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Water Waves near a Shoreline in a Flow with Vorticity: Two Classical Examples

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Abstract

The equations that describe the classical problem of water waves – inviscid, no surface tension and constant pressure at the surface – are non-dimensionalised and scaled appropriately, and the two examples: traditional gravity waves and edge waves, are introduced. In addition each type of wave is allowed to propagate over an existing flow field that is rotational and also admits a shoreline; some examples of such background flows are presented. Then, for each problem, a suitable asymptotic solution is constructed; for gravity waves, this is chosen to be that which gives a balance between nonlinearity and dispersion far from the shore (so that a soliton-type problem is recovered there), and then the behaviour of this solution is examined as the shoreline is approached. Sufficiently close to the shore, the asymptotic expansion is not valid, resulting in the formulation of a new, scaled problem. It is then shown – not surprisingly – that the wave, close inshore, is dominated by nonlineaity, with the amplitude of the wave growing according to Green's law. The problem of edge waves is formulated in a similar fashion, although the relevant scales are different; in particular, the background flow must be roughly of the same size as the edge wave itself, for a self-consistent asymptotic theory of the type presented here. The development follows closely that used in the absence of a background flow, but with the background flow now appearing in the solution to leading order. This has the effect of distorting, for example, the run-up pattern of the edge waves at the shoreline, to the extent that, under certain conditions, the two solutions of the earlier theory can now be replaced by one (unique) solution.

1. Introduction

The extension of classical, simple models in fluid mechanics to encompass more realistic flow scenarios has been an enduring challenge. An example of this, of particular interest here, is how the familiar problems that lead to a theory for nonlinear surface gravity waves, or for edge waves, can be developed to accommodate a general background flow that is described by some distribution of vorticity. Further, in the case of edge waves, any such flow must also allow the existence of a shoreline i.e. the free surface and the bottom profile must intersect, thereby producing a run-up pattern on the beach. In order to permit a comparison of these two types of waves – one incoming towards a shore, say, and the other propagating *along* the shoreline – we shall superimpose each on the same background state: a vortical flow-field with a shoreline.

The two wave-types that we discuss here each have a long and worthy history, although we shall describe this only in outline in this paper – many texts and research papers give far more information for the interested reader. In the first, which was initially analysed in any detail by Stokes, we have a plane wave propagating over finite, but constant, depth; in the small-amplitude approximation, this is the familiar Stokes wave; see [16]. Following this seminal work, one direction eventually led Korteweg and de Vries, [13], to consider the approximation that represented a balance between small amplitude and weak dispersion; to leading order (in some sense, which we shall carefully explain later), this produces the now-very-famous Korteweg-de Vries (KdV) equation for surface waves. (This work was prompted, in part, by the debate in the 1870s – mainly between Rayleigh and Boussinesq – over the existence and properties of John Scott Russell's observations of solitary waves.) In the decades since the introduction of the KdV equation, we have seen the developments that have become 'soliton theory' i.e. inverse scattering transform theory; these have, in turn, led to many additional and deep ideas, not to mention the appearance of vastly many equations – some with significant practical applications – that can be solved by these soliton techniques. Finally, there have been attempts to embed these soliton results within more complete and accurate models for wave propagation, even though this is often at the expense of generating non-soliton-type problems. Indeed, typically, the end result is an approximate system that can be characterised as some perturbation of a standard soliton problem.

In the context of a plane gravity wave, incoming towards a shoreline over an existing flow with vorticity, both the background flow and the variable depth will distort any soliton or solitary wave (i.e. a solution based on the conventional KdV equation). It is already known that a background flow that admits a general distribution of vorticity, but over constant depth, does give rise to a suitable KdV equation: only the constant coefficients of the equation are affected by the presence of the underlying flow field; see [6]. (We comment that we shall not pursue here the possibility of a critical layer appearing in the flow, but this can be investigated; see [8].) On the other hand, the inclusion of variable depth does give rise to a distortion of the corresponding KdV problem, producing one that, at best, can be interpreted as a perturbation of the classical KdV equation. At worst, a KdV-type equation is recovered, but one that contains variable coefficients, and then, for general depth profiles, no headway is possible within classical soliton theory; for further general background to these types of problems, see [9]. The combination of both a background flow and variable depth, in the context of a KdV approximation, is described in [10], where a problem of flow, such as that over a weir, is discussed; this has some connection with the results that we shall present here.

The second problem, whose study was also initiated by Stokes (see [15]), but which has had a slightly more chequered history, is that of edge waves. These are waves that, in the shoreline context, propagate along the beach (and for which a non-zero beach slope is an essential requirement). For many years, these waves were thought to be merely mathematical curiosities (but with many intriguing properties; see, for example, [5, 17, 18] and [12] for an overview). However, over the last two decades or so, edge waves have been recognised as providing an important mechanism in erosion processes near a shoreline, by contributing to the movement of sand and pebbles along the shore. A recent development, [11], based on an important presentation, [2], of observations first made in [14, 19], showed that a scaling consistent with a transformation of Gerstner's exact, non-trivial solution of the classical water-wave problem, [7], gave a new form of the solution for the edge wave over variable depth. This analysis produced, at leading order in a suitable asymptotic formulation, a fully nonlinear theory (with an exact solution) for the edge wave; this recovered both the essential features of the edge wave, and also the run-up pattern typically seen when such waves are present. The inclusion, within this version of the edge-wave problem, of some pre-existing background vorticity can be expected to distort the edge-wave profile and, probably, the shape of the run-up pattern; we shall explore these possibilities.

The plan, therefore, is to present the general equations for the classical water-wave problem (expressed in a suitable non-dimensional form), and then the relevant and possible background states consistent with these equations. Although for each problem these states take the same form, the scalings (and sizes) turn out to be different. Then, in the two cases of interest, this background state is appropriately perturbed to produce either the familiar gravity wave approaching a shore, or the corresponding problem for edge waves propagating along the shoreline. The essential technique that we adopt in order to accomplish all this is the familiar one based on the construction of asymptotic expansions in a suitable parameter. In the former case, it is to be expected that even a KdV-type theory – and we will arrange for this to be the appropriate underlying problem – will not be valid as the depth continues to decrease as the beach is reached; the corresponding problem close inshore will carefully described. On the other hand, the latter problem (for edge waves) is dominated by the existence of a beach and a shoreline, so in this case we might expect only relatively minor adjustments due to the presence of the underlying and pre-existing flow field.

