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Geometric Differences between the Burgers and the Camassa-Holm Equations

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Abstract

The Burgers equation and the Camassa-Holm equations can both be recast as the Euler equation for a right-invariant metric on the diffeomorphism group of the circle, the L^2-metric for Burgers and the H^1-metric for Camassa-Holm. Their geometric behaviors are however very different. We present a survey of this geometrical approach and discuss these differences.

1 Introduction

The idea of studying geodesic flow in order to analyze the motion of inertial continuum mechanical systems is due to Arnold [2]. He showed that the Euler equations of the motion of a rigid body and the Euler equations of hydrodynamics (with fixed boundary) can both be obtained as the geodesic equations of a one-sided invariant Riemannian metric on a Lie group. In each case the metric corresponds to the kinetic energy of the system and is given by an inner product on the Lie algebra of the group.

For the rigid body the group is the rotation group SO(3) and the inner product on the Lie algebra $\mathfrak{so}(3) \simeq \mathbb{R}^3$ (the space of angular velocities) is given by

$$< \omega, \eta >= A\omega \cdot \eta$$

where A is the *inertia tensor* of the rigid body.

For hydrodynamics, the group is the infinite-dimensional Lie group SDiff(D) of smooth volume- and orientation-preserving diffeomorphisms of the fluid domain D. The Lie algebra SVect(D) of SDiff(D) consists of all divergence-free vector fields tangent to the boundary of D and is equipped with the L^2 inner-product

$$< u, v >= \int_D (u \cdot v)\, d\mu.$$

This structure is the prototype for the mathematical treatment of many important physical systems. Other equations from mathematical physics were found to have an interpretation as geodesic flows on diffeomorphism groups (see for example [21, 22, 31, 32]).

However, Arnold's initial paper remains essentially formal. Subsequent analytical work have been achieved later. Ebin and Marsden [13] enlarged the actual configuration space of smooth diffeomorphisms to diffeomorphisms of Sobolev class H^s, making possible further analytical studies. This work was then extended by Shkoller in [34] to the *mean motion of an ideal fluid* where the L^2-metric on $\mathrm{SDiff}(D)$ is replaced by the H^1-metric.

The aim of this paper is to present a detailed overview of two such systems arising in fluid mechanics: the inviscid *Burgers* equation and the *Camassa-Holm* equation. Both of these systems appear as the Euler equation on the diffeomorphism group of the circle $\mathrm{Diff}(\mathbb{S}^1)$ for some right-invariant metric, the L^2-metric for Burgers' equation and the H^1-metric for Camassa-Holm's equation. It is interesting to consider both of these metrics as particular members of the more general family of H^k-metrics ($k \in \mathbb{N}$) where $k = 0$ correspond to the L^2-metric. As we shall see, the geometric behavior of these two systems $k = 0$ and $k \geq 1$ appear to be very different.

Besides the motivation of understanding Burgers' and Camassa-Holm's equations from a geometrical point of view, there is another reason to study Euler equations for H^k-metrics on $\mathrm{Diff}(\mathbb{S}^1)$. The rigorous analytical analysis of Euler equations on smooth diffeomorphisms groups is a difficult problem. $\mathrm{Diff}(\mathbb{S}^1)$ is the simplest of the diffeomorphisms groups and it is expected that understanding some mechanisms within this setting can give insight to deal with more ambitious situations.

Before proceeding with the special case of Euler's equation on $\mathrm{Diff}(\mathbb{S}^1)$ in Section 3, we first review some fundamental aspects of the general Euler equation on an abstract Lie group in Section 2. Section 4 is devoted to the study of short-time existence for geodesics of the H^k-metrics on the diffeomorphism group of the circle. Section 5 is about the regularity of the Riemannian exponential map. A final section recapitulates the main results and deals with the case of the Virasoro group and the KdV equation: a case which has been considered prior to the case of $\mathrm{Diff}(\mathbb{S}^1)$ but which, form a didactic point of view, has to be studied afterwards because of additional technical difficulties.

2 Euler equation on an abstract Lie group

The configuration space of the motions of a rigid body around its center of mass can be identified with group of rotations SO(3). A *motion* of the body is represented by a (parameterized) curve lying in the group SO(3). The "velocity vector", given by the derivative of the motion, lies in the tangent space to the group element. It can be pulled back to the Lie algebra $\mathfrak{so}(3)$ (identified with the 3-space $\mathbb{R}^3$) by left translation and defines the *angular velocity* ω. The kinetic energy corresponds to a *left-invariant* Riemannian metric on SO(3). Its value at the identity element is given by

$$K = \frac{1}{2} A\omega \cdot \omega, \qquad \omega \in \mathbb{R}^3$$

where A is the *inertia tensor* of the body Σ

$$A\omega = \int_{\Sigma} r \wedge (\omega \wedge r)\, \rho dr.$$

If there are no external forces, the motions are the *extremals of the kinetic energy*, that is the geodesics of the left-invariant metric on $SO(3)$ defined by the inertia operator A.

Equations for angular velocities were derived by Euler [14] in 1765 and Lagrange [25] in 1788

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\omega_2\omega_3, \qquad \dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_1\omega_3, \qquad \dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2,$$

where ω_k are the angular velocities around principal axis and I_k are the eigenvalues of the inertia tensor A.

The derivation of the general Euler equation for an arbitrary Lie group G given in this section is for a *right-invariant metric*. It follows one given by Euler and Lagrange for the rigid body (see [23] for more details).

2.1 Right-invariant metric on a Lie group

A right-invariant metric on a Lie group G is completely determined by its value at the unit element e of the group, or in other words, by an inner product on its Lie algebra $\mathfrak{g}$. This inner product can be expressed in terms of a symmetric linear operator

$$A : \mathfrak{g} \to \mathfrak{g}^*,$$

that is $(Au, v) = (Av, u)$ for all $u, v \in \mathfrak{g}^*$, where the round brackets stand for the pairing of elements of the dual spaces $\mathfrak{g}$ and $\mathfrak{g}^*$. We call A the *inertia operator*, in reminiscence of the motion of the rigid body.

