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USING PARAMETRIC FUNCTIONS TO SOLVE SYSTEMS OF LINEAR FUZZY EQUATIONS WITH A SYMMETRIC MATRIX

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A method to solve linear fuzzy equations with a symmetric matrix is proposed. Ignoring the symmetry leads to an overestimation of the solution. Our method to find the solution of a system of linear fuzzy equations takes the symmetry of the matrix into account and is based on parametric functions. It is a practical algorithm using parametric functions in which the variables are given by elements of the support of the fuzzy coefficients of the system.
Key words: fuzzy number, system of linear fuzzy equations, symmetric matrix, parametric function

1. Introduction

In this paper we search for a proper solution to systems of linear fuzzy equations. In many applications a solution to such systems has to be found. For instance, the finite element method is a well established and a widely used technique for the numerical simulation of different processes and phenomena in structures. The method was initially developed for structural mechanics applications in civil and mechanical structures. Nowadays the applications area is however extremely wide with problems of heat transport, fluid flow, electromagnetism, ... The classical finite element method is a deterministic procedure: the structure is characterised by nominal values of geometrical and material properties. The two major steps of the method are the construction of a system of linear equations and solving the obtained system. The result of the analysis is also deterministic. In practice however it is very difficult and in many cases even impossible to define correct and unique input data. Fuzzy arithmetic may provide a solution for those cases. So in the finite element method with fuzzy parameters, a system of linear fuzzy equations has to be solved. In this paper we search for a solution of the matrix equation:

$$\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}},$$

for $\tilde{\mathbf{x}} = [\tilde{x}_k]_{n\times 1}$ where $\tilde{A} = [\tilde{a}_{ij}]_{n\times n}$ is a matrix with fuzzy numbers as entries and $\tilde{\mathbf{b}} = [\tilde{b}_k]_{n\times 1}$ is a vector of fuzzy numbers. Differently expressed,

$$\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j = \tilde{b}_i, \quad \text{for } 1 \leq i \leq n,$$

where fuzzy multiplication and addition based on the extension principle of Zadeh are used. Taking the α-levels of these equations we obtain systems of linear interval equations:

$$\sum_{j=1}^{n}[(\underline{a}_{ij})_\alpha, (\overline{a}_{ij})_\alpha][(\underline{x}_j)_\alpha, (\overline{x}_j)_\alpha] = [(\underline{b}_i)_\alpha, (\overline{b}_i)_\alpha],$$
$$\text{for } 0 < \alpha \leq 1 \text{ and } 1 \leq i \leq n,$$

$$(\tilde{\mathbf{x}}_e)_i(x) = \sup\{\alpha \mid \alpha \in [0,1] \text{ and } x \in [(\underline{x}_i)_\alpha, (\overline{x}_i)_\alpha]\},$$
$$\forall x \in \mathbb{R}.$$

This solution is denoted as $\tilde{\mathbf{x}}_e$ as it is the exact solution of the system; when it is reentered into the system the equations are satisfied. However, these interval equations are hard to solve exactly and often $(\underline{x}_j)_\alpha$ and $(\overline{x}_j)_\alpha$ do not generate a fuzzy number.[1] This is based on a earlier result that solutions to systems of linear interval equations are not necessarily intervals.[2] Consequently the exact solution does not exist and therefore the search for an alternative solution has a solid ground. There are already some alternative approaches known in literature. Fuller[3] considers a system of linear fuzzy equations with Lipschitzian fuzzy numbers. He assigns a degree of satisfaction to each equation in the system and then calculates a measure of consistency for the whole system. Abramovich et al.[4] try to minimize the deviation of the left hand side from the right hand side of the system with LR-type fuzzy numbers. Both methods try to approximate the exact solution, i.e., they try to minimize the error when one reenters the solution into the system. In practice however it is more convenient to base the solution on the 'United Solution Set' used for systems of linear interval equations $[A]x = [b]$ (see Refs. 5, 6, 7, 8):

$$\begin{aligned}[x]_{\exists\exists} &= \{x \in \mathbb{R} \mid (\exists A \in [A])(\exists b \in [b])(Ax = b)\} \\ &= \{x \in \mathbb{R}^n \mid [A]x \cap [b] \neq \emptyset\}.\end{aligned}$$

In that way a safety margin for the solution is considered, because all possible solutions of systems in the support of the fuzzy parameters of the system are taken into account. A consequence of using the algebraic or exact solution is that the solution becomes less fuzzy as the coefficients in the matrix become more fuzzy. Moreover, if the support of the left hand side of the system $\tilde{A}\tilde{\mathbf{x}}$ is too big, then the algebraic or exact solution doesn't exist. This is counterintuitive. If the system becomes more fuzzy, the solution should also be fuzzier. Therefore it is more natural to base the solution on the 'United Solution Set'. As described in Ref. 9 the fuzzy finite element method is also based on solving all possible systems. The 'United Solution Set' provides a more probabilistic approach of the problem: the result determines the probability that a certain crisp solution is the 'right' solution of the system. Buckley and Qu[1] have based their solution on the 'United Solution Set'. We follow their line of reasoning, although the solution can be adjusted a little bit and we consider the symmetry of the matrix. A practical algorithm to obtain this solution, where we take the symmetry into account, is proposed here. The original method based on parametric functions for solving systems of linear fuzzy equations with non-symmetric matrices is described in Refs. 10 and 11.

2. Preliminaries

First we recall some definitions concerning fuzzy numbers.[12] Let A be a fuzzy set on $\mathbb{R}$. Then A is called convex if

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min(A(x_1), A(x_2)),$$

for all x_1, $x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$. If for $x \in \mathbb{R}$ it holds that $A(x) = 1$, then we call x a modal value of A. The support of A is defined as

$$\operatorname{supp} A = \{x \mid x \in \mathbb{R} \text{ and } A(x) > 0\}.$$

For all $\alpha \in [0,1]$, the α-level is defined as the set:

$$A_\alpha = \begin{cases} \{x \mid x \in \mathbb{R} \text{ and } A(x) \geq \alpha\}, & \text{if } \alpha > 0, \\ \overline{\{x \mid x \in \mathbb{R} \text{ and } A(x) > 0\}}, & \text{if } \alpha = 0. \end{cases}$$

A mapping $f : \mathbb{R} \to \mathbb{R}$, or in particular $f : \mathbb{R} \to [0,1]$, is called upper-semicontinuous when f is right-continuous where f is increasing, and left-continuous where f is decreasing.

Definition 1 *A fuzzy number is defined as a convex upper-semicontinuous fuzzy set on $\mathbb{R}$ with a unique modal value and bounded support.*[12]

From now on fuzzy numbers will be denoted by a lowercase letter with a tilde, e.g. $\tilde{a}$, and a vector of fuzzy numbers will be denoted as $\tilde{\mathbf{b}}$. Sometimes we will denote the i-th component of $\tilde{\mathbf{b}}$ by $\tilde{b}_i$. Crisp numbers will be represented by a lowercase letter, e.g. a, and vectors of crisp numbers will be denoted as $\mathbf{b} = (b_1, b_2, \ldots, b_n)^T$. The notions of support and α-level are extended componentwise for vectors or matrices of fuzzy numbers. A triangular fuzzy number is a special case of a fuzzy number which membership function contains an increasing and decreasing linear part: let a, b, c in $\mathbb{R}$ such that $a \leq b \leq c$, then we define the triangular fuzzy number A as, for all $x \in \mathbb{R}$,

$$A(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } x \in \,]a,b], \\ \dfrac{c-x}{c-b}, & \text{if } x \in [b,c[, \\ 0, & \text{else.} \end{cases}$$

We denote $A = (a \,/\, b \,/\, c)$. The arithmetic of fuzzy numbers is based on Zadeh's extension principle. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers, then the sum of $\tilde{a}$ and $\tilde{b}$, denoted by $\tilde{a} \oplus \tilde{b}$, is given by, for all $z \in \mathbb{R}$,

$$(\tilde{a} \oplus \tilde{b})(z) = \sup_{z=x+y} \min(\tilde{a}(x), \tilde{b}(y)). \tag{1}$$

Analogous definitions follow for the fuzzy multiplication, subtraction and division. The fuzzy arithmetic based on Zadeh's extension principle (see (1)) can also be calculated by interval arithmetic applied to the α-levels.

Definition 2 *Given two intervals $[\underline{x}, \overline{x}] \subseteq \mathbb{R}$ and $[\underline{y}, \overline{y}] \subseteq \mathbb{R}$, the four elementary operations on intervals are defined by*[13]

$$[\underline{x}, \overline{x}] \text{ op } [\underline{y}, \overline{y}] = \{x \text{ op } y \mid x \in [\underline{x}, \overline{x}] \text{ and } y \in [\underline{y}, \overline{y}]\},$$

for $\text{op} \in \{+, \times, -, \div\}$.

It is well-known that $(\tilde{a} \oplus \tilde{b})_\alpha = \tilde{a}_\alpha + \tilde{b}_\alpha$ and similarly for the other operations.