2. The governing equations

We consider an incompressible, inviscid fluid which is bounded above by the free surface $z = H(x, y, t)$, and below by a fixed, impermeable bed $z = \beta(x)$ (which is given as a function of only *x*, for simplicity, here); we elect to describe the problem in rectangular Cartesian co-ordinates ($\mathbf{x} \equiv (x, y, z)$), with *z* measured vertically upwards. In the absence of waves, the flat free surface is $z = 0$, and this intersects the bed profile along $x = 0$; the fluid, undisturbed by waves, extends into $x < 0$. (We comment that when an underlying flow field, with some prescribed vorticity, is included, the free surface even in the absence of waves is not, in general, a flat surface.) Although we assume an inviscid fluid (which is an acceptable model for water waves, because it is observed that such waves form and evolve long before viscous dissipation can start to have a significant effect), the flow field may be rotational; that is, we allow some general vorticity in the flow. Also, in the absence of viscous stresses, we can impose no more than a normal stress at the surface; we take this to be simply pressure = constant (= atmospheric pressure) here, because we also ignore surface tension – which is relevant only if very short (capillary) waves are to be included in the model. The governing equations are therefore: the Euler equation, the equation for incompressible fluids, surface and bottom kinematic conditions, and a dynamic condition – constant pressure – at the surface. All these together constitute the classical water-wave problem. The equations are

$$\begin{cases} \dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\dfrac{1}{\rho}\nabla P + \mathbf{F} \quad (\text{where } \mathbf{F} \equiv (0, 0, -g)) \\ \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{2.1}$$

where $\mathbf{u} \equiv (u, v, w)$ is the velocity field, and

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

is the familiar material derivative (and t is time). The boundary conditions are

$$\left.\begin{cases} P = P_a = \text{constant} \\ w = \dfrac{\partial H}{\partial t} + u\dfrac{\partial H}{\partial x} + v\dfrac{\partial H}{\partial y} \end{cases}\right. \quad \text{both on } z = H(x,y,t) \qquad (2.2)$$

and

$$w = u\frac{\mathrm{d}\beta}{\mathrm{d}x} \text{ on } z = \beta(x), \qquad (2.3)$$

where $\rho = \text{constant}$ is the density of the water, g is the constant acceleration of gravity and P_a is the constant air pressure above the water surface. It should be noted that the second boundary condition in (2.2), and that in (2.3), ensure that the two appropriate surfaces remain boundaries of the fluid.

At this stage, it is convenient to introduce a suitable non-dimensionalisation of these equations and boundary conditions (although the versions required for our two problems differ slightly). Let λ be a typical wavelength of the waves that we shall discuss – although we are not restricted to this choice in the solution that we describe – and take h_0, correspondingly, as a typical or average depth of the water. A suitable speed scale is that associated with the (approximate) speed of propagation of waves over the depth h_0, namely, $\sqrt{gh_0}$; this, in turn, produces the time scale $\lambda/\sqrt{gh_0}$. Thus far, we have the non-dimensionalisation represented by the transformation

$$(x,y) \to \lambda(x,y),\ z \to h_0 z,\ (u,v) \to \sqrt{gh_0}(u,v),\ t \to \left(\lambda/\sqrt{gh_0}\right)t, \qquad (2.4)$$

where, for example, $x \to \lambda x$ is to be read as x (the original, dimensional variable) is replaced by λx, where x is now the non-dimensional version of x. This specification, (2.4), does not however complete the process: the non-dimensionalisation of w requires a little care. Consider, for simplicity, the two-dimensional equation of mass conservation:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

which implies the existence of a stream function, $\psi(x,z,t)$, such that

$$u = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial x}.$$

The scalings already chosen for u and z show that the non-dimensionalisation for ψ is $\psi \to \left(h_0\sqrt{gh_0}\right)\psi$, and hence that for w becomes $w \to \left(h_0\sqrt{gh_0}/\lambda\right)w$. Finally, we define the pressure as the sum of that due to the hydrostatic pressure distribution and that due to the passage of the wave:

$$P = P_a - \rho g z + \rho g h_0 p\,, \tag{2.5}$$

where p is the non-dimensional pressure perturbation. (In (2.5), z is still in the original, dimensional form.)

It is convenient to define H and β, following the scheme just described, as

$H = h_0 h(x, y, t)$ and $\beta = h_0 b(x)$,

respectively. The problem represented by equations (2.1)-(2.3), now written in non-dimensional variables, becomes

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}; \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial p}{\partial y}; \quad \delta^2 \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z}; \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,, \tag{2.6}$$

with $$p = h \text{ and } w = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} \text{ on } z = h(x, y, t) \tag{2.7}$$

and $$w = u\frac{\mathrm{d}b}{\mathrm{d}x} \text{ on } z = b(x)\,, \tag{2.8}$$

where $\delta = h_0/\lambda$ is the shallowness, or long-wavelength, parameter. Of course, a complete prescription of the problem involves the inclusion of some appropriate initial data; for the purposes of the discussion that we present here, it is sufficient to suppose that initial data exist which will give rise to the types of wave that we investigate.

At this stage, we make the final, overall adjustments to this formulation which will enable us to discuss, separately, the two problems of interest (and will also lead to the specification of the background state). So first, for the wave approaching a shoreline (from the deep ocean, say), we restrict the motion to be that of a plane wave approaching the beach; thus we choose to suppress the dependence on y, with $v \equiv 0$. Further, we shall allow the bottom profile to evolve on a suitable scale – this will provide the basis for the parameter that we use in the developments described here. We introduce $b(x) = -B(\sigma x)$, where σ is a parameter (and the minus sign is no more than a convenience). Thus we obtain, from (2.6)-(2.8), the set

$$u_t + uu_x + wu_z = -p_x\,; \; \delta^2\left(w_t + uw_x + ww_z\right) = -p_z\,; \; u_x + w_z = 0\,,$$

with $\quad p = h$ & $w = h_t + uh_x$ on $z = h(x,t)$, and $w = -\sigma B'(\sigma x)$ on $z = -B(\sigma x)$,

where we have used subscripts to denote partial derivatives and the prime denotes the derivative with respect to the argument of the function. Now we rescale the variables to remove δ^2 in favour of a new parameter ε (which will be defined below). Thus we transform according to

$$x \to \frac{\delta}{\sqrt{\varepsilon}} x\,,\ t \to \frac{\delta}{\sqrt{\varepsilon}} t\,,\ w \to \frac{\sqrt{\varepsilon}}{\delta} w\,,$$

and the rest of the variables are unchanged; this is the familiar scaling used in any comprehensive derivation of the KdV equation. Thus we obtain

$$u_t + uu_x + wu_z = -p_x\,;\ \varepsilon\left(w_t + uw_x + ww_z\right) = -p_z\,;\ u_x + w_z = 0\,,$$

with $\quad p = h\ \ \&\ \ w = h_t + uh_x$ on $z = h(x,t)$,

and $\quad w = -\dfrac{\sigma\delta}{\sqrt{\varepsilon}} B'\left(\dfrac{\sigma\delta}{\sqrt{\varepsilon}} x\right)$ on $z = -B\left(\dfrac{\sigma\delta}{\sqrt{\varepsilon}} x\right)$.

Finally, we choose ε, for given σ and δ, so that

$$\frac{\sigma\delta}{\sqrt{\varepsilon}} = \varepsilon \ \text{ i.e. } \varepsilon = (\sigma\delta)^{2/3}\,,$$

but with the requirement that $\sigma\delta \to 0$ (which may be interpreted in any way consistent with this condition); this produces the form of the equations that we shall use for our first problem:

$$\begin{cases} u_t + uu_x + wu_z = -p_x;\ \varepsilon\left(w_t + uw_x + ww_z\right) = -p_z;\ u_x + w_z = 0, \\ \text{with} \quad p = h \quad \& \quad w = h_t + uh_x \quad \text{on} \quad z = h(x,t) \\ \text{and} \quad w = -\varepsilon u B'(\varepsilon x) \quad \text{on} \quad z = -B(\varepsilon x). \end{cases} \tag{2.9}$$

For the second problem, involving a study of edge waves, we first choose to use only one scale length (which may be anything appropriate) and so $\delta = 1$. However, we make the same choice of the depth profile as used above: we set $b(x) = -B(\varepsilon x)$, for $\varepsilon \to 0$, and in this case we shall define ε as the slope at the beach i.e. $|b'(0)| = \varepsilon$. Equations (2.6)-(2.8) therefore become

$$\begin{cases} \dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p, \quad \nabla \cdot \mathbf{u} = 0 \\ \text{with} \quad p = h \quad \& \quad w = h_t + uh_x + vh_y \quad \text{on} \quad z = h(x,y,t) \\ \text{and} \quad w = -\varepsilon u B'(\varepsilon x) \quad \text{on} \quad z = -B(\varepsilon x), \end{cases} \tag{2.10}$$

where we have reverted to the familiar notation using the operators $\mathrm{D}/\mathrm{D}t$ and ∇ (with the velocity vector $\mathbf{u}$).