2.2 Eulerian velocities and momenta

Let $g(t)$ be a smooth curve on G. The *velocity vector* or *Lagrangian velocity* is defined as the derivative $\dot{g}(t)$, which lies in T_gG. Left and right *Eulerian velocities* are defined by

$$u_L = L_{g^{-1}}\dot{g}, \qquad u_R = R_{g^{-1}}\dot{g},$$

where L, R stand for left and right translations on G respectively. Similarly, we get left and right *Eulerian momenta* by

$$m_L = L_g^* m, \qquad m_R = R_g^* m.$$

where

$$m(t) =< \dot{g}(t), \cdot >_{g(t)}, \qquad m(t) \in T^*_{g(t)}G,$$

is the *co-velocity*. These four objects are related by

$$m_R = Au_R, \qquad u_R = Ad_g\, u_L, \qquad m_R = Ad_g^*\, m_L, \tag{2.1}$$

where Ad is the *adjoint action* of G on $\mathfrak{g}$ defined by

$$Ad_g\, u = L_g R_g^{-1} u, \qquad g \in G, \quad u \in \mathfrak{g}$$

and Ad^* is the *coadjoint action* of G on $\mathfrak{g}^*$ defined by

$$(Ad_g^*\, m, u) = (m, Ad_{g^{-1}}\, u) \qquad g \in G, \quad u \in \mathfrak{g}, \quad m \in \mathfrak{g}^*.$$

2.3 Euler's theorems

The invariance of the metric under right translations generates a *conserved momentum* as a result of *Noether's theorem* and leads to *Euler first theorem* which is a generalization of the *angular momentum conservation law* for the motion of a free rigid body.

Theorem 1 (Euler's first theorem)**.** *The* left Eulerian momentum *associated to a geodesic of a right-invariant metric on a Lie group G is independent of t*

$$\frac{dm_L}{dt} = 0. \tag{2.2}$$

Remark 1. In the case of the motion of an incompressible fluid, this conserved momentum corresponds to the *isovorticity*: the curl of the velocity vector field is transported along the flow.

Taking the time derivative of the third relation of (2.1) and using (2.2), we then obtain *Euler equation* on $\mathfrak{g}^*$. Its expression involves the definition of the coadjoint action $\mathrm{ad}_u^* m$ of $\mathfrak{g}$ on $\mathfrak{g}^*$, defined by

$$(\mathrm{ad}_u^* m, v) = -(m, \mathrm{ad}_u v), \qquad u, v \in \mathfrak{g}, \quad m \in \mathfrak{g}^*.$$

Theorem 2 (Euler's second theorem)**.** *The right Eulerian momentum associated to a geodesic of a right-invariant metric on a Lie group G satisfies the following evolution equation*

$$\frac{dm_R}{dt} = \mathrm{ad}_{u_R}^* m_R, \tag{2.3}$$

which is called Euler equation *on* $\mathfrak{g}^*$*.*

Remark 2. There is another interpretation and another way to derive Euler equation on $\mathfrak{g}^*$. On a general Riemannian manifold, the geodesic flow is a *Hamiltonian flow* for the *canonical symplectic structure* on the cotangent bundle. The Hamiltonian is given by the energy functional. In the special case of a Lie group G, the canonical symplectic structure is invariant (either by left or right translations). It does not get down to a symplectic structure on the quotient space $T^*G/G \cong \mathfrak{g}^*$ but the corresponding *Poisson structure* induces a natural Poisson structure on $\mathfrak{g}^*$ called the *Lie-Poisson Brackets*

$$\{f, g\}_{LP}(m) = m([d_m f, d_m g]), \qquad f, g \in C^\infty(\mathfrak{g}^*).$$

A one-sided invariant metric on G generates a reduced Hamiltonian function H_A and a reduced Hamiltonian vector field X_A on $\mathfrak{g}^*$

$$H_A(m) = \frac{1}{2}(m, A^{-1}m), \qquad X_A(m) = ad^*_{A^{-1}m} m, \qquad m \in \mathfrak{g}^*$$

where A is the *inertia operator*.

2.4 Euler equation on $\mathfrak{g}$

To obtain a contravariant formulation of (2.3), that is an equation on $\mathfrak{g}$ rather than $\mathfrak{g}^*$, we need the adjoint (relative to the metric A) of ad, that is a bilinear operator $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

$$< \mathrm{ad}_u v, w >=< B(w,u), v >, \qquad u,v,w \in \mathfrak{g}. \tag{2.4}$$

Equation (2.3) can then be rewritten as a quadratic differential equation on $\mathfrak{g}$,

$$\dot{u}_R = -B(u_R, u_R), \tag{2.5}$$

which we call *Euler equation* on $\mathfrak{g}$ associated to the *right-invariant metric* defined by the inertia operator A.

Remark 3. Integration of the geodesic equations is reduced to two successive quadratures

$$\dot{u}_L = -B(u_R, u_R), \qquad \dot{g} = R_g\, u_R.$$

When the metric is bi-invariant, $B(u,u) = 0$, and the geodesics through the unit element are just one-parameter subgroups of G.

2.5 Sectional curvatures

The Levi-Civita connection of a right-invariant metric is also right-invariant. It is thus defined by its value at the unit element

$$(\nabla_{\xi_u} \xi_v)(e) = \frac{1}{2}[u,v] - \frac{1}{2}\big(B(u,v) + B(v,u)\big),$$

where ξ_u and ξ_v are the right-invariant vector fields generated by u and v. The sectional curvatures of such a metric are also right-invariant and also determined by their value at the unit element of the group. They have been computed by Arnold [2].

Theorem 3 (Arnold, 1966)**.** *The sectional curvature C_{uv} for a* right-invariant *metric on a Lie group G at the unit element e for the* 2*-plane defined by orthonormal vectors $u,v \in \mathfrak{g}$ is given by*

$$C_{uv} =< \delta,\delta > +2 < \alpha,\beta > -3 < \alpha,\alpha > -4 < B_u, B_v > \tag{2.6}$$

where

$$2\alpha = [u,v], \qquad 2\beta = B(u,v) - B(v,u), \qquad 2\delta = B(u,v) + B(v,u)$$

and

$$2B_u = B(u,u), \qquad 2B_v = B(v,v).$$

3 Euler equations on $\mathrm{Diff}(\mathbb{S}^1)$

In this section, we extend the theory presented in the preceding section for a finite dimensional Lie group G to the group $\mathrm{Diff}(\mathbb{S}^1)$ of smooth, orientation preserving diffeomorphisms of the circle. This group is naturally equipped with a *Fréchet manifold* structure. More precisely, we can cover $\mathrm{Diff}(\mathbb{S}^1)$ with charts taking values in the *Fréchet vector space*[1] $\mathrm{C}^\infty(\mathbb{S}^1)$ and in such a way that the change of charts are smooth maps (see [11] or [13] for more details).

