rangle$ and, subsequently, to the well-known closure problem [20]). The Reynolds equations governing the mean velocity field are

$$\frac{\partial \langle u_i \rangle}{\partial t} + \langle u_j \rangle \frac{\partial \langle u_i \rangle}{\partial x_j} + \frac{\partial \langle p \rangle}{\partial x_i} = \nu \frac{\partial^2 \langle u_i \rangle}{\partial x_{jj}^2} - \frac{\partial \langle u_i' u_j' \rangle}{\partial x_j}$$

$$\frac{\partial \langle u_k \rangle}{\partial x_k} = 0,$$

where $R_{ij} = -\langle u_i' u_j' \rangle$ is the new unknown additional functions in the Reynolds equations. To find the evolution equations for this term we can apply again Eulerian averaging and as a result, the equations for $\langle u_i' u_j' \rangle$ are

$$\frac{\partial \langle u_i' u_j' \rangle}{\partial t} + \langle u_k \rangle \frac{\partial \langle u_i' u_j' \rangle}{\partial x_k} + \left(\langle u_i' u_k' \rangle \frac{\partial \langle u_j \rangle}{\partial x_k} + \langle u_j' u_k' \rangle \frac{\partial \langle u_i \rangle}{\partial x_k} \right) + \frac{\partial \langle u_i' u_j' u_k' \rangle}{\partial x_k} =$$
$$-\rho^{-1} \left(\langle u_i \frac{\partial p'}{\partial x_j} \rangle + \langle u_j \frac{\partial p'}{\partial x_i} \rangle \right) - \nu \left(\langle u_i \frac{\partial^2 u_j}{\partial x_{kk}^2} \rangle + \langle u_j \frac{\partial^2 u_i}{\partial x_{kk}^2} \rangle \right),$$

where $\langle u_i' u_j' u_k' \rangle$ is the third-order moment (or the so-called correlation tensor). The procedure to write equations for the new correlation functions may be continued for finding higher-order correlation tensors and the infinite system obtained is called by the Keller-Friedman chain. The Reynolds equations admit the Lie group of symmetries G^{ac} in the following modified form

$$\bar{x}_i = e^a x_i, \quad \bar{t} = e^{ac} t, \quad \langle \bar{u}_i \rangle = e^{a-ac} \langle u_i \rangle, \quad \bar{\nu} = e^{a(2-c)} \nu,$$
$$\langle \bar{p} \rangle = e^{2a(1-c)} \langle p \rangle, \quad \langle \overline{u_j' u_i'} \rangle = e^{2(a-ac)} \langle u_j' u_i' \rangle.$$

As the first application of the results from the previous section, we derive the relationship between mean velocity distribution $\langle u \rangle$ and the the distance y from the wall in the case of a steady-state planar turbulent flow using the Reynolds equations. We recall that scaling laws (invariants of the corresponding symmetry groups) are the cornerstones of the statistical theory of turbulence, the best known self-similar states are found in the intermediate region in wall-bounded turbulence, whose mean structure has been widely thought to be well described by the von Kármán–Prandtl universal logarithmic law of the wall. This law have been put to use in applications, for practical reasons it is important to know how the time averaged velocity varies as the distance y from the wall increases. During the last sixty years two contrasting laws for mean velocity distribution in the so-called intermediate region (where viscosity is small but finite) could be found in the literature: the first is the power law. Engineers determined empirically (in the early years of turbulent research) that

$$\langle u_1 \rangle = A x_2^n \tag{5.1}$$

where the power n and the coefficient A are depend slightly on Re and were determined from experiment. Here the planar wall-bounded shear steady-state flow is considered where all statistical quantities depend only on the wall-normal coordinates x_2. The second law found in the literature is the universal logarithmic law

$$\langle u_1 \rangle = u_* \left(\kappa^{-1} \ln \frac{u_* x_2}{\nu} + C \right), \tag{5.2}$$

where u_* is a given external parameter (friction velocity), κ and C are "universal" constants, independent of the Reynolds number. The friction velocity is defined by the integrated form of the leading order of the momentum equation in stream-wise direction according to

$$\nu \frac{\partial \langle u_1 \rangle}{\partial x_2} - \langle u_1' u_2' \rangle = \frac{\tau_w}{\rho} = u_*^2. \tag{5.3}$$

u_*^2 may be considered as a given external parameter, such as a boundary condition. The von Kármán assumed that the close to the wall right outside the viscous sub-layer the wall shear u_* (friction) velocity is only parameter determining the flow.