3. Solving systems of linear fuzzy equations

First of all, we require that the matrix $\tilde{A}$ of fuzzy numbers is regular in the sense that the inverse matrix of A exists for all $a_{ij} \in \text{supp}(\tilde{a}_{ij})$ with $a_{ij} = a_{ji}$ for all $(i, j) \in \{1, \ldots, n\}^2$.

Buckley and Qu[1] proposed to construct a set of all crisp solutions corresponding to the crisp systems formed by the elements in a certain α-level. The requirement for their solution is stronger: the matrix $\tilde{A}$ has to be regular so all inverse matrices of A within the support of the fuzzy coefficients have to exist. They define the solution by, for all $\alpha \in [0, 1]$,

$$\begin{aligned}\Omega(\alpha) = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n \text{ and } (\exists A = [a_{ij}]_{n\times n} \in \mathbb{R}^{n\times n})\\ (\exists \mathbf{b} = [b_k]_{n\times 1} \in \mathbb{R}^n)((\forall (i, j, k) \in \{1, 2, \ldots, n\}^3)\\ (a_{ij} \in (\tilde{a}_{ij})_\alpha \text{ and } b_k \in (\tilde{b}_k)_\alpha) \text{ and } A\mathbf{x} = \mathbf{b})\}\end{aligned}$$

and for all $\mathbf{x} \in \mathbb{R}^n$,

$$\tilde{\mathbf{x}}_B(\mathbf{x}) = \sup\{\alpha \mid \alpha \in [0, 1] \text{ and } \mathbf{x} \in \Omega(\alpha)\}.$$

We see that $\tilde{\mathbf{x}}_B$ is defined as a fuzzy set on $\mathbb{R}^n$ and not as a vector of fuzzy numbers. The solution $\tilde{\mathbf{x}}_B(\mathbf{x})$ expresses to what extent the crisp vector $\mathbf{x}$ is a solution of the system of linear fuzzy equations $\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$. We prefer to define the solution as a vector of fuzzy numbers to avoid information loss. Therefore we give a membership degree to every component of the solution vector and then $(\tilde{\mathbf{x}}_B)_i(x)$ expresses the degree to which x belongs to the fuzzy set $(\tilde{\mathbf{x}}_B)_i$, independent of $(\tilde{\mathbf{x}}_B)_j$, for all $j \neq i$. We thus define for all $x \in \mathbb{R}$ and for all $i \in \{1, 2, \ldots, n\}$

$$\begin{aligned}(\tilde{\mathbf{x}}_B)_i(x) = \sup\{\alpha \mid \alpha \in [0, 1] \text{ and}\\ (\exists \mathbf{x} \in \Omega(\alpha))(x = x_i)\},\end{aligned}$$

where x_i denotes the i-th component of $\mathbf{x}$. This method is purely theoretical: in fact all crisp systems are solved. When all these systems have to be solved, the computation time will be large. An other drawback of this method proposed by Buckley and Qu is that the symmetry isn't taken into account. The symmetry can be taken into account by instead of considering all possible matrices where the fuzzy numbers are replaced by real values in their support, considering only the symmetric matrices. When the symmetry isn't considered in the solution of systems of linear equations, the solution is overestimated. There is artificial uncertainty added to the solution. Moreover, when the symmetry is ignored the chance is much greater that there are singular matrices in the matrix $\tilde{A}$. This is illustrated in Example 1. Example 2 illustrates that the solution obtained when the symmetry is taken into account is a subset of (and therefore more precise than) the solution obtained when the symmetry isn't taken into account.

Example 1 *Assume*

$$\tilde{A} = \begin{pmatrix} 2 & (-1 \,/\, 0 \,/\, 2) \\ (-1 \,/\, 0 \,/\, 2) & -1 \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The matrix $\tilde{A}$ contains singular matrices: the matrix $A = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}$ and the matrix $A = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}$ for example. With the first matrix the solution of the real system is the straight line $2x_1 - x_2 = 1$. The system corresponding to the second matrix has no solution. This system of linear fuzzy equations doesn't satisfy the requirement for a solution to exist. The fuzzy matrix $\tilde{A}$ is however symmetric and when only symmetric matrices are considered if the fuzzy numbers are replaced by real values in its supports, there are no singular matrices, because $|A| = -2 - k^2 \leq -2$ with $k \in \text{supp}(-1 \,/\, 0 \,/\, 2)$.

Example 2 *Assume*

$$\tilde{A} = \begin{pmatrix} 5 & (-1 \,/\, 0 \,/\, 2) \\ (-1 \,/\, 0 \,/\, 2) & 3 \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We see that $A_{k,l} = \begin{pmatrix} 5 & k \\ l & 3 \end{pmatrix}$ *for* $k, l \in \text{supp}(-1 \,/\, 0 \,/\, 2)$ *has no singular matrices because the determinant is* $15 - kl \neq 0$.

When the symmetry isn't taken into account the set of all possible solutions S is equal to:

$$\begin{aligned} &= \left\{ \frac{1}{|A_{k,l}|} \operatorname{adj}(A_{k,l}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;\middle|\; k, l \in \overline{\text{supp}(-1 \,/\, 0 \,/\, 2)} \right\} \\ &= \left\{ \frac{1}{15 - kl} \begin{pmatrix} 3 & -k \\ -l & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;\middle|\; k, l \in \overline{\text{supp}(-1 \,/\, 0 \,/\, 2)} \right\} \\ &= \left\{ \frac{1}{15 - kl} \begin{pmatrix} 3 - k \\ 5 - l \end{pmatrix} \;\middle|\; k, l \in [-1, 2] \right\}. \end{aligned}$$

The solution of the system where the symmetry is not taken into account is then:

$$\text{supp}\,\tilde{\mathbf{x}} = \begin{pmatrix} \left[\frac{1}{17}, \frac{2}{7}\right] \\ \left[\frac{3}{17}, \frac{3}{7}\right] \end{pmatrix},$$

$$\tilde{\mathbf{x}}_{0.5} = \begin{pmatrix} \left[\frac{4}{31}, \frac{14}{59}\right] \\ \left[\frac{8}{31}, \frac{22}{59}\right] \end{pmatrix},$$

$$\tilde{\mathbf{x}}_{1} = \begin{pmatrix} \frac{1}{5} \\ \frac{1}{3} \end{pmatrix}.$$

When we take the symmetry into account the set of all possible solutions S_{sym} is equal to:

$$\begin{aligned} &= \left\{ \frac{1}{15 - k^2} \begin{pmatrix} 3 & -k \\ -k & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;\middle|\; k \in \overline{\text{supp}(-1 \,/\, 0 \,/\, 2)} \right\} \\ &= \left\{ \frac{1}{15 - k^2} \begin{pmatrix} 3 - k \\ 5 - k \end{pmatrix} \;\middle|\; k \in [-1, 2] \right\}. \end{aligned}$$

The solution of the system of linear fuzzy equations with a symmetric coefficient matrix is then

$$\text{supp}\,\tilde{\mathbf{x}}_{sym} = \begin{pmatrix} \left[\frac{1}{11}, \frac{2}{7}\right] \\ \left[\frac{1}{-2\sqrt{10} + 10}, \frac{3}{7}\right] \end{pmatrix},$$

$$(\tilde{\mathbf{x}}_{sym})_{0.5} = \begin{pmatrix} \left[\frac{1}{7}, \frac{14}{59}\right] \\ \left[\frac{2}{7}, \frac{22}{59}\right] \end{pmatrix},$$

$$(\tilde{\mathbf{x}}_{sym})_{1} = \begin{pmatrix} \frac{1}{5} \\ \frac{1}{3} \end{pmatrix}.$$

When the symmetry isn't taken into account, an overestimation is made:

$$\tilde{\mathbf{x}}_{sym} \subseteq \tilde{\mathbf{x}}.$$

This overestimation due to not considering the symmetry of the matrix is clear for systems of linear interval equations when the 'United Solution Set' is calculated:

$$\begin{aligned} &[\mathbf{x}]^{sym}_{\exists\exists} \\ &= \{\mathbf{x} \in \mathbb{R}^n \mid (\exists A^s \in [A]_{sym})(\exists \mathbf{b} \in [\mathbf{b}])(A^s \mathbf{x} = \mathbf{b})\} \\ &= \{\mathbf{x} \in \mathbb{R}^n \mid [A]_{sym} \mathbf{x} \cap [\mathbf{b}] \neq \emptyset\} \\ &\subseteq \\ &[\mathbf{x}]_{\exists\exists} \\ &= \{\mathbf{x} \in \mathbb{R}^n \mid (\exists A \in [A])(\exists \mathbf{b} \in [\mathbf{b}])(A \mathbf{x} = \mathbf{b})\} \\ &= \{\mathbf{x} \in \mathbb{R}^n \mid [A] \mathbf{x} \cap [\mathbf{b}] \neq \emptyset\}, \end{aligned}$$

where $[A]_{sym}$ only considers the symmetric matrices and all these matrices are contained in the interval matrix $[A]$.

