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On the Fluid Motion in Standing Waves

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Abstract

This paper concerns linear standing gravity water waves on finite depth. We obtain qualitative and quantitative understanding of the particle paths within the wave.

1 Introduction

The study of permanent progressive water waves is a classical topic within mathematical analysis, and its challenging problems have continued to draw attention up to this day [2, 8, 9, 10, 17, 26, 31]. As the name suggests, standing waves do not propagate, but arise—for example—as the superposition of progressive waves traveling in opposite direction. A typical instance of this are waves reflected against a vertically flat rock formation. Standing waves can also be observed by setting water in a container in motion. For some historical background on the mathematical theory of standing waves we refer to [12, 22, 28].

The particle path approach led to the only existing explicit solution for gravity waves with a non-flat surface [16, 29] (see also the discussion in [4] and the adaption to edge waves in [3]). These solutions are rotational, with a particular distribution of vorticity. Recently, new findings have shed light on the behavior of particles within exact and linear progressive waves in general [5, 6, 11, 18, 19]. Those show in what way the classical first order truncation—describing the trajectories as closed ellipses—is an approximation. Typically one obtains that the trajectories are not closed, and that the particles display a forward drift (even so for linear waves). The effects of vorticity do influence this pattern in different ways [14, 15], but near the flat bed the non-closed oval paths are preserved. Since standing waves arise as the result of two such waves, it is a natural question to ask what happens in this case. Just as the limit of L-periodic traveling waves, being solitary waves [1], do not preserve the particle paths of periodic waves [7], one expects the particles within a standing wave to behave differently than within a periodic steady wave. That notion is supported by physical measurements, indicating that the particle trajectories corresponding to standing waves resemble parabolas [30]. This paper confirms that hypothesis, providing detailed information on the behavior for every particle within the fluid.

Another impetus comes from the recent investigations [20, 21, 22, 28]. There, the authors establish the existence of standing gravity waves as solutions of the exact water wave problem. The intractability of the exact problem however makes it generally very hard to find exact properties of its solutions. Since near the laminar flows a good approximation is given by the corresponding linear system, that is the natural starting point for further understanding of the exact waves. In this investigation we also show in what way linear waves are "accurate" for small amplitudes, and inaccurate for larger waves.

The paper has the following disposition: Section 2 describes the physical background leading up to the mathematical water wave problem. We rescale, linearize and present a solution of the linear problem. Section 3 contains the main results, Lemma 3.1 and Theorem 3.5. Via standard methods for ordinary differential equations we deduce the Lagrangian solutions in terms of closed expressions. This admits for the presentation of some properties of standing waves. Finally, in Section 4 we illustrate our results with the aid of numerical examples. Note that since in this case we were able to deduce exact solutions, the figures presented do not represent approximations through numerical schemes, but plots of closed functions.

2 Preliminaries

Choose Cartesian coordinates (x, y) with the y-axis pointing vertically upwards. so that the origin lies on the flat bed below the crest. Let $(u(t, x, y),\ v(t, x, y))$ be the velocity field of the flow, let $h > 0$ be the depth below the mean water level $y = h$, and let $y = h+\eta(t, x)$ be the water's free surface. We assume that gravity is the restoring force once a disturbance was created, neglecting the effects of surface tension. Homogeneity (constant density) is a physically reasonable assumption for gravity waves [25], and it implies the equation of mass conservation

$$u_x + v_y = 0 \tag{2.1a}$$

throughout the fluid. Appropriate for gravity waves is the assumption of inviscid flow [25], so that the equation of motion is Euler's equation

$$\begin{aligned} u_t + uu_x + vu_y &= -P_x, \\ v_t + uv_x + vv_y &= -P_y - g, \end{aligned} \tag{2.1b}$$

where $P(t, x, y)$ denotes the pressure and g is the gravitational constant of acceleration. The free surface decouples the motion of the water from that of the air so that, ignoring surface tension, the dynamic boundary condition

$$P = P_0 \qquad \text{on} \quad y = h + \eta(t, x), \tag{2.1c}$$

must hold, where P_0 is the constant atmospheric pressure [23] . Moreover, since the same particles always form the free surface, we have the kinematic boundary condition