In the case when the evolution of a flow is fixed by the averaged Euler equations these laws were unified [13] in the framework of Symmetry Analysis. The following four symmetries consisting of two scaling symmetries

$$\overline{X}_{s_1} = x_2 \frac{\partial}{\partial x_2} + \langle u_1 \rangle \frac{\partial}{\partial \langle u_1 \rangle} + 2R_{ij} \frac{\partial}{\partial R_{ij}}, \ \overline{X}_{s_2} = -\langle u_1 \rangle \frac{\partial}{\partial \langle u_1 \rangle} - 2R_{ij} \frac{\partial}{\partial R_{ij}},$$

the Galilei transformation in x_1 and the translation invariance in x_2 directions

$$\overline{X}_{u_1} = \frac{\partial}{\partial \langle u_1 \rangle}, \quad \overline{X}_{x_2} = \frac{\partial}{\partial x_2}$$

are admitted by the two-point correlation equations for plane shear flows in the so-called outer region of a boundary layer which is obtained by taking the limit $\nu \to 0$ in the equations (see [13]). Using the superposition principle for the above-mentioned symmetries, we consider the symmetry operator

$$\overline{X} = k_{s_1} \overline{X}_{s_1} + k_{s_2} \overline{X}_{s_2} + k_{u_1} \overline{X}_{u_1} + k_{x_2} \overline{X}_{x_2} \tag{5.4}$$

where k_{s_i} are constant and repeating the results [13], we can find easily invariants of the symmetry operator (5.4).

Indeed, in the case of a planar shear flow, the characteristic equations to determine invariants are

$$\frac{dx_2}{k_{s_1} x_2 + k_{x_2}} = \frac{d\langle u_1 \rangle}{(k_{s_1} - k_{s_2}) \langle u_1 \rangle + k_{u_1}} = \ldots \quad . \tag{5.5}$$

For the different combination of parameter k_{s_1} and k_{s_2} a variety of scaling laws for the flow can be written as follows:
a) $k_{s_1} = k_{s_2}$.
Then we obtain from (5.5) that

$$\langle u_1 \rangle = \frac{k_{u_1}}{k_{s_1}} \ln \left(x_2 + \frac{k_{x_2}}{k_{s_1}} \right) + C \tag{5.6}$$

where C is a constant. Therefore (5.6) takes the form of a logarithmic law of the von Kármán–Prandtl type for $k_{x_2} = 0$ and the function (5.6) is invariant under the symmetry transformation

$$\bar{x}_2 = e^{k_{s_1}} (x_2 + k_{x_2}/k_{s_1}) - k_{x_2}/k_{s_1}, \quad \langle \bar{u}_1 \rangle = \langle u_1 \rangle + k_{\bar{u}_1}.$$

b) $k_{s_1} \neq k_{s_2}$.
This leads us to an algebraic velocity law of the following form

$$\langle u_1 \rangle = C \left(x_2 + \frac{k_{x_2}}{k_{s_1}} \right)^{1-k_{s_2}/k_{s_1}} + \frac{k_{u_1}}{k_{s_2} - k_{s_1}}. \tag{5.7}$$

This function is invariant under the symmetry transformation

$$\begin{aligned} \bar{x}_2 &= e^{k_{s_1}}(x_2 + k_{x_2}/k_{s_1}) - k_{x_2}/k_{s_1}, \\ \langle \bar{u}_1 \rangle &= e^{k_{s_1}-k_{s_2}}\left(\langle u_1 \rangle - k_{\bar{u}_1}/(k_{s_2} - k_{s_1})\right) + k_{u_1}/(k_{s_2} - k_{s_1}). \end{aligned}$$

Therefore (5.7) takes the form of an algebraic law but differ from the law suggested in [14], [16] by their derivation and application area. The constants involve into these formulas may be determined from numerical experiments [13].