The intention is to construct solutions of each set (2.9) and (2.10), based on suitable asymptotic expansions in ε, which represent perturbations of a background state – and this state we now describe.

3. The background state

The background state is an exact solution of the governing equations that is steady (time independent) and, in this context, depends on only $X = \varepsilon x$ and z. For the problem given by the set (2.9), we write this state as

$$u = U(X, z; \varepsilon),\ w = \varepsilon W(X, z; \varepsilon),\ p = P(X, z; \varepsilon),\ h = H(X; \varepsilon), \qquad (3.1)$$

which therefore satisfies the problem described by

$$\begin{cases} UU_X + WU_z = -P_X; \quad \varepsilon^3 (UW_X + WW_z) = -P_z; \quad U_X + W_z = 0, \\ \text{with} \qquad P = H \quad \& \quad W = UH' \quad \text{on} \quad z = H(X;\varepsilon) \\ \text{and} \qquad W = -UB' \quad \text{on} \quad z = -B(X). \end{cases} \qquad (3.2)$$

On the other hand, the problem of edge waves described by (2.10) – for reasons that will become evident later – requires a slight adjustment to the system given in (3.2). In this case, we must work with a background flow that possesses a suitable weak (in the sense of $\varepsilon \to 0$) vorticity; this is defined by a further transformation on (3.2) (but also note that ε^3 there is replaced by ε^2 because $\delta = 1$):

$$(U,W) \to \sqrt{\varepsilon}(U,W); \quad (P,H) \to \varepsilon(P,H) \qquad (3.3)$$

to give the set

$$\begin{cases} UU_X + WU_z = -P_X; \quad \varepsilon^2 (UW_X + WW_z) = -P_z; \quad U_X + W_z = 0, \\ \text{with} \qquad P = H \quad \& \quad W = \varepsilon UH' \quad \text{on} \quad z = \varepsilon H(X;\varepsilon) \\ \text{and} \qquad W = -UB' \quad \text{on} \quad z = -B(X). \end{cases} \qquad (3.4)$$

This implies that we can obtain solutions of (3.4), with error $O(\varepsilon^2)$, by rescaling solutions of (3.2) according to (3.3).

The possibility, and nature, of solutions of the set (3.2) (and, equivalently, of (3.4)) are discussed at some length in [3], where a careful derivation and proofs are presented. In particular, it is shown that flow that possesses non-zero vorticity and a shoreline must have a non-flat free surface. Although it is possible to describe the general structure, and conditions for the existence, of relevant solutions of (3.2) for arbitrary ε, explicit, simple solutions are not available. However, we shall be concerned with the case described by $\varepsilon \to 0$, and then we can write down various solutions of (3.2), with an error $O(\varepsilon^3)$ (and it is straightforward to confirm that the higher-order correction terms merely contribute small adjustments in a uniformly valid asymptotic description of the background state). The general discussion and underlying principles resulting in the construction of solutions need not be rehearsed here – the details can be found in [3] – but we do present three simple, although important and illuminating, examples.

From (3.2), with $\varepsilon \to 0$, we obtain the reduced problem for the background state as

$$\begin{cases} UU_X + WU_z = -P_X; \quad P_z = 0; \quad U_X + W_z = 0, \\ \text{with} \quad P = H \quad \& \quad W = UH' \quad \text{on} \quad z = H(X;\varepsilon) \\ \text{and} \quad W = -UB' \quad \text{on} \quad z = -B(X), \end{cases} \tag{3.5}$$

where the shoreline (in the absence of waves) is at $z = H = -B = 0$, which we fix to be at $X = 0$. We note that the vorticity $\boldsymbol{\omega} \equiv \left(0, U_z - \varepsilon^2 W_X, 0\right) \to \left(0, U_z, 0\right)$ as $\varepsilon \to 0$. The constant-vorticity solution, $\boldsymbol{\omega} \equiv (0, 2k, 0)$, of (3.5) is

$$\begin{cases} U = 2kz + k(B-H), \quad W = -k(B'-H')z + k(BH)' \\ \text{with} \quad P \equiv H = -\left(B + \dfrac{1}{k^2}\right) + \dfrac{1}{k^2}\sqrt{1+2k^2 B}, \end{cases} \tag{3.6}$$

for given $B(X)$. (In this case, the stream function is $\Psi = kz^2 + k(B-H)z - kBH$, with $U = \Psi_z$, $W = -\Psi_X$, $z \in [-B, H]$.) An example of the free surface, and internal streamlines, is shown in figure 1 (for a suitable choice of $B(X)$).

A simple example of a flow with variable vorticity is provided by the stream function

$$\Psi = \frac{\alpha}{\sinh[\ell(H+B)]}\left\{\sinh[\ell(H+B)] - \sinh[\ell(z+B)] + \sinh[\ell(z-H)]\right\}, \tag{3.7}$$

where ℓ and α are constants; the free surface is described by the solution of the equation

$$\alpha^2 \ell^2 \left(\frac{\cosh[\ell(H+B)] - 1}{\sinh[\ell(H+B)]} \right)^2 = -2H, \tag{3.8}$$

for given $B(X)$. In this case, the vorticity is

$$\boldsymbol{\omega} \equiv \left(0, -\alpha\ell^2 \frac{\cosh\left[\ell\left(z + \frac{1}{2}B - \frac{1}{2}H\right)\right]}{\cosh\left[\frac{1}{2}\ell(H+B)\right]}, 0 \right), \quad z \in [-B, H]. \tag{3.9}$$

Two examples of the free surface for this variable vorticity are shown in figure 2 (for the same choice of $B(X)$ as used for figure 1); an associated vorticity distribution is given in figure 3.

Although it is of less interest in the context of flow fields – and waves – near a beach, we comment that there are also solutions that exhibit isolated regions of vorticity, outside which the vorticity is zero. This requires, however, a special topography: a region of finite extent between two points of equal depth (at, say, $X = X_1$, $X = X_2$). Further, we require $B'(X_1) = B'(X_2) = B''(X_1) = B''(X_2) = 0$ with $B(X_1) = B(X_2) = B_0$ and $B > B_0$ for $X \in (X_1, X_2)$; an example of such a stream function is

$$\Psi = \begin{cases} \dfrac{B_0}{\pi}\sqrt{2\left[B(X) - B_0\right]}\sin\left[\dfrac{\pi}{B_0}\left(z + B(X)\right)\right], & X_1 \le X \le X_2, \quad -B \le z \le H \\ \\ 0 \quad \text{otherwise,} \end{cases} \tag{3.10}$$

and the corresponding free surface is

$$H(X) = \begin{cases} B_0 - B(X), & X_1 \le X \le X_2 \\ \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

In figure 4, we present an example of streamlines for such an isolated region of vorticity, for a suitable choice of $B(X)$.