Since the composition and the inverse are smooth maps for this structure we say that $\mathrm{Diff}(\mathbb{S}^1)$ is a *Fréchet-Lie group* [17]. Its "*Lie algebra*" $\mathrm{Vect}(\mathbb{S}^1)$ is isomorphic to $\mathrm{C}^\infty(\mathbb{S}^1)$ with the Lie bracket given by

$$[u, v] = u_x v - u v_x.$$

3.1 H^k metrics

A *right-invariant* metric on $\mathrm{Diff}(\mathbb{S}^1)$ is defined by an inner product $\mathbf{a}$ on the Lie algebra of the group $\mathrm{Vect}(\mathbb{S}^1) = \mathrm{C}^\infty(\mathbb{S}^1)$. In this article, we study the case where $\mathbf{a}$ is the H^k-*Sobolev* inner product ($k \geq 0$),

$$\mathbf{a}_k(u, v) = \int_{\mathbb{S}^1} \left(uv + u_x v_x + \cdots + u_x^{(k)} v_x^{(k)} \right) dx \qquad u, v \in \mathrm{C}^\infty(\mathbb{S}^1). \tag{3.1}$$

This inner product generates, by translation on each tangent space, a *weak-Riemannian metric* on $\mathrm{Diff}(\mathbb{S}^1)$. It is called "weak" because, even if it is non-degenerate, it is not an isomorphism between the tangent space and the co-tangent space. Besides, the topology induced by the associated norm is not complete and is weaker than the initial C^∞ topology.

In finite dimensional Riemannian geometry, the *geodesic pseudo-distance* $d(x, y)$, defined as the infimum of the lengths of all piecewise C^1 curves from x to y, is in fact a distance, that is $d(x, y) > 0$ if $x \neq y$. This may not be the case for a weak-Riemannian metric. For the H^k metric on $\mathrm{Diff}(\mathbb{S}^1)$, Michor and Mumford [29] proved the following result.

Theorem 4 (Michor & Mumford 2005)**.** *The* Riemannian pseudo-distance d_k *induced by the* H^k*-metric on* $\mathrm{Diff}(\mathbb{S}^1)$ *is a distance for* $k \geq 1$. *It vanishes identically for* $k = 0$ *(*L^2 *metric).*

Sketch of proof. For any piecewise C^1 curve φ in $\mathrm{Diff}(\mathbb{S}^1)$ we have

$$L_k(\varphi) = \int_0^1 \left\| \varphi_t \circ \varphi^{-1} \right\|_{H^k} dt \geq \int_0^1 \left\| \varphi_t \circ \varphi^{-1} \right\|_{H^1} dt = L_1(\varphi), \qquad k \geq 1.$$

Hence, it is enough to show that the pseudo-distance d_1 induced by the H^1-metric is a distance on $\mathrm{Diff}(\mathbb{S}^1)$. Let $\varphi_1 \in \mathrm{Diff}(\mathbb{S}^1)$ be a diffeomorphism which is different from the

[1] A topological vector space E has a canonical *uniform structure*. When this structure is *complete* and when the topology of E may be given by a countable family of *semi-norms*, we say that E is a Fréchet vector space. In a Fréchet space, such classical results like the *Cauchy-Lipschitz theorem* or the *local inverse theorem* are no longer valid in general as they are in on Banach manifold. The typical example of a Fréchet space is the space of smooth functions on a compact manifold where semi-norms are just the C^k-norms ($k = 0, 1, \ldots$).

identity. We are going to show that the H^1-length of any path $\varphi(t,\cdot)$ joining Id and φ_1 is bounded from below by some positive constant independent of the path. First, notice that since $\varphi_1 \neq Id$, the smooth periodic function $f = (\varphi_1)_x - 1$ does not vanish identically and hence

$$\begin{aligned} \int_{\mathbb{S}^1} f(\varphi_1^{-1}(y))\,dy - \int_{\mathbb{S}^1} f(x)\,dx &= \int_{\mathbb{S}^1} f(x)\,(\varphi_1)_x\,(x)\,dx - \int_{\mathbb{S}^1} f(x)\,dx \\ &= \int_{\mathbb{S}^1} \left[(\varphi_1)_x - 1\right]^2\,dx > 0. \end{aligned}$$

Let

$$a(t) = \int_{\mathbb{S}^1} f(\psi(t,x))\,dx,$$

where $\psi(t,\cdot) = \varphi^{-1}(t,\cdot)$. We have

$$\dot{a} = \int_{\mathbb{S}^1} (f' \circ \psi)\psi_t\,dx = -\int_{\mathbb{S}^1} (f' \circ \psi)\psi_x u\,dx = -\int_{\mathbb{S}^1} (f \circ \psi)_x u\,dx,$$

where $u = \varphi_t \circ \varphi^{-1}$ because $\psi_t = -\psi_x u$. Integrating by parts we get

$$\dot{a} = \int_{\mathbb{S}^1} (f \circ \psi) u_x\,dx$$

and hence

$$|\dot{a}| \leq \max\big(\,|f|\,\big) \int_{\mathbb{S}^1} |(u)_x|\;dx \leq C\,\|u\|_{H^1}\,.$$

Therefore

$$0 < |a(1) - a(0)| \leq C L_1(\varphi),$$

and since C does not depend on the path, we are done.

To show that the pseudo-distance d_0 vanishes identically on $\mathrm{Diff}(\mathbb{S}^1)$, we introduce the subset $\mathcal{H}$ of all diffeomorphisms $\varphi_1 \in \mathrm{Diff}(\mathbb{S}^1)$ with the following property: For each $\varepsilon > 0$ there exists a smooth curve $\varphi(t,\cdot)$ from the identity to φ_1 in $\mathrm{Diff}(\mathbb{S}^1)$ with energy

$$E_0(\varphi) = \int_{\mathbb{S}^1} \|u\|_{L^2}^2\;dt \leq \varepsilon.$$

Since $L_0(\varphi)^2 \leq E_0(\varphi)$, it is enough to show that $\mathcal{H} = \mathrm{Diff}(\mathbb{S}^1)$. We claim that $\mathcal{H}$ is a normal subgroup of $\mathrm{Diff}(\mathbb{S}^1)$ [29]. Moreover, since $\mathrm{Diff}(\mathbb{S}^1)$ is a *simple group* (*i.e* has no nontrivial normal subgroup) [16], it is enough to show that $\mathcal{H}$ is not reduced to $\{Id\}$, which is done in [29], using compression waves. ∎