Since the algebraic law was characterized by a maximum of symmetry transformation, a new one can be obtained from the following assumption
c) $k_{s_1} = 0$.
Implementation this expression into the invariant surface condition leads to a new exponential velocity profile

$$\langle u_1 \rangle = C \exp\left(-\frac{k_{s_2}}{k_{x_2}} x_2 \right) + \frac{k_{u_1}}{k_{s_2}}. \tag{5.8}$$

The function obtained is invariant under the symmetry transformation

$$\begin{aligned} \bar{x}_2 &= x_2 + k_{x_2}, \\ \langle \bar{u}_1 \rangle &= e^{-k_{s_2}}(\langle u_1 \rangle - k_{u_1}/k_{s_2}) + k_{u_1}/k_{s_2}. \end{aligned}$$

In normalized and nondimensional form, (5.8) can be rewritten in the boundary layer form

$$\frac{\langle u_\infty \rangle - \langle u_1 \rangle}{u_*} = c^* \exp\left(-\beta \frac{x_2}{\Delta} \right), \tag{5.9}$$

where $\Delta = \delta_1 \langle u_\infty \rangle / u_*$ is the Rotta-Causer length scale, $\langle u_\infty \rangle$ is the stream mean velocity in the limit $x_2 \to \infty$. The assumption *c*) characterizes a violation of symmetry according to the scale transformation generated by the infinitesimal operator $\overline{X}$ suppresing scalability of the length scale.

The penultimate case of a turbulent shear flow results from the violation of symmetries by the assumptions
d) $k_{s_1} = 0$ and $k_{s_2} = 0$.

For this case we obtain the linear scaling law

$$\langle u_1 \rangle = \frac{k_{u_1}}{k_{x_2}} x_2 + C. \tag{5.10}$$

This equation is invariant under the symmetry transformation

$$\bar{x}_2 = x_2 + k_{x_2}, \quad \langle \bar{u}_1 \rangle = \langle u_1 \rangle + k_{u_1}.$$

The Couette flow is a well documented example for a plane turbulent flow with a linear velocity law. This flow is determined by an external velocity scale as well as by an external velocity law.

Therefore, we demonstrated (in a compressed form) how to derive a family of scaling laws (invariants) by the knowledge of symmetries, see for details [11]-[13]. Moreover, this analysis showed us what kind of scaling laws may be obtained. Verification of the scaling laws by using experimental and DNS data are discussed in details in [11]-[13].

The next step is to include the Reynolds number dependence into consideration. The above scaling laws symmetries were obtained by using the symmetries of the Euler equations. We note that the use of symmetries of the Navier-Stokes equations do not enables us to introduce the Reynolds number dependence into scaling laws. Moreover, these symmetries cannot be used to derive the so-called algebraic scaling law which is realized in experiments. The crucial point for understanding of Reynolds number dependence is that viscosity is only significant for small scale turbulence at the order of the Kolmogorov length scale i.e. into the inner region (wall region) of a turbulent flow.

To involve the Reynolds number dependence into the scaling laws we rewrite the Reynolds equations in the following form

$$\frac{\partial\langle u_i\rangle}{\partial t} + \langle u_j\rangle\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle p\rangle}{\partial x_i} = 2\nu\frac{\partial}{\partial x_j}S_{ij} + \frac{\partial}{\partial x_j}R_{ij} \tag{5.11}$$

$$\frac{\partial\langle u_k\rangle}{\partial x_k} = 0, \tag{5.12}$$

where S_{ij} is the strain rate of the mean velocity field which is defined by

$$S_{ij} = \frac{1}{2}\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right).$$

In the case of a planar steady-state shear flow (i.e. it solely depends on the x_2 coordinate) equations (5.11),(5.12) admit again the Lie group of symmetries G^{ac} in the following modified form

$$\bar{x}_2 = e^a x_2, \quad \langle\bar{u}_1\rangle = e^{a-ac}\langle u_1\rangle, \quad \bar{\nu} = e^{a(2-c)}\nu,$$
$$\overline{R}_{12} = e^{2(a-ac)}R_{12}, \quad \overline{S}_{12} = e^{ac}S_{12}.$$

The group G^{ac} generates the one-parametric groups

$$G^a: \qquad \bar{x}_2 = e^a x_2, \quad \langle\bar{u}_1\rangle = e^a\langle u_1\rangle, \quad \bar{\nu} = e^{2a}\nu,$$
$$\overline{R}_{12} = e^{2a}R_{12}, \quad \overline{S}_{12} = e^a S_{12}.$$

and

$$G^s: \qquad \bar{x}_2 = x_2, \quad \langle\bar{u}_1\rangle = e^{-s}\langle u_1\rangle, \quad \bar{\nu} = e^{-s}\nu,$$
$$\overline{R}_{12} = e^{-2s}R_{12}, \quad \overline{S}_{12} = e^{-s}S_{12},$$