For systems of linear fuzzy equations, there is an overestimation when the symmetry isn't considered because for every considered α-level the solution set is an overestimation:

$$\begin{aligned}
&\Omega(\alpha)^{sym}\\
&= \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n \text{ and } (\exists A^s = [a_{ij}]_{n\times n} \in \mathbb{R}^{n\times n})\\
&\quad (\exists \mathbf{b} = [b_k]_{n\times 1} \in \mathbb{R}^n)((\forall (i,j,k) \in \{1,2,...,n\}^3)\\
&\quad (a_{ij} \in (\tilde{a}_{ij})_\alpha \text{ and } b_k \in (\tilde{b}_k)_\alpha \text{ and } a_{ij} = a_{ji})\\
&\quad \text{and } A^s\mathbf{x} = \mathbf{b})\}\\
&\subseteq\\
&\Omega(\alpha)\\
&= \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n \text{ and } (\exists A = [a_{ij}]_{n\times n} \in \mathbb{R}^{n\times n})\\
&\quad (\exists \mathbf{b} = [b_k]_{n\times 1} \in \mathbb{R}^n)((\forall (i,j,k) \in \{1,2,...,n\}^3)\\
&\quad (a_{ij} \in (\tilde{a}_{ij})_\alpha \text{ and } b_k \in (\tilde{b}_k)_\alpha) \text{ and } A\mathbf{x} = \mathbf{b})\}.
\end{aligned}$$

In this paper we propose a practical algorithm to compute the solution where the symmetry is also taken into account. Instead of solving all these crisp systems with a symmetric matrix, we determine parametric functions with elements of the support of the fuzzy numbers as variables.

3.1. *Systems with one fuzzy coefficient*

We first consider the case that we have to solve a system of linear fuzzy equations in which exactly one of the coefficients is a fuzzy number and the other coefficients are crisp. The approach is different for a fuzzy non-diagonal or a fuzzy diagonal element of the matrix or a fuzzy component of the vector $\tilde{\mathbf{b}}$.

3.1.1. *The fuzzy coefficient is a diagonal element of the matrix $\tilde{A}$*

We first consider the case that we have to solve a system of linear fuzzy equations in which exactly one of the coefficients on the diagonal of the matrix is a fuzzy number and the other coefficients are crisp. Without loss of generality we may assume that $\tilde{a}_{11}$ is a fuzzy number. So we consider the following matrix equation:

$$\begin{pmatrix} \tilde{a}_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \dots & a_{nn}\end{pmatrix} \begin{pmatrix} \tilde{x}_1\\ \tilde{x}_2\\ \vdots\\ \tilde{x}_n\end{pmatrix} = \begin{pmatrix} b_1\\ b_2\\ \vdots\\ b_n\end{pmatrix}, \qquad (2)$$

where $\tilde{a}_{11}$ is a fuzzy number and $a_{ij} \in \mathbb{R}$, for all $(i,j) \in \{1,\dots,n\}^2 \setminus \{(1,1)\}$, and $b_k \in \mathbb{R}$, for all $k \in \{1,\dots,n\}$. In order to obtain the solution $\tilde{\mathbf{x}}^{sym}$ of (2), we have to solve the crisp systems

$$A^s(a_{11})\mathbf{x} = \mathbf{b},$$

where

$$A^s(a_{11}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \dots & a_{nn}\end{pmatrix},$$

$$\mathbf{x} = \begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} b_1\\ b_2\\ \vdots\\ b_n\end{pmatrix},$$

for all $a_{11} \in [\underline{a}_{11}, \overline{a}_{11}] = \text{supp}(\tilde{a}_{11})$. We can solve all of these systems through Cramer's rule thanks to the non-singularity of every crisp matrix $A(a_{11})$, for all $a_{11} \in \text{supp}(\tilde{a}_{11})$. So we can write the solution for every component as a quotient of two determinants:

$$x_j = \frac{\begin{vmatrix} a_{11} & \cdots & \overset{\overset{j}{\downarrow}}{b_1} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ a_{n1} & \cdots & b_n & \cdots & a_{nn}\end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \dots & a_{nn}\end{vmatrix}}.$$

The determinant of a matrix A is denoted as $|A|$. By expanding the determinants in the numerator and the denominator along the first row, we can write each component of the solution using parameters c_{1j}, c_{2j}, c_3 and c_4:

$$x_j = f_j(a_{11}) = \frac{a_{11}c_{1j} + c_{2j}}{a_{11}c_3 + c_4}. \qquad (3)$$

Due to this result, every solution can be written using parametric functions with variable a_{11}. Note that c_{1j} and c_{2j} are dependent of j due to the fact that the j-th column in the numerator contains the components of $\mathbf{b}$. On the other hand, the denominator is the same for all $j \in \{1,\dots,n\}$, so c_3 and c_4 are independent of j.

So we propose the following method to solve (2). First we compute the determinants of the matrices

$A(\underline{a}_{11})$ and $A(\overline{a}_{11})$. The parameters c_3 and c_4 are obtained by solving the following system of linear crisp equations:

$$\begin{cases} |A(\underline{a}_{11})| = \underline{a}_{11}c_3 + c_4 \\ |A(\overline{a}_{11})| = \overline{a}_{11}c_3 + c_4. \end{cases}$$

We find

$$\begin{cases} c_3 = \dfrac{|A(\overline{a}_{11})| - |A(\underline{a}_{11})|}{\overline{a}_{11} - \underline{a}_{11}}, \\ c_4 = |A(\overline{a}_{11})| - \overline{a}_{11}c_3. \end{cases} \tag{4}$$

We solve the crisp systems

$$A(\underline{a}_{11})\mathbf{x} = \mathbf{b}, \tag{5}$$

$$A(\overline{a}_{11})\mathbf{x} = \mathbf{b}, \tag{6}$$

and denote by $\underline{\mathbf{x}} = (\underline{x}_1, \ldots, \underline{x}_n)^T$ and $\overline{\mathbf{x}} = (\overline{x}_1, \ldots, \overline{x}_n)^T$ the solutions of (5) and (6) respectively. Then, for all $j \in \{1, \ldots, n\}$, we obtain c_{1j} and c_{2j} by solving the following system of crisp equations:

$$\begin{cases} \underline{x}_j|A(\underline{a}_{11})| = \underline{a}_{11}c_{1j} + c_{2j}, \\ \overline{x}_j|A(\overline{a}_{11})| = \overline{a}_{11}c_{1j} + c_{2j}. \end{cases}$$

We obtain

$$\begin{cases} c_{1j} = \dfrac{\overline{x}_j|A(\overline{a}_{11})| - \underline{x}_j|A(\underline{a}_{11})|}{\overline{a}_{11} - \underline{a}_{11}}, \\ c_{2j} = \overline{x}_j|A(\overline{a}_{11})| - \overline{a}_{11}c_{1j}. \end{cases} \tag{7}$$

Note that f_j is continuous as the denominator is never 0 because of the regularity of the matrix $\tilde{A}$. As a consequence $(\tilde{x}_j)_\alpha = f_j((a_{11})_\alpha)$ (see Ref. 14) and therefore $\tilde{a}_{11}$ is a fuzzy number so $\tilde{x}_j$ is a fuzzy number. Note also that it is possible that $f_j(a_{11}) = f_j(a'_{11})$ for two different $a_{11}, a'_{11} \in \text{supp}(\tilde{a}_{11})$; this happens e.g. when $a_{12} = a_{13} = \ldots = a_{1n} = b_1 = 0$, since then $f_j(a_{11}) = \frac{c_{1j}}{c_3}$, for all $a_{11} \in \text{supp}(\tilde{a}_{11})$. Finally we define $\tilde{\mathbf{x}}^{sym} = (\tilde{x}_1, \ldots, \tilde{x}_n)^T$ and we call $\tilde{\mathbf{x}}^{sym}$ the solution of the system (2).

Consequently, all possible solutions for the crisp systems $A^s(a_{11})\mathbf{x} = \mathbf{b}$, for all $a_{11} \in \text{supp}(\tilde{a}_{11})$, can be obtained using (3). We define for all $j \in \{1, \ldots, n\}$ the fuzzy number $\tilde{x}_j^{sym}$ as:

$$\tilde{x}_j^{sym}(x) = \sup\{\tilde{a}_{11}(a_{11}) \mid a_{11} \in \text{supp}(\tilde{a}_{11}) \text{ and } x = f_j(a_{11})\},$$

for all $x \in f_j(\text{supp}(\tilde{a}_{11}))$, and

$$\tilde{x}_j^{sym}(x) = 0,$$

for all $x \in \mathbb{R} \setminus f_j(\text{supp}(\tilde{a}_{11}))$.