Finally, as we have already commented, we may construct solutions of the set (3.4) by rescaling solutions of (3.2). Thus, in the case of constant vorticity described in (3.6), we choose to write $\ell = \sqrt{\varepsilon}L$ and then, invoking the transformation (3.3), we have a solution of (3.2):

$$\begin{cases} U = 2Lz + L(B - \varepsilon H), \quad W = -Lz(B' - \varepsilon H') + \varepsilon L(BH)', \quad P = H, \\ \\ \text{where} \quad \varepsilon H = -\left(B + \dfrac{1}{\varepsilon L^2}\right) + \dfrac{1}{\varepsilon L^2}\sqrt{1 + 2\varepsilon L^2 B} \quad \text{and so } H \sim -\tfrac{1}{2}L^2 B^2, \end{cases} \tag{3.12}$$

for given $B(X)$, all defined for $-B(X) \le z \le H(X)$. The solution that corresponds to (3.7) and (3.8), for variable vorticity, is obtained by setting $\alpha = \sqrt{\varepsilon}A$ (with (3.3) and $\Psi \to \sqrt{\varepsilon}\Psi$) to yield

$$
\begin{cases}
\Psi = \dfrac{A}{\sinh[\ell(B+\varepsilon H)]}\left\{\sinh[\ell(B+\varepsilon H)] - \sinh[\ell(z+B)] + \sinh[\ell(z-\varepsilon H)]\right\} \\
\text{with} \qquad A^2\ell^2\left(\dfrac{\cosh[\ell(B+\varepsilon H)]-1}{\sinh[\ell(B+\varepsilon H)]}\right)^2 = -2H \\
\text{and so} \qquad H \sim -\dfrac{1}{2}A^2\ell^2\left(\dfrac{\cosh(\ell B)-1}{\sinh(\ell B)}\right)^2.
\end{cases} \tag{3.13}
$$

With the description of the background state in place, we may proceed to consider perturbations of these states that admit wave-like solutions.

4. Nonlinear, dispersive waves approaching a shoreline

The first problem that we address is that of a plane gravity wave, moving over a vortical, variable-depth flow, which approaches a beach. This problem makes use of the equations given in the set (2.9), together with a background state described by a solution to the set (3.2) or, with an error $\mathrm{O}(\varepsilon^3)$, to the set (3.5). To proceed, we assume that there exists a solution of (2.9) for which the perturbation can be expressed in terms of a suitable characteristic variable (for waves moving to the right over variable depth) and a corresponding (slow) evolution variable; these are conveniently defined by

$$
\xi = -t + \frac{1}{\varepsilon}\int_{-X_0}^{X} \frac{\mathrm{d}X'}{c(X')} \quad \text{and our original } X = \varepsilon x, \tag{4.1}
$$

respectively. Here, we have yet to determine $c(X')$, and we have elected to consider the problem in which there is, we shall suppose, constant depth for some $X < 0$, which is where the wave is initiated; $X = -X_0$ is in this region of constant depth. We now seek a solution of (2.9) written in the form, which follows our earlier convention in the use of '$\rightarrow$',

$$
\begin{cases}
u \rightarrow U(X,z;\varepsilon) + \varepsilon u(\xi,X,z;\varepsilon), \quad w \rightarrow \varepsilon\big(W(X,z;\varepsilon) + w(\xi,X,z;\varepsilon)\big) \\
p \rightarrow P(X,z;\varepsilon) + \varepsilon p(\xi,X,z;\varepsilon) \\
\text{with} \qquad h \rightarrow H(X;\varepsilon) + \varepsilon h(\xi,X;\varepsilon).
\end{cases} \tag{4.2}
$$

We see that *u*, *w*, *p* and *h* are now used to represent the perturbations of the background state. The equations for this perturbation, given the background state, are therefore

$$\left\{ \begin{array}{c} -u_\xi + (U+\varepsilon u)\left(\frac{1}{c}u_\xi + \varepsilon u_X\right) + \varepsilon(W+w)u_z + \varepsilon u U_X + wU_z = -\frac{1}{c}p_\xi - \varepsilon p_X; \\ \varepsilon\left\{-w_\xi(U+\varepsilon u)\left(\frac{1}{c}w_\xi + \varepsilon w_X\right) + \varepsilon(W+w)w_z + \varepsilon^2 uW_X\right\} = -p_z; \\ \frac{1}{c}u_\xi + w_z + \varepsilon u_X = 0 \end{array} \right. \tag{4.3}$$

with the boundary conditions

$$\left\{ \begin{array}{l} P + \varepsilon p = H + \varepsilon h \quad \& \quad W + w = h_\xi + (U+\varepsilon u)\left(H' + \frac{1}{c}h_\xi + \varepsilon h_X\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{both on} \quad z = H + \varepsilon h \\ \text{and} \qquad\qquad w = -\varepsilon u B' \quad \text{on} \quad z = -B(X). \end{array} \right. \tag{4.4}$$

The two boundary conditions at the free surface, $z = H + \varepsilon h$, are mapped to the corresponding conditions on the known surface $z = H(X;\varepsilon)$; this is equivalent to generating Taylor expansions about $z = H$, valid asymptotically as $\varepsilon \to 0$.

It is now a routine exercise to seek a solution of the set (4.3), with boundary conditions (4.4), in the form of an asymptotic expansion in ε; we write

$$q(\xi, X, z; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n q_n(\xi, X, z), \tag{4.5}$$

where q (and correspondingly q_n) represents each of u, w and p; the asymptotic expansion for h is then

$$h(\xi, X; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n h_n(\xi, X). \tag{4.6}$$

At leading order in ε, we find that

$$\left\{ \begin{array}{l} u_0 = \left(\dfrac{1}{U(X,z) - c(X)} - U_z \displaystyle\int_{-B}^{z} \dfrac{\mathrm{d}z'}{[U(X,z') - c(X)]^2} \right) h_0(X); \\ \\ w_0 = -\left(1 - \dfrac{U(X,z)}{c(X)}\right) h_{0\xi}(X) \displaystyle\int_{-B}^{z} \dfrac{\mathrm{d}z'}{[U(X,z') - c(X)]^2}; \quad p_0 = h_0(X), \end{array} \right. \tag{4.7}$$

(for $z \in [-B, H]$), with

$$\int_{-B(X)}^{H(X)} \frac{\mathrm{d}z'}{[U(X,z')-c(X)]^2} = 1, \tag{4.8}$$

which is a version of the Burns condition, [1]; this determines $c(X)$, given $U(X,z)$, $B(X)$ and $H(X)$. It is convenient to introduce a compact notation for the various integrals that appear here, so we define

$$I_n(X,z) = \int_{-B(X)}^{z} \frac{\mathrm{d}z'}{[U(X,z')-c(X)]^n} \quad (\text{for } n = 2, 3, 4, \dots) \tag{4.9}$$

and $I_{ns}(X) = I_n(X, H(X))$ is used to denote evaluation at the surface, $z = H(X)$; the Burns condition then becomes $I_{2s} = 1$. We note that, at this order of approximation, we have interpreted the background state as a solution of the set (3.5), and so there is an error $\mathrm{O}(\varepsilon^3)$ implied here (although we could, formally, elect to use an exact solution of (3.2) e.g. $H = H(X;\varepsilon)$). At this order, the first approximation to the surface wave, $h_0(\xi, X)$, is unknown.