These groups are realized as a closer of the orbits [9] of G^{ac} for $c \to 0$, $c = -s/a$ and $a \to 0$ respectively and generate the infinitesimal operators

$$Z_{s_1} = x_2\frac{\partial}{\partial x_2} + \langle u_1\rangle\frac{\partial}{\partial\langle u_1\rangle} + 2R_{12}\frac{\partial}{\partial R_{12}} + S_{12}\frac{\partial}{\partial S_{12}} + 2\nu\frac{\partial}{\partial\nu} \tag{5.13}$$

and

$$Z_{s_2} = -\langle u_1\rangle\frac{\partial}{\partial\langle u_1\rangle} - 2R_{12}\frac{\partial}{\partial R_{12}} - S_{12}\frac{\partial}{\partial S_{12}} - \nu\frac{\partial}{\partial\nu}. \tag{5.14}$$

The infinitesimal operator $\overline{X}_{u_1}$ is extended to (considering R_{12} as a new dependent variable)

$$Z_3 = \frac{\partial}{\partial\langle u_1\rangle} + \frac{\partial}{\partial R_{12}} \tag{5.15}$$

and the operator $\overline{X}_{x_2}$ is admitted in the same form.

The first case to be analyzed is that of the classical logarithmic-law-of-the-wall. We recall that von Kármán's key assumption was, that close to the wall, just beyond the viscous sub-layer, the friction velocity u_* (see, the formula (5.2) is the only flow determining parameter. We show how to use the symmetries admitted by the Reynolds equations to derive that u_* is a really flow determining parameter and the above-mentioned von Kármán's assumption is fulfilled. For this aim, it will be convenient to rewrite the Reynolds equations (5.11), (5.12) in the form

$$\frac{\partial\langle u_i\rangle}{\partial t} + \langle u_j\rangle\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle p\rangle}{\partial x_i} = \frac{\partial}{\partial x_j}w_{ij} \tag{5.16}$$

$$\frac{\partial\langle u_k\rangle}{\partial x_k} = 0, \tag{5.17}$$

where $w_{ij} = R_{ij} + 2\nu S_{ij}$. Further, for the present case of shear planar steady-state flows for which all statistical variables only depends on the x_2-coordinate, the infinitesimal operator Z_3 can be rewritten in $\partial/\partial\langle u_1\rangle$, $\partial/\partial w_{12}$ variables and as a result, Z_3 is transformed to

$$Z_{u_1,w_{12}} = \frac{\partial}{\partial\langle u_1\rangle} + \frac{\partial}{\partial w_{12}}. \tag{5.18}$$

The operators Z_{s_1} and Z_{s_2} take the form (at the same definitions)

$$Z_{s_1} = x_2\frac{\partial}{\partial x_2} + \langle u_1\rangle\frac{\partial}{\partial\langle u_1\rangle} + 2w_{12}\frac{\partial}{\partial w_{12}} + \nu\frac{\partial}{\partial\nu} \tag{5.19}$$

and

$$Z_{s_2} = -\langle u_1\rangle\frac{\partial}{\partial\langle u_1\rangle} - 2w_{12}\frac{\partial}{\partial w_{12}}. \tag{5.20}$$

Here we use that

$$\frac{\partial}{\partial R_{12}} = \frac{\partial}{\partial w_{12}}, \quad \frac{\partial}{\partial S_{12}} = 2\nu\frac{\partial}{\partial w_{12}}, \quad \frac{\partial}{\partial\nu} = 2S_{12}\frac{\partial}{\partial w_{12}}$$

due to the formula $w_{12} = R_{12} + \nu S_{12}$ and that the viscosity ν can be considered as a transformed parameter.

Let us consider the symmetry operator of the flow under consideration

$$Z = k_{s_1}Z_{s_1} + k_{s_2}Z_{s_2} + k_{u_1,w_{12}}Z_{u_1,w_{12}} + k_{x_2}\overline{X}_{x_2} \tag{5.21}$$

and suppose that
a_1) $k_{s_1} = k_{s_2}$.