$$v = \eta_t + u\eta_x \qquad \text{on} \quad y = \eta(t, x). \tag{2.1d}$$

The fact that water cannot penetrate the rigid bed at $y = 0$ yields the kinematic boundary condition

$$v = 0 \qquad \text{on} \quad y = 0. \tag{2.1e}$$

2.1 Nondimensionalization

We now introduce a non-dimensionalization of the variables. As above, h is the average height above the bottom, and we let a denote the typical *amplitude*, and λ the typical *wavelength*. Take $\sqrt{gh}$ as the scale of the horizontal velocity; that is the approximate speed of irrotational long waves [23]. We then make the transformations

$$x \mapsto \frac{x}{\lambda}, \quad y \mapsto \frac{y}{h}, \quad t \mapsto \frac{\sqrt{gh}\,t}{\lambda}, \quad u \mapsto \frac{u}{\sqrt{gh}}, \quad v \mapsto \frac{\lambda\, v}{h\sqrt{gh}}, \quad \eta \mapsto \frac{\eta}{a}.$$

After applying those mappings, define furthermore a new pressure function $p = p(t, x, y)$ by the equality

$$P = P_0 + gh(1 - y) + ghp.$$

Here P_0 is the constant atmospheric pressure, and $gh(1 - y)$ is the hydrostatic pressure distribution, describing the pressure change within a stationary fluid. The new variable p thus measures the pressure perturbation induced by the wave. The water wave problem (2.1) then reads

$$u_x + v_y = 0, \tag{2.2a}$$

$$u_t + uu_x + vu_y = -p_x, \tag{2.2b}$$

$$v_t + uv_x + vv_y = -\frac{\lambda^2}{h^2} p_y, \tag{2.2c}$$

valid in the fluid domain $0 < y < 1 + \frac{a}{h}\eta$, and

$$v = \frac{a}{h}\,(\eta_t + u\eta_x), \tag{2.2d}$$

$$p = \frac{a}{h}\,\eta, \tag{2.2e}$$

valid at the surface $y = 1 + \frac{a}{h}\eta$, in conjunction with the boundary condition (2.1e) on the flat bed $y = 0$. Here appear naturally the parameters

$$\varepsilon := \frac{a}{h}, \qquad \delta := \frac{h}{\lambda},$$

called the *amplitude parameter*, and the *shallowness parameter*, respectively. Since the shallowness parameter is a measure of the length of the wave compared to the depth, small δ models long waves of small amplitude or, equivalently, shallow water waves. The amplitude parameter measures the relative size of the wave, so small ε is customarily used to model a small disturbance of the underlying flow (cf. [23]).

2.2 The linearization

To enable the study of explicit solutions, we shall linearize around a laminar irrotational flow, i.e. still water. For such flows we have a flat surface, $y = 1$ (in non-dimensional variables), corresponding to $\eta = 0$. We now write a general solution as a perturbation of such a vanishing solution, i.e.

$$u = \varepsilon\tilde{u}, \quad v = \varepsilon\tilde{v}, \quad p = \varepsilon\tilde{p}. \tag{2.3}$$

Recall that small ε corresponds to waves whose amplitude is small in comparison with the depth. Since the surface is described by $1+\varepsilon\eta$, η should thus be of unit size. Dropping the tildes, we obtain

$$u_x + v_y = 0, \tag{2.4a}$$
$$u_t + \varepsilon(uu_x + vu_y) = -p_x, \tag{2.4b}$$
$$v_t + \varepsilon(uv_x + vv_y) = -\frac{p_y}{\delta^2}, \tag{2.4c}$$

valid in the fluid domain $0 < y < 1 + \varepsilon\eta$, and

$$v = \eta_t + \varepsilon u \eta_x, \tag{2.4d}$$
$$p = \eta, \tag{2.4e}$$

at the approximate surface $y = 1$. Moreover, $v = 0$ on the flat bed $y = 0$. The linearization is attained by formally letting $\varepsilon \to 0$, and it is given by