At the next order, which is considerably more involved (but, nevertheless, fairly routine to analyse), we find, for example

$$p_1 = h_1 + \left(\int_z^H \left(1 - \frac{U(X,z')}{c(X)} \right) I_2(X,z')\,\mathrm{d}z' \right) h_{0\xi\xi},$$

and eventually we obtain the equation for h_0 (leaving the equation for h_1 to be determined at the next order):

$$2c\sqrt{-I_{3s}}\left(c\sqrt{-I_{3s}}\,h_0 \right)_X + 3I_{4s} h_0 h_{0\xi} + \frac{1}{c^2} J_s h_{0\xi\xi\xi} = 0. \tag{4.10}$$

This takes the form of a Korteweg-de Vries (KdV) equation, but with variable coefficients by virtue of $I_{ns}(X)$ and

$$J_s(X) = \int_{-B(X)}^{H(X)} \int_{z}^{H(X)} \int_{-B(X)}^{z'} \frac{[U(X,z')-c(X)]^2}{[U(X,z)-c(X)]^2 [U(X,z'')-c(X)]^2} \mathrm{d}z''\mathrm{d}z'\mathrm{d}z\,; \tag{4.11}$$

related problems that give rise to similar variants of the KdV equation can be found in [6, 9, 10]. Thus, as we intended, we have used scales – that is, we have selected an appropriate region of physical space defined in terms of the parameters – for which we have a leading-order balance between nonlinearity and dispersion. Although our KdV-type equation, (4.10), cannot be solved in any general sense (because of the variable coefficients), we may surmise that waves satisfying this equation will propagate towards the shore, and that these can be expected to exhibit solitary- or soliton-like characteristics. (Certainly, in the case of constant depth and zero vorticity, we recover the classical KdV equation for water waves:

$$2h_{0X} + 3h_0 h_{0\xi} + \tfrac{1}{3} h_{0\xi\xi\xi} = 0,$$

for which the whole panoply of soliton theory is applicable; for example, see [4].) Any such solution, we observe, is valid where $\xi = \mathrm{O}(1)$ (a coordinate that follows the wave) and $X = \mathrm{O}(1)$; we are not concerned here with any difficulties that might be encountered as $|X| \to \infty$, an issue that must be addressed in more conventional KdV applications. Indeed, the wave is initiated in a region where $X = \mathrm{O}(1)$, and we are particularly interested in the development of a solution – expressed by the asymptotic expansion – as $X \to 0$, which is in the neighbourhood of the shoreline (defined more precisely by the region where the local depth $D = B + H \to 0$).

It is an altogether straightforward exercise to provide the details of this problem for the simple choices of background flow; for example, the constant-vorticity solution (given in (3.6)) yields, first,

$$c(X) = \sqrt{D + k^2 D^2} \quad (D(X) = B(X) + H(X)) \tag{4.12}$$

from the Burns condition, (4.8), for right-running (incoming) waves. Then the appropriate form of the KdV equation, (4.10), becomes

$$2\left(\frac{c}{D}\right)^{3/2}\left(\frac{c^{3/2}}{D^{1/2}} h_0\right)_X + \left(\frac{3+4k^2 D}{D}\right) h_0 h_{0\xi} + \left(\frac{D^3}{3c^2(c+kD)^2}\right) h_{0\xi\xi\xi} = 0, \tag{4.13}$$

where $c(X)$ is that given in (4.12). An avenue of further exploration might be to construct numerical solutions – the only viable approach in this situation – of (4.13), perhaps using solitary-wave initial data at $X = -X_0$, for various choices of k (which represents the given constant vorticity) and of $B(X)$, with

$$D(X) = B(X) + H(X) = \frac{1}{k^2}\left(-1 + \sqrt{1 + 2k^2 B(X)}\right); \tag{4.14}$$

see (3.6). However, the main interest here is to examine the nature of the problem for $D \to 0$, corresponding to the shoreline where $B \to 0$.

It is particularly straightforward to find the asymptotic form of (4.13), as $D \to 0$, together with the corresponding results for other terms in the expansion (such as *H*, *U*, *u*, etc.). Indeed, we find that

$$\begin{cases} c \sim \sqrt{D},\ I_{3s} \sim -1/\sqrt{D},\ I_{4s} \sim 1/D \quad (\text{as } D \to 0) \\ \\ H \sim -\tfrac{1}{2}k^2B^2 \quad (\text{as } B \to 0) \end{cases} \tag{4.15}$$

so that the KdV equation, (4.13), takes the approximate form as $D \to 0$:

$$2D^{1/4}\left(D^{1/4}h_0\right)_X + \frac{3}{D}h_0h_{0\xi} + \frac{D}{3}h_{0\xi\xi\xi} = 0, \tag{4.16}$$

which is more conveniently expressed in terms of $D^{1/4}h_0 = A_0(\xi,\chi)$, with $\chi = \int^X D^{-7/4}(X')\,\mathrm{d}X'$, to give

$$2A_{0\chi} + 3A_0A_{0\xi} + \tfrac{1}{3}\Delta^{9/4}A_{0\xi\xi\xi} = 0, \tag{4.17}$$

where $\Delta(\chi) = D\left[X(\chi)\right]$. This last form of the KdV equation, (4.17), demonstrates that, as $D \to 0$, the amplitude of the wave is dominated by the $D^{-1/4}$ growth, usually known as Green's law; the dispersion effects, we note, diminish in this same limit. Further, this also shows that the asymptotic expansion for the surface wave, for example (and the others follow the same pattern), based on the first two terms in the expansion, possesses the property

$$h \sim -\tfrac{1}{2}k^2B^2 + \varepsilon D^{-1/4}A_0$$

as $D \to 0$ (and so $B \to 0$); this can be recast in terms of D:

$$h \sim -\tfrac{1}{2}k^2D^2 + \varepsilon D^{-1/4}A_0, \tag{4.18}$$

because $B = D - H \sim D + \tfrac{1}{2}k^2B^2 \sim D + \tfrac{1}{2}k^2D^2$ as $D \to 0$. This asymptotic expansion is therefore not uniformly valid as $D \to 0$ where $D = \mathrm{O}(\varepsilon^{4/9})$, which is the region close inshore that we shall need to explore in a little more detail. However, this conclusion is based on the specific result: a constant-vorticity background state. What is the situation for more general background flows?