Then (5.21) determines the following invariants

$$I_1 = \langle u_1 \rangle - \bar{\kappa}_1 \ln \left(x_2 + \frac{k_{x_2}}{k_{s_1}} \right), \quad I_2 = \frac{\left(x_2 + \frac{k_{x_2}}{k_{s_1}} \right)}{\nu}, \quad I_3 = \bar{\kappa}_1 \ln \nu - w_{12} - a,$$

where $\bar{\kappa}_1 = k_{u_1,w_{12}}/k_{s_1}$, a is an arbitrary constant and we can find the representation for a mean velocity $\langle u_1 \rangle$ in the following invariant form

$$I_1 + I_3 = \langle u_1 \rangle - \bar{\kappa}_1 \ln e^{\bar{\kappa}_1^{-1}(w_{12}+a)} I_2 \equiv \langle u_1 \rangle - \bar{\kappa}_1 \ln e^{\bar{\kappa}_1^{-1}(u_*^2+a)} I_2. \tag{5.22}$$

This formula shows that u_* is the flow determining parameter and the above-mentioned von Kármán's assumption is fulfilled. Moreover, since $I_1 + I_3$ is an invariant of the group transformation generated by the infinitesimal operator (5.21) the function

$$I_{a_1} = \langle u_1 \rangle - \bar{\kappa}_1 \ln e^{\bar{\kappa}_1^{-1}(u_*^2+a)} I_2 \tag{5.23}$$

gives us a scaling law for the mean velocity $\langle u_1 \rangle$ under the action of the infinitesimal operator (5.21).

Let us fix the value of this parameter u_*, assume that $\bar{\kappa}_1 = u_*$ and apply the Taylor formula for

$$e^{\bar{\kappa}_1^{-1}(w_{12}+a)} = 1 + \bar{\kappa}_1^{-1} w_{12} + \bar{\kappa}_1^{-1} a + o(w_{12}).$$

Then we can calculate the value of this function of the first-order of approximation for $w_{12} = u_*^2$ and $a = -\bar{\kappa}_1$. As a result, we obtain that

$$I_1 + I_3 \approx \langle u_1 \rangle - u_* \ln \left[u_* \left(x_2 + \frac{k_{x_2}}{k_{s_1}} \right) \nu^{-1} \right],$$

where the right-hand part is similar to the classical logarithmic law of the wall.

b_1) $k_{s_1} \neq k_{s_2}$.

As above we can calculate the following invariants of the operator (5.21) under the specifications b_1)

$$\langle u_1 \rangle = I_1 \bar{\kappa}_2 \left(x_2 + \frac{k_{x_2}}{k_{s_1}} \right)^{1-\frac{k_{x_2}}{k_{s_1}}} + \frac{k_{u_1,w_{12}}}{k_{s_2} - k_{s_1}},$$

$$I_2 = \left(x_2 + \frac{k_{x_2}}{k_{s_1}} \right) \nu^{-1},$$

$$I_3 = \frac{\nu}{w_{12} + \frac{k_{u_1,w_{12}}}{2(k_{s_1}-k_{s_2})}} \left(w_{12} + \frac{k_{u_1,w_{12}}}{2(k_{s_1} - k_{s_2})} \right)^{1-\frac{k_{s_1}}{2(k_{s_1}-k_{s_2})}}.$$

where $\bar{\kappa}_2 = \left(k_{s_1}^{(k_{s_1}-k_{s_2})/k_{s_2}} \right) / (k_{s_1} - k_{s_2})$. Using the invariants I_2, I_3 and assuming $w_{12} = u_*^2$, we can write the following

$$\langle u_1 \rangle = I_1 \bar{\kappa}_2 \nu^{1-\frac{k_{x_2}}{k_{s_1}}} \left(\frac{x_2 + \frac{k_{x_2}}{k_{s_1}}}{\nu} \right)^{1-\frac{k_{x_2}}{k_{s_1}}} + \frac{k_{u_1,w_{12}}}{k_{s_2} - k_{s_1}},$$

$$I_3 = \frac{\nu}{u_*^2 + \frac{k_{u_1,w_{12}}}{2(k_{s_1}-k_{s_2})}} \left(u_*^2 + \frac{k_{u_1,w_{12}}}{2(k_{s_1} - k_{s_2})} \right)^{1-\frac{k_{s_1}}{2(k_{s_1}-k_{s_2})}}.$$

or

$$\langle u_1 \rangle = B(\nu, u_*, k_{s_1}, k_{s_2}, k_{u_1,w_{12}}) \left(\frac{x_2 + \frac{k_{x_2}}{k_{s_1}}}{\nu} \right)^{1-\frac{k_{x_2}}{k_{s_1}}} + \frac{k_{u_1,w_{12}}}{k_{s_2} - k_{s_1}}, \tag{5.24}$$

where B is a function dependent on the variable ν, the external parameter u_* and k_{s_1}, k_{s_2}, $k_{u_1,w_{12}}$. Since the present case is identified by a maximum of symmetries, it may be localized in regions where any symmetry breaking influence, such as a wall, is negligible. Hence we propose the algebraic law to be located in the center of a pressure of a pressure driven turbulent channel flow. Since the flow configuration between the two parallel walls admits a reflection symmetry with respect to the center-line we find that the term $k_{u_1,w_{12}}/(k_{s_2} - k_{s_1})$ may only represent a maximum value of the velocity $\langle u_1 \rangle$ on the center line, see for details [13].