$$u_x = -v_y, \qquad u_t = -p_x, \qquad \delta^2 v_t = -p_y, \tag{2.5a}$$

valid in $0 < y < 1$, and

$$v = \eta_t, \qquad p = \eta, \tag{2.5b}$$

for $y = 1$. A special class of waves are the standing waves for which the surface is periodic separately in the (t,x)-variables. A natural *Ansatz* is therefore $\eta(t,x) := \cos(\mu t)\cos(2\pi x)$. This leads to the solution

$$\begin{aligned} u(t,x,y) &= 2\pi\delta C \sin(\mu t) \sin\left(2\pi x\right) \cosh\left(2\pi\delta y\right), \\ v(t,x,y) &= -2\pi C \sin(\mu t) \cos\left(2\pi x\right) \sinh\left(2\pi\delta y\right), \\ p(t,x,y) &= \mu\delta C \cos(\mu t) \cos\left(2\pi x\right) \cosh(2\pi\delta y), \end{aligned} \tag{2.6}$$

where

$$C := \frac{\mu}{2\pi \sinh(2\pi\delta)} := \frac{1}{\sqrt{2\pi\delta \sinh(2\pi\delta)\cosh(2\pi\delta)}}.$$

The corresponding approximation to the original system (2.1) is

$$\begin{aligned} u(t,x,y) &= \frac{a\omega}{\sinh(hk)} \sin\left(\omega t\right) \sin\left(kx\right) \cosh\left(ky\right), \\ v(t,x,y) &= -\frac{a\omega}{\sinh(hk)} \sin\left(\omega t\right) \cos\left(kx\right) \sinh\left(ky\right), \\ P(t,x,y) &= P_0 + g(h-y) \\ &\quad + \frac{ag}{\cosh(hk)} \cos\left(\omega t\right) \cos\left(kx\right) \cosh\left(ky\right), \\ \eta(t,x) &= h + a\cos\left(\omega t\right) \cos\left(kx\right). \end{aligned} \tag{2.7}$$

Here

$$k := \frac{2\pi}{\lambda} \quad \text{and} \quad \omega := \sqrt{gk \tanh hk} \tag{2.8}$$

are the *wave number* and the *frequency*, respectively. The second equality in (2.8) can be taken either as a definition, or as the dispersion relation determining the *frequency* ω. The size of the disturbance is proportional to a in the whole quadruple (η, u, v, p), so this solution satisfies the exact equation with an error which is $O(a^2)$ as $a \to 0$.

A few comments are in order about the solution (2.7). Looking at the form of η and v we see that there is no vertical motion along the lines $x = (n + 1/2)\pi/k$, $n \in \mathbb{Z}$. These are the *nodal lines*. In contrast, the vertical motion of the surface is greatest at the *antinodes*, $x = n\pi/k$, $n \in \mathbb{Z}$. Note also that there is no horizontal motion underneath the antinodes. While the above solution is *unimodal*, it is in principle possible to imagine a *multimodal* solution consisting of a linear combination of solutions of the form (2.7), with $k = mk_0$ and $\omega = n\omega_0$ for $m, n \in \mathbb{Z}$. It is known, however, that for "most" parameter values the solution of the dispersion relation (2.8) is unique (see [28] for the details). This is in contrast to the deep water setting, where multimodal solutions of the linear problem are abundant and where such solutions have been shown to exist in the exact water wave problem [20].

3 Main results

In this section we investigate the particle trajectories for the standing wave (2.7). We find explicit formulas for the trajectories, from which it is possible to get a very clear picture of the fluid motion. Our main results are presented in Theorem 3.5 at the end of the section.

The system (2.7) represents a solution, but in terms of paths $(x(t), y(t))$ it is still very implicit. Indeed, it describes the system of ordinary differential equations

$$\begin{aligned} \dot{x}(t) &= c\omega \sin(\omega t) \sin(kx) \cosh(ky), \\ \dot{y}(t) &= -c\omega \sin(\omega t) \cos(kx) \sinh(ky), \end{aligned} \tag{3.1}$$

for $c := a/\sinh hk$. This, however, can be explicitly solved. To state our findings, define $\mathrm{am}(s, k)$ as the inverse of the (incomplete) elliptic integral of the first kind,

$$F(\phi, k) := \int_0^\phi \frac{1}{\sqrt{1 - k^2 \sin^2(u)}}\, du, \qquad -\pi \le \phi \le \pi,$$

i.e. $\mathrm{am}(F(\phi, k), k) = \phi$, for $-\pi \le \phi \le \pi$ (see [24]). The Jacobi elliptic function am is defined in this way for $-2K \le s \le 2K$ where

$$K := K(k) := \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2(u)}}\, du,$$

is the complete elliptic integral of the first kind.