It is surprisingly straightforward to estimate the behaviour of the various functions and integrals in the case: $H = -B + D$, as $D \to 0$, for any background flow. We find that (on noting that $D \to 0$ corresponds to $B \to 0$ and $H \to 0$ near a shoreline)

$$c \sim U_B + \sqrt{D},\ I_{3s} \sim -1/\sqrt{D},\ I_{4s} \sim 1/D,\ J_s \sim \tfrac{1}{3}D^2$$

(and all these results agree with the special case quoted above, with U_B being U evaluated on $z = -B$). Indeed, for an arbitrary vorticity distribution, it is possible to find the general solution of equations (3.5) valid as $B \to 0$ (and hence as $D \to 0$). To accomplish this, we introduce $\zeta = z/B(X)$ and then solve for Ψ, constructing an asymptotic form driven by $B \to 0$; this gives

$$\Psi \sim \gamma B^2\left(\zeta + \zeta^2\right) \text{ and } H \sim -\tfrac{1}{2}\gamma^2 B^2, \qquad (4.19)$$

where γ is an arbitrary constant. Then we find that

$$U \sim \gamma B(1+2\zeta) \text{ and } W \sim -\gamma BB'\zeta, \qquad (4.20)$$

both expressed in terms of ζ; thus $U_B \sim -\gamma B \sim -\gamma D$ and so $c \sim \sqrt{D}$ as $D \to 0$. The vorticity in the background flow (at this order) is necessarily constant close to the shoreline:

$$U_z \sim 2\gamma \text{ as } B \to 0. \qquad (4.21)$$

The upshot is that the description of the problem, for $D \to 0$, embodied in equations (4.16)-(4.18), is the appropriate description for any vorticity (other that zero vorticity, for which $\Psi \equiv 0$). The breakdown of the asymptotic expansion, represented by the condition

$$D = \mathrm{O}(\varepsilon^{4/9}), \qquad (4.22)$$

is therefore generic (and this, we note, corresponds to $B = \mathrm{O}(\varepsilon^{4/9})$); this provides the basis for a suitably rescaled version of the problem now valid close inshore. (An examination of the behaviour of the magnitudes of the next terms in the expansions of each function, as $D \to 0$, shows that this breakdown encompasses all possible non-uniformities in the expansions, for reasonable initial data.)

5. KdV gravity waves very near a shoreline

The behaviour of the solution, as $D \to 0$ (or, equivalently, as $B \to 0$), both for the background state and its perturbation, shows that we must rescale according to the scheme

$$\left\{\begin{array}{l} H = \varepsilon^{8/9}\hat{H},\, B = \varepsilon^{4/9}\hat{B}, U = \varepsilon^{4/9}\hat{U}, W = \varepsilon^{8/9}\hat{W},\, P = \varepsilon^{8/9}\hat{P} \\ \text{with} \\ c = \varepsilon^{2/9}\hat{c},\, h = \varepsilon^{-1/9}\hat{h},\; p = \varepsilon^{-1/9}\hat{p}, u = \varepsilon^{-1/3}\hat{u},\, w = \varepsilon^{-1/9}\hat{w}, \end{array}\right. \qquad (5.1)$$

and $z = \varepsilon^{4/9}Z$. Under this scaling transformation, the free surface becomes $Z = \hat{H} + \hat{h}$ (so that the background state and its perturbation are the same size), and the bottom is simply $Z = -\hat{B}$. The full set of equations, with the scaling (5.1), are readily written down, but they are lengthy and – it might appear – overly complicated. It is sufficient to outline the main results that we obtain here (mainly because there are no surprises); thus we seek an asymptotic solution based on expansions of the form

$$\hat{q}(\xi, X, Z; \varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^{2n/9} \hat{q}_n(\xi, X, Z), \tag{5.2}$$

for each of $\hat{u}$, $\hat{w}$ and $\hat{p}$ (and likewise for $\hat{h}$, with the dependence on Z omitted, and also for $\hat{c}(X;\varepsilon)$). The background is exactly as described earlier, although now rescaled according to (5.1).

We find, at leading order, that

$$\hat{u}_0 = \frac{\hat{h}_0}{\hat{c}_0}, \quad \hat{w}_0 = -\frac{1}{\hat{c}_0^2}\left(Z + \hat{B}\right)\hat{h}_{0\xi}, \quad \hat{p}_0 = \hat{h}_0 \quad \left(-\hat{B} \le Z \le 0\right), \tag{5.3}$$

with $\hat{c}_0 = \sqrt{\hat{B}(X)}$, but $\hat{h}_0(\xi, X)$ is undetermined. At the next order (i.e. $\varepsilon^{2/9}$), we obtain

$$\hat{u}_1 = \frac{\hat{h}_1}{\sqrt{\hat{B}}} - 2\gamma\hat{h}_0, \quad \hat{w}_1 = \left(\frac{\gamma}{\sqrt{\hat{B}}}\hat{h}_{0\xi} - \frac{1}{\hat{B}}\hat{h}_{1\xi}\right)\left(Z + \hat{B}\right), \quad \hat{p}_1 = \hat{h}_1, \tag{5.4}$$

with $\hat{c}_1 = 0$ and where we have used $\hat{U}_Z = 2\gamma$, the vorticity close inshore. Finally, at $\mathrm{O}(\varepsilon^{4/9})$, we generate an equation for $\hat{h}_0(\xi, X)$ (leaving $\hat{h}_1$ to be determined at the next order):

$$2\hat{B}^{5/4}\left(\hat{B}^{1/4}\hat{h}_0\right)_X + 3\hat{h}_0\hat{h}_{0\xi} = 0, \tag{5.5}$$

which corresponds precisely with the first two terms in our KdV equation, (4.16) (when we remember that $D \sim B$ as $B \to 0$). Indeed, when we introduce $\hat{B}^{1/4}\hat{h}_0 = \hat{A}_0$ and $\hat{\chi} = \int^X \hat{B}^{-7/4}(X')\,\mathrm{d}X'$, we obtain

$$2\hat{A}_{0\hat{\chi}} + 3\hat{A}_0\hat{A}_{0\xi} = 0, \tag{5.6}$$

which will, in general, represent a breaking wave (in the sense of allowing discontinuous – jump – solutions) sufficiently close to the shoreline. Further, it is immediately clear that solutions of (5.6) will match to solutions of (4.17) as $D \to 0$, the dispersive effects now being dominated by the nonlinearity of the wave. In summary, therefore, we have confirmed that a solution of our KdV equation, valid in the region defined by $X = \mathrm{O}(1)$ where $D = \mathrm{O}(1)$, will evolve into a purely nonlinear, non-dispersive wave, with an amplitude that grows like $\hat{D}^{-1/4}$ (with $\hat{D} \sim \hat{B}$) as the depth decreases. The region where this occurs has also been determined: it is where the depth is as small as $\mathrm{O}(\varepsilon^{4/9})$.

6. Edge waves propagating over a background flow

We now turn to an investigation of our second water-wave example: edge waves. The plan in this case is to take the development of this problem as described in [11], which is based on a new choice of scales suggested by the work in [2], and then superimpose this on a suitable background state. We start with the situation of an edge wave that is propagating (in the *y*-direction) in stationary water; this is described by a solution of equations (2.10), scaled according to

$(u,v,w) \to \sqrt{\varepsilon}\left(u,v,\varepsilon w\right)$ and $(p,h) \to \varepsilon(p,h)$,

together with the choice of new independent variables

$$\xi = \ell y - \omega\sqrt{\varepsilon}\, t, \quad \theta = \frac{1}{\varepsilon}\int^{X} \alpha(X';\varepsilon)\,\mathrm{d}X', \quad X = \varepsilon x; \tag{6.1}$$

we leave *z* unchanged, and α and ω (= constant) are to be determined, given the wave number ℓ. Now the inclusion of the background state would suggest that we transform according to, for example, $u \to U(X,z;\varepsilon) + \sqrt{\varepsilon}\,u(\xi,\theta,X;\varepsilon)$, but this implies, at leading order as $\varepsilon \to 0$, that $\partial u/\partial\theta = 0$; such a requirement would negate the existence of an appropriate edge-wave solution. The only consistent way forward – consistent, that is, with the solution in [11] – is to restrict the background flow-field to be no larger than the perturbation of it that describes the edge wave; this requires that the stream function representing the background state is $\mathrm{O}(\sqrt{\varepsilon})$. Then, correspondingly, we see that both *p* and *h* in the background state must be $\mathrm{O}(\varepsilon)$.