c_1) $k_{s_1} = 0$.

In this case the infinitesimal operator (5.21) generates the following invariants

$$\langle u_1 \rangle = I_1 \exp\left(-\frac{k_{s_2}}{k_{x_2}} x_2 \right) + \frac{k_{u_1,w_{12}}}{k_{s_2}}, \quad I_2 = \frac{\langle u_1 \rangle - \frac{k_{u_1,w_{12}}}{k_{s_2}}}{2w_{12} - \frac{k_{u_1,w_{12}}}{k_{s_2}}},$$

where I_2 is a function dependent on x_2. In order to find a functional form of the invariant I_1, we use that

$$\langle u_1 \rangle - \frac{k_{u_1,w_{12}}}{k_{s_2}} = I_2(x_2) \left(2w_{12} - \frac{k_{u_1,w_{12}}}{k_{s_2}} \right) = I_1 \exp\left(-\frac{k_{s_2}}{k_{x_2}} x_2 \right) \tag{5.25}$$

Let us fix the value of the parameter u_*, we can obtain that

$$I_1 = 2u_*^2 - \frac{k_{u_1,w_{12}}}{k_{s_2}} \quad \text{or} \quad I_1 = C u_*,$$

where C depends on u_*, k_{s_2} and $k_{u_1,w_{12}}$. The assumption c_1) imposes symmetry breaking on the scaling transformation such that the length-scale may not be scaled. In a boundary layer flow the symmetry breaking length-scale may only be the boundary layer thickness itself. We can account that for positive value k_{s_2}/k_{x_2} the velocity law (5.25) convergence for $x_2 \to \infty$ to a constant velocity $\langle u_\infty \rangle$ and $k_{u_1,w_{12}}/k_{s_2}$ is specified by

$$\frac{k_{u_1,w_{12}}}{k_{s_2}} = \langle u_\infty \rangle.$$

This may be applicable to an infinite or semi-infinite domain such as a boundary-layer type of flow, see [13]. Therefore we obtain an exponential velocity profile in the following form

$$\langle u_1 \rangle - \frac{k_{u_1,w_{12}}}{k_{s_2}} = u_* C(u_*, k_{s_2}, k_{u_1,w_{12}}) \exp\left(-\frac{k_{s_2}}{k_{x_2}} x_2 \right) \tag{5.26}$$

d_1) $k_{s_1} = 0$ and $k_{s_2} = 0$.

A classical shear flow is given by the symmetry breaking assumptions $k_{s_1} = 0$ and $k_{s_2} = 0$. It appears that the present case applies to the turbulent plane Couette flow.

For this case the following invariants are generated by the infinitesimal operator Z

$$I_1 = \langle u_1 \rangle - \frac{k_{u_1,w_{12}}}{k_{x_2}} x_2, \quad I_2 = \langle u_1 \rangle - w_{12}.$$

Invariant I_2 depends on x_2 and realizes an obvious functional relation between $\langle u_1 \rangle$ and u_*^2. Hence, comparing to the case d), the mean velocity may be written as

$$\langle u_1 \rangle = D \frac{u_w x_2}{h} + I_1, \tag{5.27}$$

where h and u_w are the channel width and the the velocity of the pulled wall respectively. Verification of the linear mean velocity profile (5.27) by experimental and DNS data can be found in [13].

To close this section, we make the following conclusion remarks that knowledge of symmetries of a shear planar steady-state flow allows to derive from the various combination of the infinitesimal operators Z_{s_1}, Z_{s_2}, $Z_{u_1,w_{12}}$ and $\overline{X}_{x_2}$ the so-called invariant representations (scaling laws) of the mean velocity $\langle u_1 \rangle$, and to introduce the viscosity and the friction velocity dependence into the scaling laws for the mean velocity $\langle u_1 \rangle$ explicitly.

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