3.2 Euler equations

The *topological dual* of the Fréchet space $\mathrm{Vect}(\mathbb{S}^1)$ is isomorphic to the space of distributions on the circle. We define the *regular dual* of $\mathrm{Vect}(\mathbb{S}^1)$, denoted $\mathrm{Vect}^*(\mathbb{S}^1)$, as the subspace of linear functionals with a smooth density $m \in \mathrm{C}^\infty(\mathbb{S}^1)$

$$u \mapsto \int_{\mathbb{S}^1} mu\, dx.$$

The L^2 inner product defines an isomorphism between $\mathrm{Vect}(\mathbb{S}^1)$ and its regular dual $\mathrm{Vect}^*(\mathbb{S}^1)$. The *inertia operator* A_k associated to the H^k metric (3.1) is defined as an operator from $\mathrm{Vect}(\mathbb{S}^1)$ to its regular dual $\mathrm{Vect}^*(\mathbb{S}^1)$, both spaces being identified with $\mathrm{C}^\infty(\mathbb{S}^1)$. It is given by

$$A_k = 1 - \frac{d^2}{dx^2} + \cdots + (-1)^k \frac{d^{2k}}{dx^{2k}}$$

which is a continuous, symmetric, invertible linear operator. The corresponding Euler equation is

$$u_t = -A_k^{-1} \left[2A_k(u)u_x + uA_k(u)_x \right]. \tag{3.2}$$

Two cases are of special interest: $k = 0$ and $k = 1$.

Burgers equation. For $k = 0$ (that is for the L^2-metric), the corresponding Euler equation (3.2) is the *inviscid Burgers equation* [4]

$$u_t + 3uu_x = 0, \tag{3.3}$$

also known as Hopf equation. Equation (3.3) can be studied quite explicitly [18]. All solutions of (3.3) but the constant functions have a finite life span and (3.3) is a simplified model for the occurrence of shock waves in gas dynamics.

Camassa-Holm equation. For $k = 1$ (that is for the H^1-metric), the corresponding Euler equation (3.2) is the *Camassa-Holm equation*

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0. \tag{3.4}$$

Equation (3.4) is a model for the unidirectional propagation of shallow water waves [5, 19]. It has a bi-Hamiltonian structure [15] and is completely integrable [8]. Some solutions of (3.4) exist globally in time [6], whereas others develop singularities in finite time [6, 27]. The blowup phenomenon can be interpreted as a simplified model for wave breaking – the solution (representing the water's surface) stays bounded while its slope becomes unbounded [6].

3.3 Sectional curvatures

The existence of a covariant derivative compatible with a right-invariant, weak Riemannian metric is ensured by the existence of the adjoint operator B (see [10] for details). For the H^k metric we have

$$B_k(u, v) = A_k^{-1} \left[2A_k(u)v_x + vA_k(u)_x \right].$$

The sectional curvatures are then computed using Arnold's formulas (2.6).

For $k = 0$ (Burgers equation), we get

$$C_{uv} = \int_{\mathbb{S}^1} [u, v]^2 \, dx$$

so that all sectional curvatures are *non-negative*.

For $k = 1$ (Camassa-Holm equation), we get

$$\begin{aligned} 4C_{uv} = & \int_{\mathbb{S}^1} \Big[\big(2mv_x + vm_x + 2nu_x + un_x\big) A^{-1} \big(2mv_x + vm_x + 2nu_x + un_x\big) \Big] \, dx \\ & +2 \int_{\mathbb{S}^1} \Big[\big(u_x v - uv_x\big)\big(2mv_x + vm_x - 2nu_x - un_x\big) \Big] \, dx \\ & -3 \int_{\mathbb{S}^1} \Big[A\big(u_x v - uv_x\big)\big(u_x v - uv_x\big) \Big] \, dx \\ & -4 \int_{\mathbb{S}^1} \Big[\big(2mu_x + um_x\big) A^{-1} \big(2nv_x + vn_x\big) \Big] \, dx \end{aligned}$$

where $A = I - d^2/dx^2$, $m = A(u)$, $n = A(v)$. This expression is difficult to analyze. However, for $u = \cos(px)$ and $v = \cos(qx)$, where $p \neq \pm q$ and $p, q \neq 0$, we get

$$C_{pq} = \frac{P(p, q)}{Q(p, q)}$$

where the denominator

$$Q(p, q) = 8\pi \, \left(p^2 + 1\right) \, \left(q^2 + 1\right) \, \left((q - p)^2 + 1\right) \, \left((q + p)^2 + 1\right)$$

is always positive and the numerator

$$\begin{aligned} P(p, q) = & -4\left(q^8 + p^8\right) - 4\left(q^6 + p^6\right) + 8\left(q^4 + p^4\right) + 8\left(q^2 + p^2\right) \\ & - 2\,p^4\,q^4 + 88\,p^2\,q^2 + 93\left(p^2\,q^4 + p^4\,q^2\right) + 41\left(p^2\,q^6 + p^6\,q^2\right) \end{aligned}$$

changes sign. Hence, for the Camassa-Holm equation, the sectional curvature takes positive and negative values depending on the direction plane (u, v).

4 Short-time existence of the geodesics

The fist step that is needed to make meaningful this geometrical approach is to establish *short-time existence* of the geodesics. There is no general theorem for short-time existence of evolution equations on a Fréchet space other than the *Nash-Moser theorem* [17]. However, to check that a given equation satisfies the hypothesis required by this theorem can be very difficult. A direct proof of short-time existence was provided in [11] for the right-invariant metric on $\mathrm{Diff}(\mathbb{S}^1)$ generated by the H^k Sobolev norm for $k \geq 1$. Even if the spirit of the proof is "Nash-Moser", the proof does not rely on this theorem and is much simpler.

Theorem 5 (Constantin & Kolev 2003). *Let $k \geq 1$. For all $T > 0$, there exists a neighbourhood of the origin V in* $\mathrm{Vect}(\mathbb{S}^1)$ *such that for all $u_0 \in V$, there exists a unique geodesic*

$$\varphi \in C^\infty\left([0,T); \mathrm{Diff}(\mathbb{S}^1)\right)$$

for the metric H^k, starting at $\varphi(0) = Id \in \mathrm{Diff}(\mathbb{S}^1)$ in the direction $u_0 \in T_{Id}\mathrm{Diff}(\mathbb{S}^1)$. Moreover, the solution depends smoothly on the initial data $u_0 \in \mathrm{C}^\infty(\mathbb{S}^1)$.

Remark 4. For $k = 0$, which corresponds to the *inviscid Burgers equation*, one can prove short-time existence by other means [18, 10] but the proof below does not apply. The only geodesic that can be continued indefinitely in time is that in the constant direction.