Note 1 *Assume we want to obtain an approximation of the solution by computing the solution for m elements of* $\text{supp}(\tilde{a}_{11})$. *Our method requires only the calculation of two crisp $(n \times n)$-determinants, the solving of two crisp $(n \times n)$-systems, n evaluations of (7), one evaluation of (4) and $m-2$ evaluations of (3). For large n and m we have much less computational effort than the method of Buckley and Qu, since they need to solve m crisp $(n \times n)$-systems. For these two algorithms the total operation count can be calculated. Therefore the total amount of additions, subtractions, multiplications and divisions is counted. To solve a system of linear equations, Gaussian elimination and back substitution are usually applied. First of all for the Gaussian elimination, there are $(n-i+1)(n-i)$ subtractions, $n-i$ divisions and $(n-i)(n-i+1)$ multiplications in the i-th step.* So the operation count for Gaussian elimination is:*

$$\sum_{i=1}^{n-1}(n-i)(2n-2i+3) = \frac{4n^3+3n^2-7n}{6}.$$

For back substitution, there are $n-i$ additions and subtractions, $(n-i)+1$ multiplications and divisions in the i-th step. So the operation count for back substitution is:

$$\sum_{i=1}^{n}(2n-2i+1) = n^2.$$

To solve a $(n \times n)$-system, Gaussian elimination and back substitution are applied. The operation count is then $\frac{4n^3+9n^2-7n}{6}$. The method obtained above for one fuzzy coefficient computes 2 determinants of $(n \times n)$-matrices. The operation cost for calculating a determinant is equal to the operation cost for Gaussian elimination and $n-1$ multiplications (the product of the diagonal elements after Gaussian elimination).

2 determinants of $(n \times n)$-matrices	$\frac{4n^3+3n^2-n-6}{3}$
solving 2 $(n \times n)$-systems	$2n^2$
n evaluations of (7)	$8n$
1 evaluation of (4)	5
$m-2$ evaluations of (3)	$5(m-2)n$

*Recall: $\sum_{i=1}^{n} 1 = n$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

The total operation cost is then $\frac{4}{3}n^3 + 3n^2 - \frac{7}{3}n + 5mn - 3$. *The total operation cost for the method of Buckley and Qu, since they need to solve* m *crisp* $(n \times n)$*-systems is* $\frac{m(4n^3+9n^2-7n)}{6}$. *It is easy to see that for large* n *and* m *the method described above needs less computation time than the method of Buckley and Qu.*

3.1.2. *The fuzzy coefficient is a component of the vector* $\tilde{\mathbf{b}}$

When the fuzzy number is located in the right-hand side of the system of linear fuzzy equations, i.e. when we have for instance that $\tilde{\mathbf{b}} = (\tilde{b}_1, b_2, \ldots, b_n)$, one sees immediately that $c_3 = 0$ and $c_4 = |A|$. So we only have to solve the systems

$$A\mathbf{x} = \mathbf{b}(\underline{b}_1),$$
$$A\mathbf{x} = \mathbf{b}(\overline{b}_1),$$

with $\mathbf{b}(\underline{b}_1)$ and $\mathbf{b}(\overline{b}_1)$ a crisp vector obtained by replacing the fuzzy number $\tilde{b}_1$ by the lower and upper bound resp. of its support, and

$$\begin{cases} \underline{x}_j|A| = \underline{b}_1 c_{1j} + c_{2j}, \\ \overline{x}_j|A| = \overline{b}_1 c_{1j} + c_{2j} \end{cases}$$

to find c_{1j} and c_{2j} for $j \in \{1, \ldots, n\}$. The function f_j is then given by, for all $j \in \{1, \ldots, n\}$,

$$f_j(b_1) = \frac{b_1 c_{1j} + c_{2j}}{|A|},$$

and the solution $\tilde{\mathbf{x}}^{sym} = (\tilde{x}_1, \ldots, \tilde{x}_n)^T$ is given by, for all $j \in \{1, \ldots, n\}$,

$$\tilde{x}_j^{sym}(x) = \sup\{\tilde{b}_1(b_1) \mid b_1 \in \text{supp}(\tilde{b}_1) \text{ and } x = f_j(b_1)\},$$

for all $x \in f(\text{supp}(\tilde{b}_1))$, and

$$\tilde{x}_j^{sym}(x) = 0,$$

for all $x \in \mathbb{R} \setminus f(\text{supp}(\tilde{b}_1))$.

3.1.3. *The fuzzy coefficient is a non-diagonal element of the matrix* $\tilde{A}$

Now we assume that the fuzzy coefficient is a non-diagonal element of the matrix $\tilde{A}_{sym}$. Without loss of generality, we may assume that the element on the second column and the first row and consequently on the first column and the second row is the fuzzy number in $\tilde{A}$:

$$\begin{pmatrix} a_{11} & \tilde{a}_{12} & \cdots & a_{1n} \\ \tilde{a}_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}. \quad (8)$$

To find the solution considering the symmetry of the matrix, we have to solve

$$A^s(a_{12})\mathbf{x} = \mathbf{b},$$

where

$$A^s(a_{12}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{nn} \end{pmatrix}$$

is symmetric for all $a_{12} \in]\underline{a}_{12}, \overline{a}_{12}[= \text{supp}(\tilde{a}_{12})$.

Analogous to the previous case, all these crisp systems can be solved by Cramer's rule because each real symmetric matrix $A^s(a_{12})$ for all $a_{12} \in \text{supp}(\tilde{a}_{12})$ is regular. So we can write each component as the quotient of two determinants:

$$x_j = \frac{\begin{vmatrix} a_{11} & \cdots & \overset{\overset{j}{\downarrow}}{b_1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{1n} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{nn} \end{vmatrix}}.$$

By expanding these determinants along the first row, each component of the solution can be written using 6 parameters c_{1j}, c_{2j}, c_{3j}, c_4, c_5 and c_6:

$$x_j = f_j^{sym}(a_{12}) = \frac{a_{12}^2 c_{1j} + a_{12} c_{2j} + c_{3j}}{a_{12}^2 c_4 + a_{12} c_5 + c_6}. \quad (9)$$

Each solution of $A^s(a_{12})\mathbf{x} = \mathbf{b}$ can be expressed using the parametric functions with a_{12} as variable. First the determinants of the matrices $A^s(\underline{a}_{12})$, $A^s((\tilde{a}_{12})_1)$ and $A^s(\overline{a}_{12})$ are calculated. The values of the parameters c_4, c_5 and c_6 can be found through the following system of linear equations:

$$\begin{cases} |A^s(\underline{a}_{12})| = \underline{a}_{12}^2 c_4 + \underline{a}_{12} c_5 + c_6, \\ |A^s((\tilde{a}_{12})_1)| = (\tilde{a}_{12})_1^2 c_4 + (\tilde{a}_{12})_1 c_5 + c_6, \\ |A^s(\overline{a}_{12})| = \overline{a}_{12}^2 c_4 + \overline{a}_{12} c_5 + c_6. \end{cases} \quad (10)$$

So the following formulas can be used:

$$\begin{cases} c_4 = \frac{1}{D}\big(\underline{a}_{12}(|A^s((\tilde{a}_{12})_1)| - |A^s(\overline{a}_{12})|) \\ \qquad + \overline{a}_{12}(|A^s(\underline{a}_{12})| - |A^s((\tilde{a}_{12})_1)|) \\ \qquad + (\tilde{a}_{12})_1(|A^s(\overline{a}_{12})| - |A^s(\underline{a}_{12})|)\big), \\ c_5 = -\frac{1}{D}\big(\underline{a}_{12}^2(|A^s((\tilde{a}_{12})_1)| - |A^s(\overline{a}_{12})|) \\ \qquad + \overline{a}_{12}^2(|A^s(\underline{a}_{12})| - |A^s((\tilde{a}_{12})_1)|) \\ \qquad + (\tilde{a}_{12})_1^2(|A^s(\overline{a}_{12})| - |A^s(\underline{a}_{12})|)\big), \\ c_6 = \frac{1}{D}\big(|A^s(\underline{a}_{12})|(\overline{a}_{12}^2(\tilde{a}_{12})_1 - \overline{a}_{12}(\tilde{a}_{12})_1^2) \\ \qquad + |A^s((\tilde{a}_{12})_1)|(\underline{a}_{12}^2\overline{a}_{12} - \underline{a}_{12}\overline{a}_{12}^2) \\ \qquad + |A^s(\overline{a}_{12})|((\tilde{a}_{12})_1^2\underline{a}_{12} - (\tilde{a}_{12})_1\underline{a}_{12}^2)\big), \end{cases} \tag{11}$$

with $D = \underline{a}_{12}^2(\overline{a}_{12} - (\tilde{a}_{12})_1) + (\tilde{a}_{12})_1^2(\underline{a}_{12} - \overline{a}_{12}) + \overline{a}_{12}^2((\tilde{a}_{12})_1 - \underline{a}_{12})$. Thereafter the crisp systems:

$$A^s(\underline{a}_{12})\mathbf{x} = \mathbf{b}, \tag{12}$$

$$A^s((\tilde{a}_{12})_1)\mathbf{x} = \mathbf{b}, \tag{13}$$

$$A^s(\overline{a}_{12})\mathbf{x} = \mathbf{b}, \tag{14}$$

are solved with for example Gaussian elimination. The solutions of (12), (13) and (14) are denoted as

$$\underline{\mathbf{x}} = (\underline{x}_1, \ldots, \underline{x}_n)^T,$$
$$(\mathbf{x})_1 = ((x_1)_1, \ldots, (x_n)_1)^T$$

and

$$\overline{\mathbf{x}} = (\overline{x}_1, \ldots, \overline{x}_n)^T$$

respectively. The values of the parameters c_{1j}, c_{2j} and c_{3j} are obtained as the solution of the following system of linear equations:

$$\begin{cases} \underline{x}_j|A^s(\underline{a}_{12})| = \underline{a}_{12}^2 c_{1j} + \underline{a}_{12}c_{2j} + c_{3j}, \\ (x_j)_1|A^s((\tilde{a}_{12})_1)| = (\tilde{a}_{12})_1^2 c_{1j} + (\tilde{a}_{12})_1 c_{2j} + c_{3j}, \\ \overline{x}_j|A^s(\overline{a}_{12})| = \overline{a}_{12}^2 c_{1j} + \overline{a}_{12}c_{2j} + c_{3j}. \end{cases} \tag{15}$$