Lemma 3.1. *The solutions of* (3.1) *in* $(0, \pi/k) \times (0, \infty)$ *are given by*

$$\begin{aligned} x(t) &= \frac{1}{k} \mathrm{am}(C(kc(1 - \cos(\omega t)) + s_0), C^{-1} i), \qquad C > 0, \\ y(t) &= \frac{1}{k} \operatorname{arcsinh}\left(\frac{C}{\mathrm{sn}(C(kc(1 - \cos(\omega t)) + s_0), C^{-1} i)} \right), \end{aligned} \tag{3.2}$$

in $\{0\} \times (0, \infty)$ *by*

$$y(t) = -\frac{1}{k} \ln\left(\tanh\left(C - \frac{kc}{2}\cos(\omega t)\right)\right), \qquad C > \frac{kc}{2}, \tag{3.3}$$

and in $(0, \pi/k) \times \{0\}$ *by*

$$x(t) = \frac{1}{k} \arccos\left(\frac{1 - Ce^{-2kc\cos(\omega t)}}{1 + Ce^{-2kc\cos(\omega t)}}\right), \qquad C > 0. \tag{3.4}$$

Remark 3.2. In several references the particle paths for linear waves are described as straight lines – see e.g. [13, 28]. This is indeed the solution which is obtained if (3.1) is expanded to first order in a, or, equivalently, if the equations of motion are linearized in the Lagrangian setting [28]. The solutions obtained here seem to agree better with experiments (cf. [30]). We also note that, using a different approach, particle paths were calculated up to fourth order in the amplitude for standing waves on deep water in [27]. Their result agrees qualitatively with ours, in the sense that the particles oscillate back and forth along arcs.

Proof. To find the solutions of (3.1), consider the transformation

$$t \mapsto s(t) := kc(1 - \cos(\omega t)), \qquad \mathbb{R} \to [0, 2kc].$$

We shall be looking for solutions $(X(s), Y(s)) = (kx(t), ky(t))$. Those will *a priori* be periodic in t. However, assuming the existence of such solutions, it follows by uniqueness of (3.1) that there can be no other solutions. Hence we may put

$$(kx(t), ky(t)) := (X(kc(1 - \cos(\omega t)), Y(kc(1 - \cos(\omega t))),$$

to obtain solutions globally defined in time from those defined only on $[0, 2kc]$. It consequently suffices to solve the autonomous system

$$\begin{aligned} \dot{X} &= \sin X \cosh Y, \\ \dot{Y} &= -\cos X \sinh Y, \end{aligned} \tag{3.5}$$

in the domain $[0, 2ak/\sinh hk]$. Notice that

$$\frac{2ak}{\sinh hk} \leq 2\varepsilon \sup_{\xi} \frac{\xi}{\sinh \xi} = 2\varepsilon, \tag{3.6}$$

which will eventually give us a natural bound for amplitude parameter.

We now study the system (3.5) in the domain $0 \leq X \leq \pi$, $Y \geq 0$. The vertical lines $X = 0$ and $X = \pi$ are ∞-isoclines ($\dot{X} = 0$) and the bottom $Y = 0$ is a 0-isocline ($\dot{Y} = 0$). When $0 < X < \pi$ it follows that $\dot{X} \neq 0$ and we have

$$\frac{dY}{dX} = -\frac{\tanh(Y)}{\tan(X)},$$

which can be integrated to

$$Y(X) = Y(X; C) = \operatorname{arcsinh}\left(\frac{C}{\sin(X)}\right), \tag{3.7}$$

where $C>0$ in the domain that we are considering. Thus the particle trajectories lie on the *streamlines*

$$\Gamma_C = \{(X, Y(X;C))\colon 0 < X < \pi\}.$$

Substitution of (3.7) into (3.5) yields

$$\dot{X} = \sqrt{\sin^2(X) + C^2}, \tag{3.8}$$

meaning

$$X(s) = \operatorname{am}(C(s+s_0), C^{-1}i).$$

Using (3.7) we furthermore obtain that

$$Y(s) = \operatorname{arcsinh}\left(\frac{C}{\operatorname{sn}(C(s+s_0), C^{-1}i)}\right),$$

where $\operatorname{sn}(s,k) = \sin(\operatorname{am}(s,k))$. Returning to the physical variables, this yields (3.2).