The idea of the proof given in [11] is to study the geodesic flow on each Hilbert manifold

$$\mathcal{D}^n = \left\{\varphi \text{ is an orientation-preserving, } C^1\text{-diffeomorphism of class } H^n\right\}$$

for $n \geq 2k+1$, obtaining short-time existence on a maximal interval $[0, T_n)$. Thereafter one checks that the decreasing sequence T_n does not go to 0 as $n \to +\infty$, ensuring thus short-time existence on $\mathrm{Diff}(\mathbb{S}^1) = \bigcap_{n=2k+1}^{+\infty} \mathcal{D}^n$.

Notice that it is advisable to avoid considering directly the Euler equation

$$u_t = -B_k(u,u) = -A_k^{-1}\left[2A_k(u)u_x + uA_k(u_x)\right]. \tag{4.1}$$

The reason is that A_k is a differential operator of degree $2k$, and therefore the right hand-side of (4.1) is a pseudo-differential operator of degree 1 because of the “bad term” $uA_k(u_x)$. Hence (4.1) is not an *ordinary differential equation* in $H^n(\mathbb{S}^1)$.

The following observation was used in [11] to overcome this difficulty. The operator

$$C_k(u) = A_k(uu_x) - uA_k(u_x)$$

is a quadratic differential operator of degree $2k$. Therefore, if $k \geq 1$, the right hand-side of

$$u_t + uu_x = -A_k^{-1}\left[2A_k(u)u_x - C_k(u)\right]$$

is a pseudo-differential operator of degree 0. Moreover, $u_t + uu_x$ is just $v_t \circ \varphi^{-1}$, where $v = \varphi_t = u \circ \varphi$ is the *Lagrangian velocity*. It is therefore convenient to recast the problem as

$$\begin{cases} \varphi_t = v, \\ v_t = R_\varphi \circ P_k \circ R_{\varphi^{-1}}(v), \end{cases} \tag{4.2}$$

where $P_k = -A_k^{-1} \circ Q_k$ and $Q_k(u) = 2A_k(u)u_x - C_k(u)$.

Step 1

For each $n \geq 2k+1$, system (4.2) is a smooth *ordinary differential equation* on the Hilbert manifold $\mathcal{D}^n \times H^n(\mathbb{S}^1)$. This results from the following proposition.

Proposition 1. *For each $k \geq 1$ and each $n \geq 2k+1$, the operator*

$$\tilde{P}_k(\varphi, v) = \Big(\varphi, R_\varphi \circ P_k \circ R_{\varphi^{-1}}(v)\Big)$$

is a smooth map from $\mathcal{D}^n \times H^n(\mathbb{S}^1)$ to itself.

Remark 5. We cannot conclude directly from the smoothness of P_k that $\tilde{P}_k$ is smooth because *neither* the composition nor the inversion are smooth maps on $\mathcal{D}^n$. The Hilbert manifold $\mathcal{D}^n$ is only a topological group [13].

Proof. To prove the smoothness of $\tilde{P}_k$, we write it as the composition $\tilde{P}_k = \tilde{A}_k^{-1} \circ \tilde{Q}_k$, where

$$\tilde{A}_k(\varphi, v) = \Big(\varphi,\ R_\varphi \circ A_k \circ R_{\varphi^{-1}}(v)\Big)$$

and

$$\tilde{Q}_k(\varphi, v) = \Big(\varphi,\ R_\varphi \circ Q_k \circ R_{\varphi^{-1}}(v)\Big)$$

Notice first that

$$R_\varphi \circ A_k \circ R_{\varphi^{-1}}(v) = \sum_{p=0}^{k} (-1)^p \left(v \circ \varphi^{-1}\right)^{(2p)} \circ \varphi$$

is a polynomial expression in the variables

$$\frac{1}{\varphi_x}, \varphi_{xx}, \ldots, \varphi^{(2k)}, v, v_x, \ldots, v^{(2k)}.$$

For example for $k = 1$, we get:

$$R_\varphi \circ A_1 \circ R_{\varphi^{-1}}(v) = v + v_x \frac{\varphi_{xx}}{\varphi_x^3} - v_{xx} \frac{1}{\varphi_x^2},$$

and to prove the general case, we let $a_p = (v \circ \varphi^{-1})^{(p)} \circ \varphi$, and use the recurrence relation

$$a_{p+1} = \frac{1}{\varphi_x}\, a_p'.$$

A similar reasoning for $R_\varphi \circ Q_k \circ R_{\varphi^{-1}}(v)$, where

$$Q_k(u) = 2A_k(u)u_x - \sum_{p=0}^{k} (-1)^p \sum_{i=1}^{2p} C_{2p}^i u^{(i)} u^{(2p-i+1)},$$

shows that it is also a polynomial expression in the variables

$$\frac{1}{\varphi_x}, \varphi_{xx}, \ldots, \varphi^{(2k)}, v, v_x, \ldots, v^{(2k)}.$$

To conclude that $\tilde{A}_k$ and $\tilde{Q}_k$ are smooth maps from $\mathcal{D}^n \times H^n(\mathbb{S}^1)$ to $\mathcal{D}^n \times H^{n-2k}(\mathbb{S}^1)$, we use the following known facts (see for example [1]):

1. For $n \geq 1$, $H^n(\mathbb{S}^1)$ is a Banach algebra and hence polynomial maps on $H^n(\mathbb{S}^1)$ are smooth.

2. For $n \geq 1$, the map $H^n(\mathbb{S}^1) \to H^{n-1}(\mathbb{S}^1)$, $v \mapsto v_x$ is smooth.

3. For $n \geq 1$, the map $H^n(\mathbb{S}^1) \cap \{v > 0\} \to H^n(\mathbb{S}^1)$, $v \mapsto 1/v$ is smooth.

To show that $\tilde{A}_k^{-1} : \mathcal{D}^n \times H^{n-2k}(\mathbb{S}^1) \to \mathcal{D}^n \times H^n(\mathbb{S}^1)$ is smooth, we compute the derivative of $\tilde{A}_k$ at an arbitrary point (φ, v), obtaining

$$D\tilde{A}_k(\varphi, v) = \begin{pmatrix} Id & 0 \\ * & R_\varphi \circ A_k \circ R_{\varphi^{-1}} \end{pmatrix}.$$

It is clearly a bounded linear operator in view of the preceding analysis. It is moreover a topological linear isomorphism since A_k itself is invertible. The application of the *inverse mapping theorem* [26] in Banach spaces achieves the proof. ■