With the foregoing observations in mind, we proceed by scaling equations (2.10) according to the scheme

$$(u,v,w) \to \sqrt{\varepsilon}\left(U+u, v, \varepsilon(W+w)\right), \quad (p,h) \to \varepsilon(P+p, H+h), \tag{6.2}$$

where the set $\left\{U(X,z;\varepsilon), W(X,z;\varepsilon), P(X,z;\varepsilon), H(X;\varepsilon)\right\}$ represents a solution of the background state described by equations (3.4). The equations defining the set (*u*, *v*, *w*, *p*, *h*) – the perturbation – then become

$$\begin{cases} -\omega u_\xi + \ell v u_\xi + \varepsilon u U_X + (U+u)\left(\alpha u_\theta + \varepsilon u_X\right) + \varepsilon\left(W u_z + w U_z + w u_z\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad = -\left(\alpha p_\theta + \varepsilon p_X\right); \\ -\omega v_\xi + \ell v v_\xi + (U+u)\left(\alpha v_\theta + \varepsilon v_X\right) + \varepsilon(W+w)v_z = -\ell p_\xi; \\ \varepsilon\left\{-\omega w_\xi + \ell v w_\xi + \varepsilon u W_X + (U+u)\left(\alpha w_\theta + \varepsilon w_X\right) + \varepsilon\left((Ww)_z + w w_z\right)\right\} = -p_z; \\ \qquad \alpha u_\theta + \ell v_\xi + \varepsilon\left(u_X + w_z\right) = 0, \\ \text{with} \quad P+p = H+h \quad \& \quad W+w = -\omega h_\xi + \ell v h_\xi + (U+u)\left(\varepsilon H_X + \alpha h_\theta + \varepsilon h_X\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{both on } z = \varepsilon(H+h), \\ \text{and} \qquad w = -uB' \quad \text{on} \quad z = -B(X). \end{cases} \tag{6.3}$$

We now seek an asymptotic solution of this set, (6.3), by assuming that a solution exists of the form

$$q(\xi,\theta,X,z;\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n q_n(\xi,\theta,X,z),$$

where q, and correspondingly q_n, represent each of u, v, w and p; $h(\xi,\theta,X;\varepsilon)$ is similarly expanded. Although we may, in general, also expand both $\omega(\varepsilon)$ and $\alpha(X;\varepsilon)$, this extra freedom – and minor complication – is unnecessary here. The procedure follows that described in detail in [11] (and given in outline in [12]), with the one adjustment that the background state is included here, and given.

At leading order, we generate a nonlinear system of equations:

$$-\omega u_{0\xi} + \ell v_0 u_{0\xi} + \alpha(U+u_0)u_{0\theta} = -\alpha p_{0\theta};\ -\omega v_{0\xi} + \ell v_0 v_{0\xi} + \alpha(U+u_0)v_{0\theta} = -\ell p_{0\xi};$$
$$\alpha u_{0\theta} + \ell v_{0\xi} = 0;\ p_{0z} = 0, \text{ with } p_0 = h_0 \text{ on } z = 0,$$

which constitute a version of the nonlinear, shallow-water equations. This set has an exact solution, relevant to edge waves:

$$\left\{\begin{aligned} &U + u_0 = \frac{\ell}{\omega} A_0 \mathrm{e}^{\theta} \sin\xi, \quad v_0 = -\frac{\ell}{\omega} A_0 \mathrm{e}^{\theta}\cos\xi, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{all defined for } -B \le z \le 0 \\ &p_0 = h_0 = A_0 \mathrm{e}^{\theta}\cos\xi - \frac{1}{2}\frac{\ell^2}{\omega^2} A_0^2 \mathrm{e}^{2\theta} + C_0, \end{aligned}\right. \tag{6.5}$$

where we have set $\alpha = \ell$, and $A_0(X)$ and $C_0(X)$ are yet to be determined. At the next order, we recover precisely the problem described in [11], when we choose $C_0 = H(X)$; this then represents a uniformly valid asymptotic solution provided that $A_0(X)$ is a solution of

$$A_0 B' + 2BA_0' = -\frac{\omega^2}{\ell} A_0, \tag{6.6}$$

and so

$$A_0(X) = \frac{1}{\sqrt{-B(X)}} \exp\left\{ \frac{\omega^2}{2\ell} \int^X \frac{\mathrm{d}X'}{B(X')} \right\}. \tag{6.7}$$

At a beach, we have $B(X) \sim -X$ as $X \to 0$, which gives $A_0 \sim K(-X)^{\beta}$, $\beta = \frac{1}{2}\left(\left(\omega^2/\ell\right) - 1\right)$, where K is a constant (which is fixed by specifying the amplitude of the edge wave for some $X < 0$). Further, if we ask that $A_0(X)$ and all its derivatives exist as $X \to 0$, then we must choose $\beta = n$ ($n = 0, 1, 2, \ldots$), and then $\omega^2 = (1+2n)\ell$, which is the classical dispersion relation for edge waves. (All this is developed in [11, 12] – we need only the basic results here – but we do add the observation that a non-uniformity exists in our solution as $B \to 0$, unless the case $n = 0$ is omitted.) Thus we claim that, with relatively minor adjustments, the theory of edge waves over a slowly varying depth as presented in [11], carries over to the situation where a (weak) background flow is allowed to pre-exist the passage of the edge wave – although we should note that both the background flow and its perturbation are essentially the same size.

We complete this discussion by examining one intriguing aspect of edge waves: the run-up pattern on a beach. This is described by the curve

$$z = -B = \varepsilon(H + h), \tag{6.8}$$

where

$$H + h \sim H + A_0 e^{\theta} \cos\xi - \frac{1}{2}\frac{\ell^2}{\omega^2} A_0^2 e^{2\theta} \tag{6.9}$$

and then, for $X \to 0$ (which is where the beach exists), we may take $B(X) \sim -X$ and $H \sim -\frac{1}{2}\gamma^2 X^2$ (see (4.19)). Using $A_0 \sim K(-X)^{\beta}$, the run-up pattern, (6.8) with (6.9), can be written

$$x \sim -\frac{\gamma^2}{2} X^2 + K(-X)^{\beta} e^{\ell x} \cos\xi - \frac{\ell^2}{2\omega^2} K^2 (-X)^{2\beta} e^{2\ell x}, \tag{6.10}$$

which, upon the exclusion of the solution $x = X = 0$ and electing to set $\beta = n$ ($n = 1, 2, \ldots$), can be expressed in the normalised form

$$1 + \nu X - \mu(-Y)^{n-1} e^{Y} \cos\xi + \frac{\mu^2}{2(1+2n)}(-Y)^{2n-1} e^{2Y} = 0. \tag{6.11}$$

Here, we have written $Y = \ell x$, $\mu = K\varepsilon^n / \ell^{n-1}$ and $\nu = \varepsilon^2\gamma^2 / 2\ell$; both Y and μ relate precisely to the equation discussed in [11, 12], and ν is the new parameter representing the presence of any background flow in the vicinity of the beach. This version of the run-up pattern can be examined to decide whether this captures the essential features of what is observed on beaches; this is therefore no more than an extension of the approach adopted in [11]. The solutions of (6.11), with $\nu = 0$, are discussed in the papers already cited; in particular, it is shown that there exist, for certain parameter ranges (n and μ), two possible run-up patterns. When such solutions do not exist, the pattern comprises periodic, closed regions that either do, or do not, contain water – neither of which is a possible solution. (Either there is no water extending seawards, or there is water extending to infinity inland, respectively.) An acceptable pair of solutions is shown in figure 5 – and there appears to be no mechanism for deciding which may be an appropriate solution for water waves. (Indeed, there is some evidence to suggest that either can appear on a beach, under suitable conditions.) The inclusion of the term in ν changes all this; for a given n and μ, for which two solutions exist, there is a critical value of ν above which only one appropriate solution exists; below this value, the familiar two appear (corresponding to the pair associated with $\nu = 0$). An example of this phenomenon is shown in figure 6.