To solve (3.5) when $X=0$, we first note that if Y is a solution, so is $-Y$. Looking for a negative solution gives us the possibility to integrate

$$\dot{Y} = -\sinh Y, \quad \text{to} \quad Y = \ln\left(\tanh\left(\frac{s}{2} + \tilde{C}\right)\right) < 0.$$

Mapping $Y \to -Y$, we obtain (3.3).

Similarly, if $Y=0$ in (3.5) we get

$$\dot{X} = \sin X \quad \text{meaning} \quad \ln\left(\frac{1-\cos X}{\sin X}\right) = s + \tilde{C}.$$

Since the solution of $(1-\cos x)/\sin x = \alpha$ is given by $\cos x = \frac{1-\alpha^2}{1+\alpha^2}$, this leads to (3.4). ∎

Notice that for $X \in (0,\pi)$ we have $\dot{X} \geq C > 0$. According to (3.7) this forces $Y(s)$ to blow up in finite time for any C and s_0 (this happens when $X(s) = \pi$). To remedy this situation we prove the following proposition concerning solutions (X,Y) in $(0,\pi)\times(0,\infty)$.

Lemma 3.3. *For $0 < s < 2\varepsilon$ there exists a uniform constant $Y_* < \operatorname{arcsinh}(1/\sinh(2\varepsilon))$ such that if $Y(0) < Y_*$ then $X(s) < \pi$ and $Y(s) < \infty$.*

Proof. Let $P := (X(0), Y(0)) \in \Gamma_c$, with $\pi/2 \leq X(0) < \pi$. Then $X(0) = \pi - \arcsin(C/\sinh(Y(0)))$. The time it takes the solution starting at P to reach $X=\pi$ is given by

$$\begin{aligned}
s_* &:= s(\pi) - s(X(0)) = \int_{X(0)}^{\pi} \frac{ds}{dX}\,dX = \int_{X(0)}^{\pi} \frac{dX}{\sqrt{C^2+\sin^2(X)}} \\
&= \int_0^{\pi - X(0)} \frac{dX}{\sqrt{C^2+\sin^2(X)}} = \int_0^{\frac{C}{\sinh(Y(0))}} \frac{du}{\sqrt{(1-u^2)(C^2+u^2)}} \\
&= \int_0^{\frac{1}{\sinh(Y(0))}} \frac{dv}{\sqrt{(1-C^2v^2)(1+v^2)}} \geq \int_0^{\frac{1}{\sinh(Y(0))}} \frac{dv}{\sqrt{1+v^2}} \\
&\geq \int_0^{\frac{1}{\sinh(Y_*)}} \frac{dv}{\sqrt{1+v^2}} = \operatorname{arcsinh}\left(\frac{1}{\sinh(Y_*)}\right),
\end{aligned}$$

if $Y(0) < Y_*$. As the last expression tends to infinity as $Y_* \to 0$, we find that there is a Y_* such that $s_* > 2\varepsilon$ if $Y(0) < Y_*$. In view of that $\dot{X} > 0$,the restriction that $X(0) \geq \pi/2$ is inessential as otherwise s_* is even larger. ■

Remark 3.4. Recall (3.6). As a consequence of Lemma 3.3, if we choose

$$2\varepsilon < \operatorname{arcsinh}\left(\frac{1}{\sinh(Y_*)}\right),$$

the solutions starting at a point below $Y = Y_*$ will not blow up. Since the maximum of the surface at time $t = 0$ is at $y = h + a$, this yields the condition

$$a < \frac{h}{2} \operatorname{arcsinh}\left(\frac{1}{\sinh(k(h+a))}\right) \tag{3.9}$$

in physical variables. Given k and h this condition is clearly satisfied if a is sufficiently small.