At this stage, we regard (4.2) as an ordinary differential equation on $\mathcal{D}^n \times H^n(\mathbb{S}^1)$ ($n \geq 2k+1$), with a smooth right-hand side, viewed as a map from $\mathcal{D}^n \times H^n(\mathbb{S}^1)$ to $H^n(\mathbb{S}^1) \times H^n(\mathbb{S}^1)$. The *Cauchy-Lipschitz theorem* [26] for differential equations in Banach spaces ensures that for every $\varepsilon > 0$, we can find a positive number $T_n = T_n(\varepsilon)$ such that for every u_0 in the ball $B_n(0, \varepsilon)$ in $H^n(\mathbb{S}^1)$, equation (4.2) with initial data $\varphi(0) = Id$ and $v(0) = u_0$ has a unique solution $(\varphi, v) \in C^\infty([0, T_n); \mathcal{D}^n \times H^n(\mathbb{S}^1))$. Moreover, this solution (φ, v) depends smoothly of the initial data u_0 and can be extended to some maximal existence time $T_n^* > 0$.

The existence of solutions to the geodesic equation on the enlarged configuration space $\mathcal{D}^n$ being established, the main question is how to use this to deduce the existence of geodesics on $\mathrm{Diff}(\mathbb{S}^1)$. This is precisely the point where the rigorous approach usually breaks down and one can not deduce results on the actual configuration space (see the case of the Euler equation and of other hydrodynamical equations in [3]). Next step consists in showing that it is possible to deal with the actual configuration space $\mathrm{Diff}(\mathbb{S}^1)$.

Step 2

Consider the initial value problem (4.2) with initial data $u_0 \in \mathrm{C}^\infty(\mathbb{S}^1)$. Each solution of the corresponding Cauchy problem in $\mathcal{D}^{n+1} \times H^{n+1}(\mathbb{S}^1)$ is itself a solution of the Cauchy problem in $\mathcal{D}^n \times H^n(\mathbb{S}^1)$. What could happen is that the upper bound $T_{n+1}^*(u_0)$ of the maximal existence time interval of the solution in $\mathcal{D}^{n+1} \times H^{n+1}(\mathbb{S}^1)$ is smaller than $T_n^*(u_0)$. This second step consists to show that we have in fact

$$T_n^*(u_0) = T_{n+1}^*(u_0), \qquad \forall n \geq 2k+1.$$

The key ingredient to prove it is precisely the *conservation of the momentum* (2.2).

Lemma 1. *Let $n \geq 2k+1$. If (φ, v) is the solution of* (4.2) *with initial data $u_0 \in H^n(\mathbb{S}^1)$, defined for $t \in [0, T)$, then*

$$m(t, \varphi(t,x)) \cdot \varphi_x^2(t,x) = m_0(x), \qquad t \in [0, T),$$

where $m = A_k(u)$ and $u(t,x) = v(t, \varphi^{-1}(t,x))$.

This conservation law enables us to show that if $T^*_{n+1} < T^*_n$ then the solution

$$(\varphi(t), v(t)) \in \mathcal{D}^{n+1} \times H^{n+1}(\mathbb{S}^1)$$

would converge in $\mathcal{D}^{n+1} \times H^{n+1}(\mathbb{S}^1)$ as $t \uparrow T^*_{n+1}$ which would give a contradiction. Therefore $T^*_n = T^*_{n+1}$ (see [11] for the details).

Remark 6. It was already known [6] that the maximal existence time for the Camassa-Holm equation does not depend on the degree of smoothness of $u_0 \in H^n(\mathbb{S}^1)$ $(n \geq 3)$. In fact, all solutions of the Camassa-Holm equation with initial data $u_0 \in H^n(\mathbb{S}^1)$ $(n \geq 3)$ are uniformly bounded and the only way that a solution fails to exist for all time is that the wave breaks (the solution remains bounded while its slope becomes unbounded at a finite time) [7]. Moreover, the solution is defined for all times if and only if $m_0 = u_0 - u_0''$ does not change properly sign [27].

5 The Riemannian exponential map

In classical Riemannian geometry, the *Riemannian exponential map* is a local diffeomorphism and *normal coordinates* play a very special role especially to establish *convexity results*.

On $\mathrm{Diff}(\mathbb{S}^1)$ the existence of this privileged chart is not ensured automatically. One may find useful to recall on this occasion that the *group exponential* of $\mathrm{Diff}(\mathbb{S}^1)$ is not a local diffeomorphism[2]. A remarkable result established in [10] is that for the Camassa-Holm equation and more generally [11] for H^k metrics with $k \geq 1$, the Riemannian exponential map is a smooth local diffeomorphism.

Theorem 6 (Constantin & Kolev 2003)**.** *For $k \geq 1$, the Riemannian exponential map $\mathfrak{exp}$ for the H^k-metric on* $\mathrm{Diff}(\mathbb{S}^1)$*, is a smooth local diffeomorphism near the origin.*

Remark 7. Recently, it has been proved [20] that this Riemannian exponential map is in fact an *analytical Fréchet* map.

Sketch of proof. The approach relies on two important consequences of the conservation of the momentum (Lemma 1), whenever $n \geq 2k+1$:

Claim 1 In the scale provided by the Sobolev spaces $H^n(\mathbb{S}^1)$, the geodesic $\varphi_{u_0}(t)$ issuing from the identity in the direction of u_0 inherits at each time $t > 0$ exactly the same regularity of u_0 (if $u_0 \notin H^{n+1}(\mathbb{S}^1)$, then $\varphi_{u_0}(t) \notin H^{n+1}(\mathbb{S}^1)$ for $t > 0$).

Claim 2 For $u_0 \in \mathrm{C}^\infty(\mathbb{S}^1)$ there is no function $w \in H^n(\mathbb{S}^1) \setminus H^{n+1}(\mathbb{S}^1)$ such that

$$D\mathfrak{exp}(u_0).w \in H^{n+1}(\mathbb{S}^1).$$

[2]Indeed, this map is not locally surjective. Otherwise, every diffeomorphism sufficiently near to the identity (for the C^∞ topology) would have a square root. However one can build (see [30]) diffeomorphisms arbitrary near to the identity which have exactly 1 periodic orbit of period $2n$. But the number of periodic orbits of even periods of the square of a diffeomorphism is always even. Therefore, such a diffeomorphism cannot have a square root.