We obtain:

$$\begin{cases} c_{1j} = \frac{1}{D}\big(\underline{a}_{12}((\tilde{x}_j)_1|A^s((\tilde{a}_{12})_1)| - \overline{x}_j|A^s(\overline{a}_{12})|) \\ \qquad + \overline{a}_{12}(\underline{x}_j|A^s(\underline{a}_{12})| - (\tilde{x}_j)_1|A^s((\tilde{a}_{12})_1)|) \\ \qquad + (\tilde{a}_{12})_1(\overline{x}_j|A^s(\overline{a}_{12})| - \underline{x}_j|A^s(\underline{a}_{12})|)\big), \\ c_{2j} = -\frac{1}{D}\big(\underline{a}_{12}^2((\tilde{x}_j)_1|A^s((\tilde{a}_{12})_1)| - \overline{x}_j|A^s(\overline{a}_{12})|) \\ \qquad + \overline{a}_{12}^2(\underline{x}_j|A^s(\underline{a}_{12})| - (\tilde{x}_j)_1|A^s((\tilde{a}_{12})_1)|) \\ \qquad + (\tilde{a}_{12})_1^2(\overline{x}_|A^s(\overline{a}_{12})| - \underline{x}_j|A^s(\underline{a}_{12})|)\big), \\ c_{3j} = \frac{1}{D}\big(\underline{x}_j|A^s(\underline{a}_{12})|(\overline{a}_{12}^2(\tilde{a}_{12})_1 - \overline{a}_{12}(\tilde{a}_{12})_1^2) \\ \qquad + (\tilde{x}_j)_1|A^s((\tilde{a}_{12})_1)|(\underline{a}_{12}^2\overline{a}_{12} - \underline{a}_{12}\overline{a}_{12}^2) \\ \qquad + \overline{x}_j|A^s(\overline{a}_{12})|((\tilde{a}_{12})_1^2\underline{a}_{12} - (\tilde{a}_{12})_1\underline{a}_{12}^2)\big), \end{cases} \tag{16}$$

with $D = \underline{a}_{12}^2(\overline{a}_{12} - (\tilde{a}_{12})_1) + (\tilde{a}_{12})_1^2(\underline{a}_{12} - \overline{a}_{12}) + \overline{a}_{12}^2((\tilde{a}_{12})_1 - \underline{a}_{12})$. In that way, the values of the parameters are calculated so all the possible solutions of the linear systems $A^s(a_{12})\mathbf{x} = \mathbf{b}$, for all $a_{12} \in \text{supp}(\tilde{a}_{12})$ can be calculated by (9). We define for all $j \in \{1, \ldots, n\}$ the fuzzy number $\tilde{x}_j^{sym}$ as:

$$\tilde{x}_j^{sym}(x) = \sup\{\tilde{a}_{12}(a_{12}) \mid a_{12} \in \text{supp}(\tilde{a}_{12}) \text{ and } x = f_j^{sym}(a_{12})\}, \tag{17}$$

for all $x \in f_j^{sym}(\text{supp}(\tilde{a}_{12}))$, and

$$\tilde{x}_j^{sym}(x) = 0,$$

for all $x \in \mathbb{R} \setminus f_j^{sym}(\text{supp}(\tilde{a}_{12}))$.

Note 2 *The operation count for this method for solving a system of linear fuzzy equations with one fuzzy number as non-diagonal element in $\tilde{A}$ where the symmetry is taken into account, can be calculated as follows:*

3 determinants of $(n \times n)$-matrices	$2n^3 + \frac{3}{2}n^2 - \frac{n}{2} - 3$
3 $(n \times n)$-systems	$3n^2$
n evaluations of (16)	$91n$
1 evaluation of (11)	76
$m-3$ evaluations of (9)	$11n(m-3)$

The total cost is then $2n^3 + \frac{9}{2}n^2 + \frac{115}{2}n + 11mn + 73$.

3.2. *Systems with two fuzzy coefficients*

In this section we consider a system with two fuzzy coefficients $\widetilde{Fn_1}$ and $\widetilde{Fn_2}$. The approach differs for a fuzzy diagonal or fuzzy non-diagonal element of the matrix $\tilde{A}$ or a fuzzy component of the vector $\tilde{\mathbf{b}}$.

3.2.1. *The second fuzzy coefficient is a diagonal element in $\tilde{A}$*

Assume the second fuzzy number is a diagonal element of the matrix $\tilde{A}$. We first fix the second fuzzy number on the lower bound of its support. In that way we can find the solutions for the systems corresponding to the lower line of the rectangle in Figure 1 by the method described in Section 1. Analogously we fix the second fuzzy number on the upper bound of its support and find the solutions for the upper line in Figure 1.

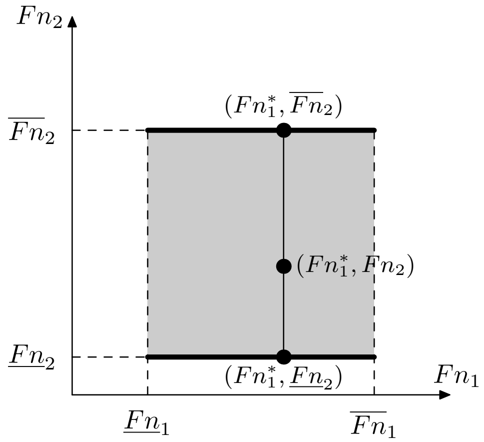

Fig. 1. Solving systems with two fuzzy coefficients where the second fuzzy number is a diagonal element in $\tilde{A}$.

Thereafter we fix arbitrarily the first fuzzy number on $Fn_1^* \in \text{supp}(\widetilde{Fn_1})$ and let $Fn_2 \in \text{supp}(\widetilde{Fn_2})$ vary. So, again, we obtain a system with only one fuzzy number, but this time the fuzzy number is $\widetilde{Fn_2}$. Thus we are looking for the solution of the crisp systems corresponding to the points on the vertical thin line in Figure 1. Similarly as we did before, we can obtain the solution of the crisp system $A^s(Fn_1^*, Fn_2)\mathbf{x} = \mathbf{b}$ as

$$x_j = f_j^*(Fn_2) = \frac{Fn_2 c_{1j}^* + c_{2j}^*}{Fn_2 c_3^* + c_4^*}, \tag{18}$$

for all $j \in \{1, \ldots, n\}$ and $Fn_2 \in \text{supp}(\widetilde{Fn_2})$. We find the parameters c_3^* and c_4^* by solving the system

$$\begin{cases} f_{denom}^l(Fn_1^*) = \underline{Fn}_2 c_3^* + c_4^*, \\ f_{denom}^u(Fn_1^*) = \overline{Fn}_2 c_3^* + c_4^*, \end{cases} \tag{19}$$

where $f_{denom}^l(Fn_1^*)$ and $f_{denom}^u(Fn_1^*)$ are the denominators of the parametric functions for respectively the lower (l) and the upper (u) bound of the support of $\widetilde{Fn_2}$, with the first fuzzy number as variable and evaluated in the fixed value Fn_1^*. Then, for all $j \in \{1, \ldots, n\}$, we obtain c_{1j}^* and c_{2j}^* by solving the following system:

$$\begin{cases} f_{num,j}^l(Fn_1^*) = \underline{Fn}_2 c_{1j}^* + c_{2j}^*, \\ f_{num,j}^u(Fn_1^*) = \overline{Fn}_2 c_{1j}^* + c_{2j}^*, \end{cases}$$

where $f_{num,j}^l(Fn_1^*)$ and $f_{num,j}^u(Fn_1^*)$ are the numerators of the j-th component of the parametric functions for respectively the lower (l) and the upper (u) bound of the support of $\widetilde{Fn_2}$, with the first fuzzy number as variable and evaluated in the fixed value Fn_1^*. Consequently, all possible solutions for the crisp systems $A(Fn_1^*, Fn_2)\mathbf{x} = \mathbf{b}$, for all $Fn_2 \in \text{supp}(\widetilde{Fn_2})$, can be obtained using (18). This approach can be used independently of the place of the first fuzzy number; it doesn't matter if the first fuzzy number is a diagonal or non-diagonal element of the matrix $\tilde{A}$ or a component of the vector $\tilde{\mathbf{b}}$.