With abuse of notation we shall write $(x, y) \in \Gamma_C$ to denote the fact that (x, y) belongs to the streamline $\sinh(ky)\sin(kx) = C$. We then have the following conclusions for the particle trajectories of standing waves.

Theorem 3.5. *Suppose that a,k, and h satisfy* (3.9)*, and let* (x, y) *be the solution given by Lemma 3.1 with* $(x(0), y(0)) = (x_0, y_0) \in [0, \pi/k) \times [0, h + a]$. *Then the general solution of* (3.1) *is given by*

$$x_n(t) := \frac{n\pi}{k} + x\left((-1)^n t\right), \quad y_n(t) := y\left((-1)^n t\right), \qquad n \in \mathbb{Z},$$

and the following properties hold:

i. (x, y) *is periodic with period* $2\pi/\omega$*, and symmetric around* π/ω.

ii. x *is increasing in the interval* $0 \leq t \leq \pi/\omega$.

iii. For a fixed value of x_0*,* $x(\pi/\omega)$ *is an increasing function of* y_0.

iv. If $(\tilde{x}, \tilde{y}) \in \Gamma_C$ *with* $\tilde{x}(0) > x_0$*, then* $\tilde{x}(\pi/\omega) > x(\pi/\omega)$.

v. For every $C \geq 0$ *there exists a point* $(x_*, y_*) \in \Gamma_C$*, with* $0 < x_* < \pi/(2k)$*, so that the distance* $x(\pi/\omega) - x_0$ *is an increasing function of* x_0 *for* $0 < x_0 < x_*$ *and decreasing for* $x_* < x_0$.

Proof. We already proved that, assuming (3.9) and $y(0) \leq a + h$, the unique and globally defined solution of the initial value problem (3.1) for $x(0) \in [0, \pi/k)$ is given by (3.2), (3.3), or (3.4), depending on initial data. It is then easy to check that (x_n, y_n) also satisfies (3.1), and by uniqueness, those are the only solutions.

Concerning properties: (i) follows from the solution formula (3.2), and since $0 \leq x \leq \pi/k$ we see from (3.1) that the sign of $\dot{x}$ changes at most at $t = \pi/\omega$, i.e. (ii) holds. To prove (iii), we use the (X, Y)-coordinates and note that for fixed X_0, a higher value of Y_0 also implies a higher value of C. Referring to (3.8) we see that the solution with the higher C will remain ahead of the other solution for small values of $s > 0$. It suffices to show that

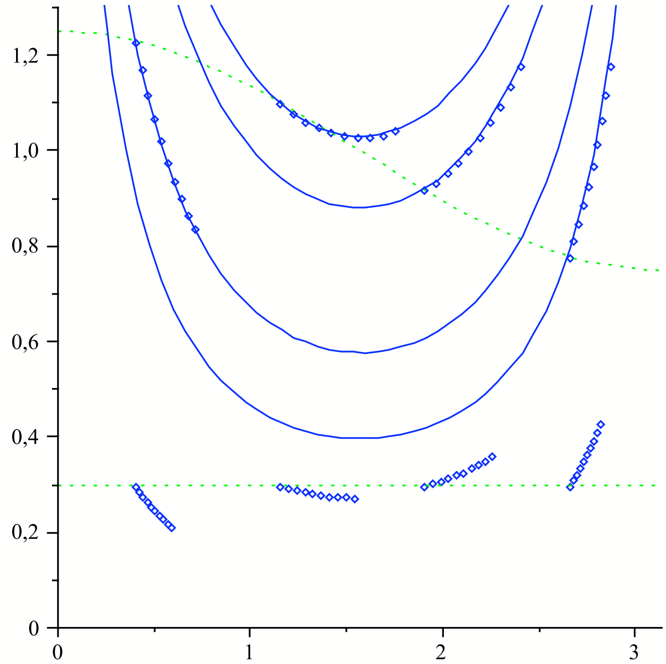

Figure 1: Streamlines $\sinh(ky)\sinh(kx) = C$ (solid curves) and particle paths (hollow circles). The dashed curves are the surface at time $t = 0$, $y = h + a\cos(kx)$, and the line $y = 0.3$.

this continues to hold for $0 < s \leq 2kc$. Assume instead that at some point the solution with the lower C catches up. Then (3.8) gives a contradiction since the solution with the higher C will have a greater value of $\dot{X}$ at the point of intersection. The statement (iv) is due to the fact that two solutions can never intersect in (t, x, y)-space.