Taking these two facts for granted, we proceed as follows. $\mathfrak{exp}$, as a map from $H^{2k+1}(\mathbb{S}^1)$ to $\mathcal{D}^{2k+1}$ is smooth and $D\mathfrak{exp}(0) = Id$. Hence, according to the *inverse mapping theorem* [26] in Banach spaces, we can find open neighborhoods $V_{2k+1} \subset H^{2k+1}(\mathbb{S}^1)$ and $O_{2k+1} \subset \mathcal{D}^{2k+1}$ of 0 and Id respectively such that $\mathfrak{exp} : V_{2k+1} \to O_{2k+1}$ is a smooth diffeomorphism. We claim that

$$\mathfrak{exp} : V_{2k+1} \cap \mathrm{C}^{\infty}(\mathbb{S}^1) \to O_{2k+1} \cap \mathrm{Diff}(\mathbb{S}^1)$$

is a smooth diffeomorphism. First, *Claim 1* ensures that this map is a bijection. We are going to show that it is smooth as well as its inverse. Let $u_0 \in V_{2k+1} \cap \mathrm{C}^{\infty}(\mathbb{S}^1)$. The regularity properties of $\mathfrak{exp}$ ensure that $D\mathfrak{exp}(u_0)$ is a bounded linear operator from $H^n(\mathbb{S}^1)$ to $H^n(\mathbb{S}^1)$ for every $n \geq 2k+1$. We now prove inductively that it is an isomorphism. For $n = 2k+1$ this is so by our choice of V_{2k+1} and O_{2k+1}. If it is true for $2k+1 \leq j \leq n$, then $D\mathfrak{exp}(u_0)$ is injective as a bounded linear map from $H^{n+1}(\mathbb{S}^1)$ to $H^{n+1}(\mathbb{S}^1)$ since its extension to H^n is injective. *Claim 2* ensures that it is a continuous linear isomorphism. Using again the *inverse mapping theorem* in Banach spaces, we claim that the map

$$\mathfrak{exp} : V_{2k+1} \cap H^n(\mathbb{S}^1) \to O_{2k+1} \cap \mathcal{D}^n$$

and its inverse

$$\mathfrak{exp}^{-1} : O_{2k+1} \cap \mathcal{D}^n \to V_{2k+1} \cap H^n(\mathbb{S}^1)$$

are smooth maps for all $n \geq 2k+1$, which achieves the proof. ∎

This theorem has an interesting corollary: the geodesic is locally the shortest path between two closeby points of $\mathrm{Diff}(\mathbb{S}^1)$.

Theorem 7 (Constantin & Kolev 2003)**.** *If $\eta, \varphi \in \mathrm{Diff}(\mathbb{S}^1)$ are closed enough then η and φ can be joined by a unique geodesic. This unique geodesic is length minimizing among all piecewise C^1-curves joining η to φ on* $\mathrm{Diff}(\mathbb{S}^1)$*.*

Remark 8. Contrary to the H^1 metric (Camassa-Holm equation), the Riemannian exponential map for the L^2 metric (Burgers equation) is not a local C^1-diffeomorphism near the origin [10]. This suggests that the geometric approach is less meaningful in this case. In fact, the notion that geodesics minimize length is lost in this case since the geodesic pseudo-distance vanishes (see [29] and Section 3.1).

6 Conclusion

In this paper, we have presented the geometric approach of Euler equations associated to the H^k metrics ($k \in \mathbb{N}$) on $\mathrm{Diff}(\mathbb{S}^1)$. From the point of view of mathematical physics, only two cases are meaningful: the L^2 metric ($k = 0$) which corresponds to *Burgers equation* and the H^1 metric which correspond to the Camassa-Holm equation. It is however fruitful to embed the problem in the whole family of H^k metrics to discover that there is a critical value $k = 0$ (Burgers) and that the rest of the family ($k \geq 1$) shares numerous properties with the Camassa-Holm equation ($k = 1$). Let's summarize the main results.

- The Riemannian pseudo-distance associated to the H^k metric on $\mathrm{Diff}(\mathbb{S}^1)$ is a distance if $k \geq 1$. It vanishes for $k = 0$.
- For $k \geq 1$, the geodesic equation is an ordinary differential equation on the enlarged configuration space $\mathcal{D}^n$ ($n \geq 2k+1$). This is false for $k = 0$.
- There is short-time existence, uniqueness and smooth dependance on the initial data of the geodesic flow for $k \geq 1$. If singularities develop, this happens at order less or equal to $2k$.
- The Riemannian exponential map is a smooth local diffeomorphism for $k \geq 1$. It is not a C^1 local diffeomorphism for $k = 0$.

One aspect of the Burgers and Camassa-Holm equations that could be expected to carry over to all H^k metrics is the existence of a bi-Hamiltonian structure [5, 8]. As we have seen, the Euler equation on the dual $\mathfrak{g}^*$ of a Lie algebra is Hamiltonian with respect to the *canonical Lie-Poisson structure*. In some cases there is another Poisson structure on $\mathfrak{g}^*$ such that the equation is Hamiltonian with respect to this second structure as well. The Burgers and Camassa-Holm equations are bi-Hamiltonian with respect to so-called *affine Lie-Poisson structures*, leading to *integrability* and the existence of an infinite number of conservation laws for these equations. It happens however that Burgers and Camassa-Holm equations are the only one in the family of H^k metrics on $\mathrm{Diff}(\mathbb{S}^1)$ which are bi-Hamiltonian with respect to an affine Lie-Poisson structure on $\mathrm{Vect}^*(\mathbb{S}^1)$ [12].

Historically, the bi-Hamiltonian formalism has been introduced[3] by Gel'fand, Dorfman, Magri and others at the end of the 1970's for the *Korteweg-de Vries equation*

$$u_t + 3uu_x - cu_{xxx} = 0, \qquad c \in \mathbb{R}. \tag{6.1}$$

Notice that the Korteweg-de Vries equation can not be recast as an Euler equation on $\mathrm{Diff}(\mathbb{S}^1)$ because the expression $3uu_x - cu_{xxx}$ is not quadratic in u. However, it has been shown [22] that it can be written as an Euler equation for the L^2 metric on the *Virasoro group*, a central extension of $\mathrm{Diff}(\mathbb{S}^1)$ by $\mathbb{R}$. This equation was already known in the seventies to be Hamiltonian with respect to both brackets on $\mathrm{C}^\infty(\mathbb{S}^1)$ defined by the operators D and $-(Dm + mD) + cD^3$.