3.2.2. *The second fuzzy number is a non-diagonal element of the matrix $\tilde{A}$*

Let us assume that the second fuzzy number is a non-diagonal element of the matrix $\tilde{A}$. We first fix the second fuzzy number on the lower and the upper bound of its support, but also on the modal value. In that way we can find the solutions for the lower, middle and upper line of Figure 1. Thereafter the parametric functions with the second fuzzy number as variable and a fixed first fuzzy number are calculated by using the earlier obtained parametric functions. We first solve the following system of linear equations:

$$\begin{cases} f_{denom}^l(Fn_1^*) = \underline{Fn_2}^2 c_4^* + \underline{Fn_2} c_5^* + c_6^*, \\ f_{denom}^m(Fn_1^*) = (\widetilde{Fn_2})_1^2 c_4^* + (\widetilde{Fn_2})_1 c_5^* + c_6^*, \\ f_{denom}^u(Fn_1^*) = \overline{Fn_2}^2 c_4^* + \overline{Fn_2} c_5^* + c_6^*, \end{cases}$$

where $f_{denom}^l(Fn_1^*)$, $f_{denom}^m(Fn_1^*)$ and $f_{denom}^l(Fn_1^*)$ are the denominators of the parametric functions for respectively the lower (l) bound of the support, the modal value (m) and the upper (u) bound of the support of $\widetilde{Fn_2}$, with the first fuzzy number as variable and evaluated in the fixed value Fn_1^*. Thereafter we calculate the parameters c_{1j}^*, c_{2j}^* and c_{3j}^* by solving the following system:

$$\begin{cases} f_{num,j}^l(Fn_1^*) = \underline{Fn}^2 c_{1j}^* + \underline{Fn} c_{2j}^* + c_{3j}^*, \\ f_{num,j}^m(Fn_1^*) = (\widetilde{Fn})_1^2 c_{1j}^* + (\widetilde{Fn})_1 c_{2j}^* + c_{3j}^*, \\ f_{num,j}^u(Fn_1^*) = \overline{Fn}^2 c_{1j}^* + \overline{Fn} c_{2j}^* + c_{3j}^*, \end{cases} \tag{20}$$

where $f_{num,j}^l(Fn_1^*)$, $f_{num,j}^m(Fn_1^*)$ and $f_{num,j}^l(Fn_1^*)$ are the numerators of the j-th component of the parametric functions for respectively the lower (l) bound of the support, the modal value (m) and the upper (u) bound of the support of $\widetilde{Fn_2}$, with the first fuzzy number as variable and evaluated in the fixed value Fn_1^*. The function $(f^*)_j^{sym}$ is then given by, for all $j \in \{1, \ldots, n\}$,

$$(f^*)_j^{sym}(Fn_2) = \frac{(Fn_2)^2 c_{1j}^* + Fn_2 c_{2j}^* + c_{3j}^*}{(Fn_2)^2 c_4^* + Fn_2 c_5^* + c_6^*},$$

for all $Fn_2 \in \text{supp}(\widetilde{Fn}_2)$.

We define for all $j \in \{1, \ldots, n\}$ the fuzzy number $\tilde{x}_j^{sym}$ as:

$$\begin{aligned}\tilde{x}_j^{sym}(x) = \sup\{\min(&\widetilde{Fn}_1(Fn_1^*), \widetilde{Fn}_2(Fn_2)) \mid \\ &Fn_1^* \in \text{supp}(\widetilde{Fn}_1) \text{ and} \\ &Fn_2 \in \text{supp}(\widetilde{Fn}_2) \text{ and} \\ &x = (f^*)_j^{sym}(Fn_2)\},\end{aligned} \quad (21)$$

for all $x \in (f^*)_j^{sym}(\text{supp}(\widetilde{Fn}_2))$, and

$$\tilde{x}_j^{sym}(x) = 0,$$

for all $x \in \mathbb{R} \setminus (f^*)_j^{sym}(\text{supp}(\widetilde{Fn}_2))$.

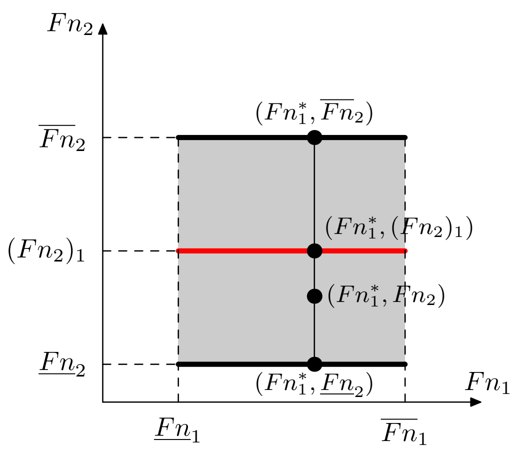

Fig. 2. Solving systems with two fuzzy coefficients where the second fuzzy number is a non-diagonal element of the matrix $\tilde{A}$.

3.2.3. *The second fuzzy number is a component of the vector* $\tilde{\mathbf{b}}$

When the second fuzzy number is located in the right-hand side of the system of linear fuzzy equations, i.e. when we have for instance that $\tilde{\mathbf{b}} = (\tilde{b}_1, b_2, \ldots, b_n)$, one sees immediately that $c_3^* = 0$ and $c_4^* = |A| = f_{denom}^l(Fn_1^*) = f_{denom}^u(Fn_1^*)$. Then we have to calculate the values for the parameters c_{1j}^* and c_{2j}^* by using the numerators of the parametric functions for the first fuzzy coefficient:

$$\begin{cases} f_{num,j}^l(Fn_1^*) = \underline{Fn_2} c_{1j}^* + c_{2j}^*, \\ f_{num,j}^u(Fn_1^*) = \overline{Fn_2} c_{1j}^* + c_{2j}^*, \end{cases}$$

for $j \in \{1, \ldots, n\}$. The function f_j^* is then given by, for all $j \in \{1, \ldots, n\}$,

$$f_j^*(Fn_2) = \frac{Fn_2 c_{1j}^* + c_{2j}^*}{f_{denom}^l(Fn_1^*)},$$

for all $Fn_2 \in \text{supp}(\widetilde{Fn}_2)$.

The solution $\tilde{x}_j^{sym}$ is obtained similarly as in (21) in all the three cases (Subsection 3.2.1, 3.2.2 and 3.2.3).

3.3. *Systems with more than two fuzzy coefficients*

Clearly, the procedure proposed in Subsection 3.2 can be extended to systems with more than two fuzzy coefficients. In Figure 3 the method is illustrated for 3 fuzzy coefficients. First we calculate all the parametric functions and solutions for the front and the back face of the cube. Thereafter we obtain the parametric functions and solutions corresponding to the lines between the front and the back face of the cube by using the parameters obtained earlier.

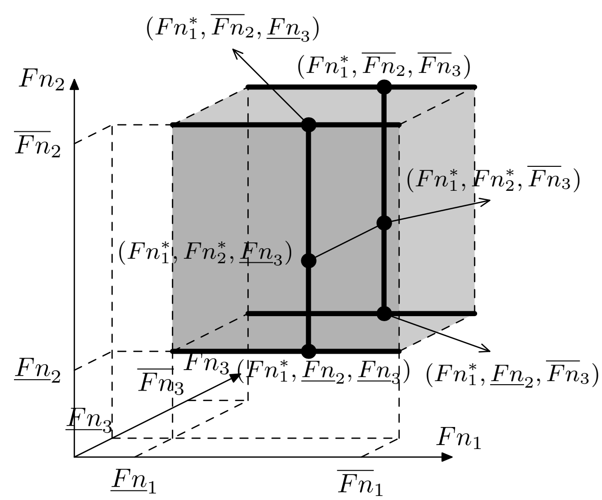

Fig. 3. Solving systems with three fuzzy coefficients

Example 3 *Let*

$$\tilde{A} = \begin{pmatrix} 1 & (0\ /\ 1\ /\ 2) \\ (0\ /\ 1\ /\ 2) & -2 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

First we obtain the determinants in which the fuzzy number is replaced by the lower bound of its support, its modal value and the upper bound of its support respectively:

$$|A^s(\underline{a}_{12})| = \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} = -2,$$
$$|A^s((\tilde{a}_{12})_1)| = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3,$$
$$|A^s(\overline{a}_{12})| = \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = -6.$$

Using (10) *we can deduce the parameters* c_4*,* c_5 *and* c_6*:*

$$\begin{cases} -2 = c_6, \\ -3 = c_4 + c_5 + c_6, \\ -6 = 4c_4 + 2c_5 + c_6 \end{cases} \iff \begin{cases} c_4 = -1, \\ c_5 = 0, \\ c_6 = -2. \end{cases}$$

Then we solve the systems in which the fuzzy number is replaced by the lower bound of its support, its modal value and the upper bound of its support respectively:

$$\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \iff \begin{cases} x_1 = 2, \\ x_2 = -\frac{3}{2}, \end{cases}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \iff \begin{cases} x_1 = \frac{7}{3}, \\ x_2 = -\frac{1}{3}, \end{cases}$$
$$\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \iff \begin{cases} x_1 = \frac{5}{3}, \\ x_2 = \frac{1}{6}. \end{cases}$$