Now to (v). We work in the (X, Y)-coordinates and denote for any $C \geq 0$ by X_C the solution of (3.8) with $X_C(0) = \pi/2$. Any other solution $X(s)$ of (3.8) for this C is then given by $X(s) = X_C(s+s_0)$ for some $s_0 \in \mathbb{R}$. The total distance traveled in the X-direction by a solution starting at $X_C(s_0)$ is given by

$$\begin{aligned} r(s_0) &:= X_C(s_0 + 2kc) - X_C(s_0) = \int_{s_0}^{s_0+2kc} \frac{dX_C}{ds}\, ds \\ &= \int_{s_0}^{s_0+2kc} \sqrt{C^2 + \sin^2(X_C(u))}\, du. \end{aligned}$$

Differentiating this expression we find that

$$r'(s_0) = \sqrt{C^2 + \sin^2(X_{\max})} - \sqrt{C^2 + \sin^2(X_0)},$$

where $X_0 = X_C(s_0)$ and $X_{\max} = X_C(s_0 + 2kc)$. This is positive as long as $\sin(X_{\max}) > \sin(X_0)$. The point x_* corresponds to the X_0 for which $\sin(X_{\max}) = \sin(X_0)$. ■

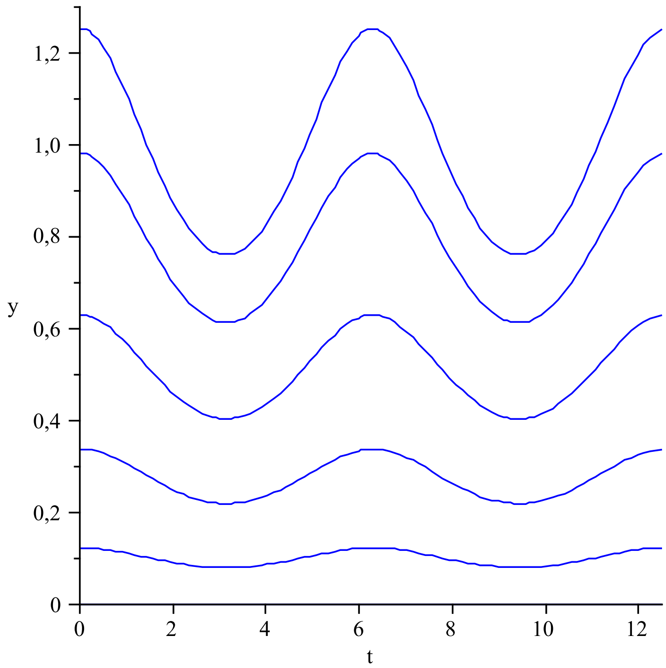

Figure 2: The progression of the particles beneath the original crest as time evolves.

4 Examples

It is a remarkable fact that all linear standing waves admit explicit solutions of the particle paths in terms of standard functions. Still, the expressions (3.2)–(3.4) are not oblique. Since the solutions are at hand it is not hard to perform the calculations. We present some figures illustrating Theorem 3.5. In these figures we have taken the parameter values $h = 1$, $k = 1$ and $a = 0.25$.

Figures 1-3 illustrate the particle paths and streamlines. In figure 1 proposition (iii) of Theorem (3.5) is apparent – for a given initial horizontal position, particles starting higher up travel further to the right. Figures 2-3 display the motion inside the axes. Notice in Figure 3 how the distance traveled first increases and then decreases as we follow the x-axis from $x = 0$ to $x = \pi$. This is consistent with proposition (v) of Theorem (3.5)

Since the kinematic boundary condition at the surface (2.1d) is linearized, the property that particles are trapped on the surface $y = \eta(t, x)$ does not hold. By following the progression of particles initially starting on the surface $y = h + a\cos(kx)$ we obtain a "Lagrangian" surface, which heuristically should stay near the surface $y = h+a\cos(\omega t)\cos(kx)$ if a is small. This is illustrated in Figure 4. Qualitatively there is a close resemblance between the two pictures. We find that the agreement is better close to the antinode $x = 0$ than near the antinode $x = \pi$, where the Lagrangian surface lies above the surface for $t = (2n + 1)\pi/\omega$, $n \in \mathbb{Z}$. If the calculation is repeated for different values of a one obtains a better agreement for smaller values, as is to be expected.