The Lie-Poisson bracket on the *regular dual* of the Virasoro algebra, $\mathrm{Vir}^* = \mathrm{C}^\infty(\mathbb{S}^1) \oplus \mathbb{R}$, is represented by matrix

$$J(m, \alpha) = \begin{pmatrix} -Dm - mD + \alpha D^3 & 0 \\ 0 & 0 \end{pmatrix}.$$

The functions $F(m, \alpha)$ on Vir^* which depend only on α are therefore *Casimir functions* (which are first integrals of all Hamiltonian vector fields) for the canonical structure on Vir^*. In particular, the Euler flow for the L^2 metric on Vir^* leaves invariant each hyperplane $\alpha = c$ (constant). The Lie-Poisson structure induces on each hyperplane $\alpha = c$ (isomorphic to $\mathrm{C}^\infty(\mathbb{S}^1)$) a Poisson structure which is represented by the operator $-(Dm + mD) + cD^3$. This gives a geometric explanation for the appearance of the operators D and $-(Dm + mD) + cD^3$ in the 1970's. For $c = 0$, we recover the canonical Poisson structure on $\mathrm{Vect}(\mathbb{S}^1)$ and the Burgers equation.

[3] See the review [33].

A similar approach can be pursued for the *general Camassa-Holm equation*

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + cu_{xxx} = 0, \qquad c \in \mathbb{R}$$

which can be obtained as the Euler equation for the H^1 right-invariant metric on the Virasoro group [21].

For the Virasoro group, short-time existence of the geodesic flow for the H^k metric was established in [9] using the approach given in Section 4. The Riemannian exponential map is however a smooth local diffeomorphism only for $k \geq 2$ in that case cf. [9].

References

[1] Adams R A, Sobolev spaces, Academic Press, New York-London, 1975.

[2] Arnold V I, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, *Ann. Inst. Fourier* **16** (1966), 319–361.

[3] Arnold V I and Khesin B, Topological methods in hydrodynamics, Springer-Verlag, New York, 1998.

[4] Burgers J M, A mathematical model illustrating the theory of turbulence, in Advances in Applied Mechanics, Academic Press New York, 1948, 171–199.

[5] Camassa R and Holm D D, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, **71** (1993), 1661–1664.

[6] Constantin A and Escher J, Well-posedness global existence and blow-up phenomena for a periodic quasi-linear hyperbolic equation, *Comm. Pure Appl. Math.*, **51** (1998), 475–504.

[7] Constantin A and Escher J, On the blow-up rate and the blow-up set of breaking waves for a shallow water equation, *Math. Z.*, **233** (2000), 75–91.

[8] Constantin A and McKean H P, A shallow water equation on the circle, *Comm. Pure Appl. Math.*, **52** (1999), 949–982.

[9] Constantin A, Kappeler T, Kolev B and Topalov P, On geodesic exponential maps of the Virasoro group, *Ann. Global Anal. Geom.* **31** (2007), 155–180.

[10] Constantin A and Kolev B, On the geometric approach to the motion of inertial mechanical systems, *J. Phys. A* **35** (2002), R51–R79.

[11] Constantin A and Kolev B, Geodesic flow on the diffeomorphism group of the circle, *Comment. Math. Helv.* **78** (2003), 787–804.

[12] Constantin A and Kolev B, Integrability of invariant metrics on the diffeomorphism group of the circle, *J. Nonlinear Sci.* **16** (2006), 109–122.

[13] Ebin D G and Marsden J, Groups of diffeomorphisms and the notion of an incompressible fluid, *Ann. of Math.* **92** (1970), 102–163.

[14] Euler L, Theoria motus corporum solidorum sei rigidorum, 1765.

[15] Fokas A S and Fuchssteiner B, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Phys. D*, **4** (1981), 47–66.

[16] Guieu L and Roger C, Algèbre de Virasoro: aspects géométriques et algébriques, Soc. Math. France, 2005.

[17] Hamilton R S, The inverse function theorem of Nash and Moser, *Bull. Amer. Math. Soc.* **7** (1982), 65–222.

[18] Hörmander L, Lectures on Nonlinear Hyperbolic Differential Equations, Springer-Verlag, Berlin, 1997.

[19] Johnson R S Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.* **455** (2002), 63–82.

[20] Kappeler T and Loubet E and Topalov P Analyticity of Riemannian exponential maps on $Diff(T)$, *J. Lie Theory* **17** (2007), 481–503.

[21] Khesin B and Misiołek G, Euler equations on homogeneous spaces and Virasoro orbits, *Adv. Math.* **176** (2003), 116–144.

[22] Khesin B and Ovsienko V, The super Korteweg-de Vries equation as an Euler equation, *Funktsional. Anal. i Prilozhen.* **21** (1987), 81–82.

[23] Kolev B, Lie groups and mechanics: an introduction, *J. Nonlinear Math. Phys.* **11** (2004), 480–498.

[24] Kolev B, Bi-Hamiltonian systems on the dual of the Lie algebra of vector fields of the circle and periodic shallow water equations, *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **365** (2007), 2333–2357.

[25] Lagrange J L, Mécanique analytique, Chez la veuve Desaint, 1788.

[26] Lang S, Fundamentals of differential geometry, Springer-Verlag, New York, 1999.

[27] McKean H P, Breakdown of a shallow water equation, *Asian J. Math.* **2** (1998), 867–874.

[28] Michor P W, Some geometric evolution equations arising as geodesic equations on groups of diffeomorphisms including the Hamiltonian approach, in Phase space analysis of partial differential equations, volume 69 of *Progr. Nonlinear Differential Equations Appl.*, Birkhäuser Boston, 2006, 133–215.

[29] Michor P W and Mumford D, Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms, *Doc. Math.* **10** (2005), 217–245.

[30] Milnor J, Remarks on infinite-dimensional Lie groups in Relativity, groups and topology, II (Les Houches, 1983), North-Holland, Amsterdam, 1984, 1007–1057.

[31] Misiołek G, A shallow water equation as a geodesic flow on the Bott-Virasoro group, *J. Geom. Phys.* **24** (1998), 203–208.

[32] Ovsienko V and Roger C, Looped cotangent Virasoro algebra and non-linear integrable systems in dimension $2 + 1$, *Comm. Math. Phys.* **273** (2007), 357–378.

[33] Praught J and Smirnov R G, Andrew Lenard: a mystery unraveled, *SIGMA Symmetry Integrability Geom. Methods Appl.* **1** (2005), Paper 005, 7 pp.

[34] Shkoller S, Geometry and curvature of diffeomorphism groups with H^1 metric and mean hydrodynamics, *J. Funct. Anal.* **160** (1998), 337–365.