Thus we calculate the values of the parameters c_{11}*,* c_{21}*,* c_{31} *and* c_{12}*,* c_{22}*,* c_{32} *by solving* (15)*:*

$$\begin{cases} 2(-2) = c_{31}, \\ \frac{7}{3}(-3) = c_{11} + c_{21} + c_{31}, \\ \frac{5}{3}(-6) = 4c_{11} + 2c_{21} + c_{31} \end{cases} \iff \begin{cases} c_{11} = 0, \\ c_{21} = -3, \\ c_{31} = -4, \end{cases}$$
$$\begin{cases} -\frac{3}{2}(-2) = c_{31}, \\ -\frac{1}{3}(-3) = c_{12} + c_{22} + c_{32}, \\ \frac{1}{6}(-6) = 4c_{12} + 2c_{22} + c_{32} \end{cases} \iff \begin{cases} c_{12} = 0, \\ c_{22} = -2, \\ c_{32} = 3. \end{cases}$$

The parametric functions are then given by

$$x_1 = f_1^{sym}(a) = \frac{-4-3a}{-2-a^2},$$
$$x_2 = f_2^{sym}(a) = \frac{3-2a}{-2-a^2},$$

where a *varies in* $]0,2[\; = \mathrm{supp}(0 \;/\; 1 \;/\; 2)$*. The parametric function of the first component reaches its maximum for* $\frac{-4}{3} + \frac{\sqrt{34}}{3} \in\;]0,2[$ *(see Figure 4). Hence the parametric function of this system is not monotonous. The first component of the solution is given by*

$$\tilde{x}_1(x) = \sup\big(\{0\} \cup \{\tilde{a}(a) \mid a \in \mathrm{supp}(\tilde{a}) = \,]0,2[\text{ and } x = \tfrac{-4-3a}{-2-a^2}\}\big).$$

We have that

$$x = \frac{-4-3a}{-2-a^2} \iff -xa^2 + 3a - 2x + 4 = 0 \iff a = \frac{-3 \pm \sqrt{-8x^2+16x+9}}{-2x}$$

and

$$a \in\,]0,2[\implies x \in \big]\min(f_1^{sym}(0), f_1^{sym}(2)), f_1^{sym}\big(\tfrac{-4}{3} + \tfrac{\sqrt{34}}{3}\big)\big[= \big]\tfrac{5}{3}, \tfrac{-9\sqrt{34}}{-68+8\sqrt{34}}\big[.$$

So we obtain (see Figure 4):

$$\tilde{x}_1(x) = \begin{cases} \max\{\tilde{a}(\frac{3+\sqrt{-8x^2+16x+9}}{2x}), \tilde{a}(\frac{3-\sqrt{-8x^2+16x+9}}{2x})\}, \\ \qquad \text{if } x \in\,]2, \frac{-9\sqrt{34}}{-68+8\sqrt{34}}], \\ \tilde{a}(\frac{3+\sqrt{-8x^2+16x+9}}{2x}), \;\; \text{if } x \in\,]\frac{5}{3}, 2], \\ 0, \;\; \text{else} \end{cases}$$
$$= \begin{cases} 2 - \frac{3+\sqrt{-8x^2+16x+9}}{2x}, \;\; \text{if } x \in\,]\frac{5}{3}, 2] \\ \qquad (\Rightarrow a \in [\frac{3}{2}, 2[), \\ \max\{\frac{3-\sqrt{-8x^2+16x+9}}{2x}, 2 - \frac{3+\sqrt{-8x^2+16x+9}}{2x}\}, \\ \qquad \text{if } x \in\,]2, \frac{7}{3}] \; (\Rightarrow a \in [1, \frac{3}{2}[\cup]0, \frac{2}{7}]), \\ \max\{\frac{3-\sqrt{-8x^2+16x+9}}{2x}, \frac{3+\sqrt{-8x^2+16x+9}}{2x}\}, \\ \qquad \text{if } x \in\,]\frac{7}{3}, \frac{-9\sqrt{34}}{-68+8\sqrt{34}}] \; (\Rightarrow a \in\,]\frac{2}{7}, 1[), \\ 0, \;\; \text{else} \end{cases}$$
$$= \begin{cases} \frac{4x-3-\sqrt{-8x^2+16x+9}}{2x}, \;\; \text{if } x \in\,]\frac{5}{3}, \frac{7}{3}], \\ \frac{3+\sqrt{-8x^2+16x+9}}{2x}, \;\; \text{if } x \in\,]\frac{7}{3}, \frac{-9\sqrt{34}}{-68+8\sqrt{34}}], \\ 0, \;\; \text{else} \end{cases}$$

using the fact that

$$\frac{3-\sqrt{-8x^2+16x+9}}{2x} > 2 - \frac{3+\sqrt{-8x^2+16x+9}}{2x} \iff 0 < x < \frac{3}{2}.$$

The membership function for the first component of the solution is shown in Figure 6. The second component of the solution is:

$$\tilde{x}_2(x) = \sup\big(\{0\} \cup \big\{\tilde{a}(a) \mid a \in \operatorname{supp}(\tilde{a}) =]0,2[\text{ and } x = \tfrac{3-2a}{-2-a^2}\big\}\big).$$

Here we have that

$$x = \frac{3-2a}{-2-a^2} \iff -xa^2 + 2a - 2x - 3 = 0$$
$$\iff a = \frac{2 \pm \sqrt{-8x^2 - 12x + 4}}{2x}$$

and

$$a \in]0,2[\implies x \in]f_2^{sym}(0), f_2^{sym}(2)[=]-\tfrac{3}{2}, \tfrac{1}{6}[.$$

So we obtain (see Figure 5):

$$\tilde{x}_2(x) = \begin{cases} \frac{2-\sqrt{-8x^2-12x+4}}{2x}, & \text{if } x \in]-\frac{3}{2}, -\frac{1}{3}], \\ 2 - \frac{2-\sqrt{-8x^2-12x+4}}{2x}, & \text{if } x \in]-\frac{1}{3}, \frac{1}{6}[, \\ 0, & \text{else} \end{cases}$$
$$= \begin{cases} \frac{2-\sqrt{-8x^2-12x+4}}{2x}, & \text{if } x \in]-\frac{3}{2}, -\frac{1}{3}], \\ \frac{4x-2+\sqrt{-8x^2-12x+4}}{2x}, & \text{if } x \in]-\frac{1}{3}, \frac{1}{6}[, \\ 0, & \text{else.} \end{cases}$$

The membership degree of the second component is shown in Figure 7.

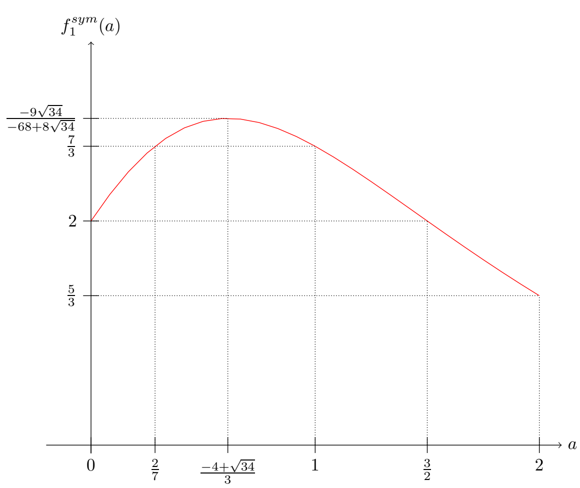

Fig. 4. The first parametric function, f_1^{sym}.

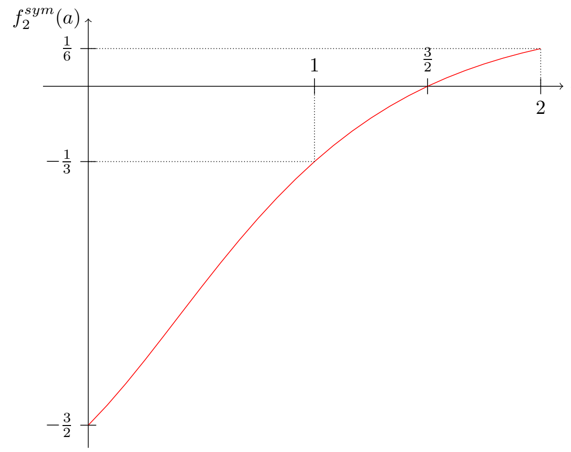

Fig. 5. The second parametric function, f_2^{sym}.

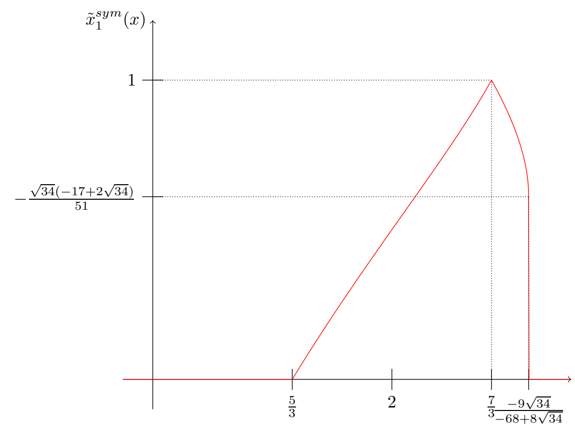

Fig. 6. The first component of the solution, $\tilde{x}_1^{sym}$.