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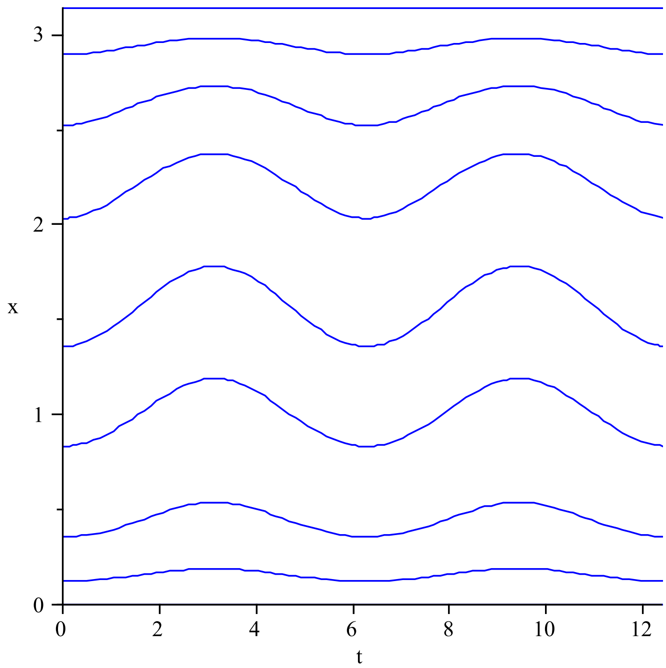

Figure 3: The progression of the particles in $(0, \pi)$ at the flat bed as time evolves.

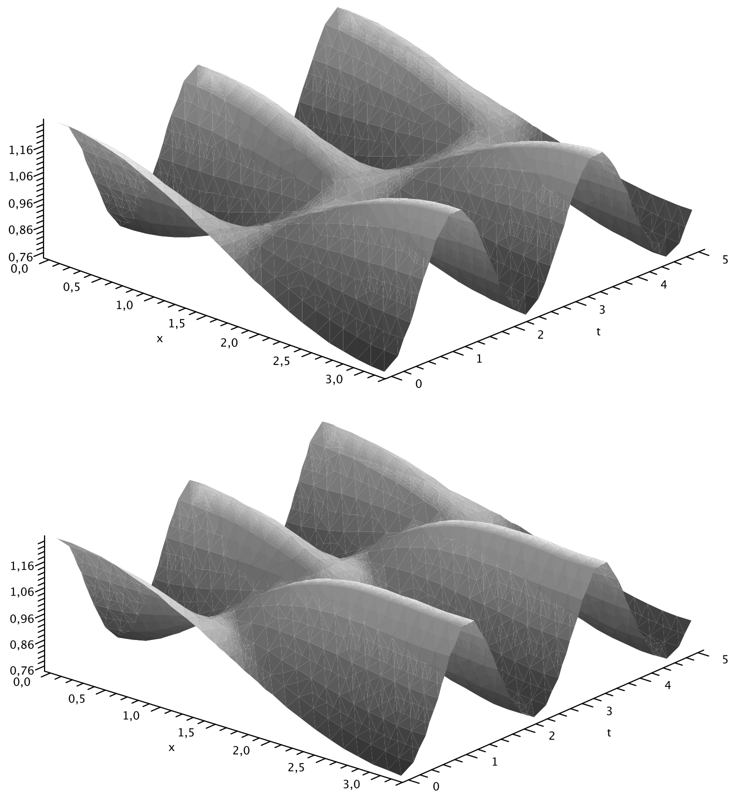

Figure 4: Above: The evolution of the surface $y = h + a\cos(\omega t)\cos(kx)$. Below: The evolution of a layer of particles starting on the surface at time $t = 0$.

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