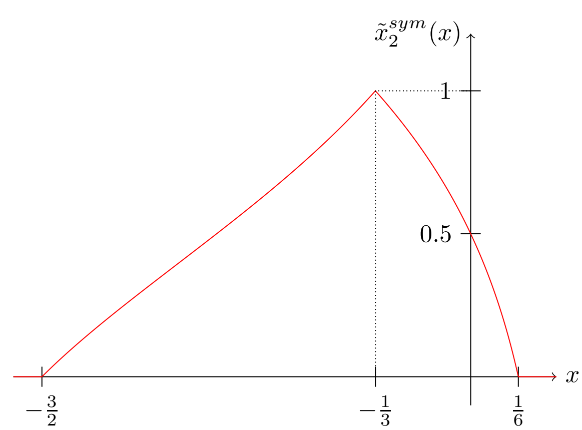

Fig. 7. The second component of the solution, $\tilde{x}_2^{sym}$.

Example 4 *Consider the following system of linear fuzzy equations:*

$$\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}},$$

where $\tilde{A} =$

$$\begin{pmatrix} 1 & (3/4/5) & (4/5/6) & 0 \\ (3/4/5) & -4 & (1/4/6) & (0/1/3) \\ (4/5/6) & (1/4/6) & 2 & 5 \\ 0 & (0/1/3) & 5 & 3 \end{pmatrix},$$

$$\tilde{\mathbf{b}} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

This system has a symmetric matrix $\tilde{A}$. The solution for this system is obtained on the one hand without taking the symmetry into account and on the other hand taking the symmetry into account. The difference in solution for both approaches is shown in Figure 8. When the symmetry is not taken into account, the solution is an overestimation.

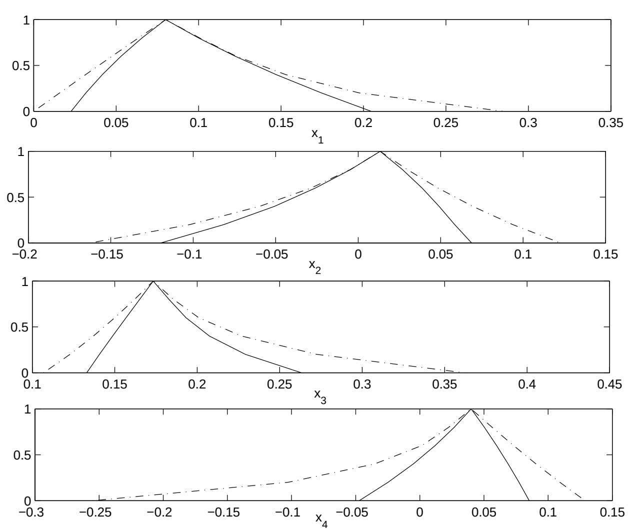

Fig. 8. Solutions for the system of Example 4 with the symmetry of the matrix $\tilde{A}$ taken into account (full line) and without taking the symmetry into account (interrupted line).

Now we can calculate the total operation count for systems of linear fuzzy equations where only non-diagonal elements of the matrix $\tilde{A}$ are fuzzy numbers. First the number of parametric functions has to be calculated. We denote the number of fuzzy numbers in the system as K. For $K = 1$ only one parametric function, for which the only fuzzy number varies in its support, has to be calculated. For $K = 2$ we first calculate the parametric functions for the three lines, the lower, middle and upper line of the rectangle (that indicates the cartesian product of the supports of the two fuzzy numbers) where the second fuzzy number is replaced by the lower bound of its support, the modal value and the upper bound of its support respectively. Thereafter the parametric functions for the perpendicular lines where the first fuzzy number is replaced by a discrete value in its support, are calculated. In total $m + 3$ parametric functions are calculated. For $K = 3$ we calculate the parametric functions for the front, middle and back face of the cube and thereafter the parametric functions for the perpendicular lines where the first and second fuzzy number are fixed on a discrete value of their supports $\text{supp}(\widetilde{Fn}_1) \times \text{supp}(\widetilde{Fn}_2)$, are calculated. The total number of parametric functions we obtain is $m^2 + 3(m + 3) = m^2 + 3m + 9 = \frac{m^3-3^3}{m-3}$. For K fuzzy numbers, the parametric functions corresponding to the front, middle and back $(K-1)$-dimensional hypercube are calculated and thereafter we calculate the parametric functions for the perpendicular lines corresponding to the discrete values in the $(K-1)$-dimensional hypercube. In total we calculate $3(\frac{m^{K-1}-3^{K-1}}{m-3}) + m^{K-1} = \frac{m^K-3^K}{m-3}$ parametric functions.

The total cost to solve a system of linear fuzzy equations with K fuzzy numbers as non-diagonal elements in the matrix $\tilde{A}$ where the symmetry is taken into account, can then be calculated as follows:

3^K determinants of of $(n \times n)$-matrices	$3^K \frac{4n^3+3n^2-n-6}{6}$
3^K $(n \times n)$-systems	$3^K n^2$
$\frac{m^K-3^K}{m-3}(n+1)$ evaluations equivalent to (20)	$\frac{m^K-3^K}{m-3}(n+1)76$
$n(m^K - 3^K)$ evaluations equivalent to (9)	$11n(m^K - 3^K)$

The total operation cost is $\frac{3^{K-1}(4n^3+3n^2-n-6)}{2} + 3^K n^2 + 76\frac{m^K-3^K}{m-3}(n+1) + 11n(m^K - 3^K)$. The total operation cost for the method of Buckley and Qu is $\frac{m^K(4n^3+9n^2-7n)}{6}$, since m^K $(n \times n)$-systems have to be solved. It is easy to see that for large n, m and K the method described above needs less computation time than the method of Buckley and Qu.

4. Conclusion

In this paper we have proposed a method to solve linear $(n \times n)$-systems in which some (or all) coefficients are fuzzy and in which we take the symmetry of the matrix $\tilde{A}$ into account. While in the method of Buckley and Qu for every element in the support of each fuzzy number the corresponding crisp $(n \times n)$-system is solved, in our method only the crisp $(n \times n)$-systems corresponding to the bounds of each support and the modal value of each fuzzy number must be solved, and the other necessary solutions for the combinations of the lower and the upper bounds of the considered α-level are obtained by evaluating parametric functions. By considering the symmetry of the matrix $\tilde{A}$, the obtained solution is a better solution, as there is no overestimation. By applying the parametric functions to find all possible solutions, we see that the computation time is considerably reduced w.r.t. the method of Buckley and Qu.

Acknowledgment

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1. J. J. Buckley and Y. Qu. Solving systems of linear fuzzy equations. *Fuzzy Sets and Systems*, 43:33–43, 1991.
2. D. M. Gay. Solving interval linear equations. *SIAM Journal on Numerical Analysis*, 19:858–870, 1982.
3. R. Fuller. On stability in possibilistic linear equality systems with lipschitzian fuzzy numbers. *Fuzzy Sets and Systems*, 34:347–353, 1990.
4. F. Abramovich, M. Wagenknecht, and Y. I. Khurgin. Solution of LR-type fuzzy systems of linear algebraic equations. *Busefal*, 35:86–99, 1988.
5. A. Neumaier. *Interval methods for systems of equations*, volume 37 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1990.
6. T. Ohta, T. Ogita, S. M. Rump, and S. Oishi. Numerical verification method for arbitrarily ill-conditioned linear systems. *Transactions on the Japan Society for Industrial and Applied Mathematics (Trans. JSIAM)*, 15:269287, 2005.
7. S. M. Rump and T. Ogita. Super-fast validated solution of linear systems. *Journal of Computational and Applied Mathematics (JCAM)*, 199:199–206, 2007.
8. S. M. Rump. Verified solution of large linear and nonlinear systems. In H. Bulgak and C. Zenger, editors, *Error Control and adaptivity in Scientific Computing*, page 279298. Kluwer Academic Publishers, 1999.
9. D. Moens and D. Vandepitte. Fuzzy finite element method for frequency response function analysis of uncertain structures. *AIAA Journal*, 40:126–136, 2002.
10. A. Vroman, G. Deschrijver, and E. E. Kerre. Solving systems of linear fuzzy equations by parametric functions. *IEEE Transactions on Fuzzy Systems*, 15:370–384, 2007.
11. A. Vroman, G. Deschrijver, and E. E. Kerre. Solving systems of linear fuzzy equations by parametric functions - An improved algorithm. *Fuzzy Sets and Systems*, 158:1515–1534, 2007.
12. E. E. Kerre. *Fuzzy Sets and Approximate Reasoning*. Xian Jiaotong University Press, Xian, People's Republic of China, 1999.
13. R. Moore. *Interval Arithmetic*. Prentice-Hall, Englewood Cliffs, NJ, USA, 1966.
14. H. T. Nguyen. A note on the extension principle for fuzzy sets. *Journal Mathematical Analysis and Applications*, 64:369–380, 1978.