7. Conclusions

The classical problem of water waves has been presented via two different sets of scalings on the standard governing equations and boundary conditions. These have been selected to enable us to describe both the familiar problem of gravity waves and that of edge-wave propagation. In each case, we have shown how these problems can be formulated to allow the waves to move over a pre-existing flow field that both possesses non-zero vorticity and admits a shoreline. Some examples of such background flows have been presented. Furthermore, each problem for the motion of the waves is, even at leading order, appropriately nonlinear. In the first case, the scalings have been chosen to recover the relevant KdV-type equation for the gravity wave that is approaching a beach; thus the wave exhibits, at a reasonable distance from the shore, nonlinearity and dispersion, adapted to accommodate the (slow) variation in depth and background flow. The resulting asymptotic solution has been shown – not surprisingly – to break down (the expansion is not uniformly valid, and this is the case for any non-zero vorticity) as the beach is approached. The problem has been suitably rescaled in the neighbourhood of the beach, resulting in the wave now being dominated, at leading order, by the nonlinear effects. The solution in this region matches to an appropriate solution of the KdV-type equation; in general, the solution will now take the form of a 'breaking' wave close inshore, with an amplitude that grows according to Green's law.

The corresponding problem for edge waves requires a slightly more careful formulation, although the underlying principles are the same. In this case, the background flow had to be the same size – in the parametric sense – as the edge wave. However, once this selection is incorporated, it was demonstrated that the development of the solution-technique followed that of [11] (which presented the problem of the edge wave in the presence of a slowly-varying depth, but with a zero background state). The leading-order description of the edge wave mirrors very closely that already given in [11]: there is an appropriate exact solution that recovers all the essential features of the edge wave, but this is now combined with a contribution from the background flow. This new ingredient enables a significant re-interpretation of the edge-wave solution previously obtained, in particular as it relates to the run-up pattern, as we shall comment below.

The main results from our analysis can be summarised as follows. For the gravity wave, we have derived a variable-coefficient KdV equation

$$2D^{1/4}\left(D^{1/4}h_0\right)_X + \frac{3}{D}h_0 h_{0\xi} + \frac{D}{3}h_{0\xi\xi\xi} = 0, \tag{7.1}$$

and one avenue of investigation, perhaps worthy of some effort, is to obtain (numerical) solutions for $h_0(\xi, X)$. This would enable the effects of various background flows, and choices of variable depth, to be itemised and studied; we comment that the process of increasing amplitude as the depth decreases, close to the shoreline, has already been discussed in our asymptotic solution. Although this falls a little short of a fundamentally new result – something close to this is given in [10] – equation (7.1) does encapsulate a number of important properties: nonlinear, dispersive (soliton-type) wave propagation with variable depth in a flow with vorticity. Nevertheless, our work has now shown that we can allow the background flow-field to represent depth profiles, and flows, relevant to beaches – a possibility not envisaged before.

On the other hand, the equation

$$1+\nu X-\mu(-Y)^{n-1}\mathrm{e}^{Y}\cos\xi+\frac{\mu^{2}}{2(1+2n)}(-Y)^{2n-1}\mathrm{e}^{2Y}=0, \qquad (7.2)$$

for the run-up pattern produced by an edge wave (and, by implication, other properties of an edge wave) is new. This work has built on the equation previously obtained in the case $\nu=0$ (which was first reported in [11]) and, for $\nu\neq 0$, seems to go some way towards addressing a problem encountered in this earlier work. Although it was argued (in [11]) that solutions corresponding to the observed run-up patterns are recovered for $\nu=0$, such solutions always come in pairs; there is no immediate and obvious mechanism for selecting one rather than the other. However, the inclusion of a background flow (and near a shoreline, this is generic) offers a way forward: for a given n and μ, there is a critical value of ν above which only one relevant solution exists. Further, if this solution is identified, and traced back as ν decreases, then one of the pair is selected. Although, in the work reported here, we have only begun the investigation of the problem with a background flow and of the solutions of equation (7.2), this would seem to be an area that is worthy of further investigation.

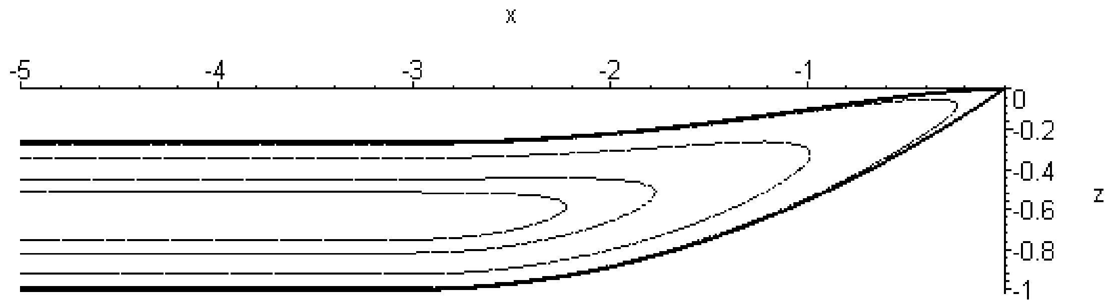

Figure 1: An example of a flow field with constant vorticity, showing the surface streamline and bottom topography (heavy lines), and some internal streamlines. The bottom profile is $B(X)=1$ in $X<-3$ and $B(X)=1-(X+3)^2/9$ in $-3\le X\le 0$.

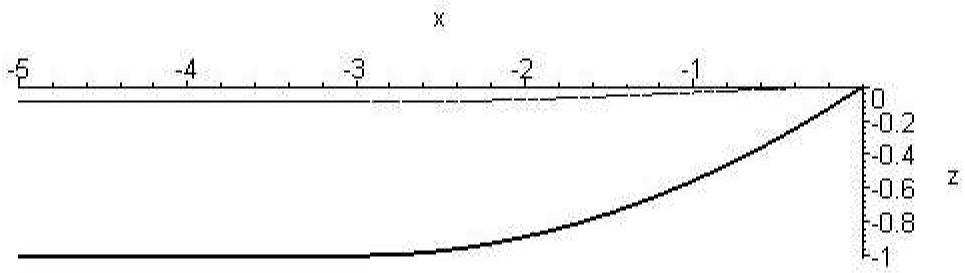

Figure 2a

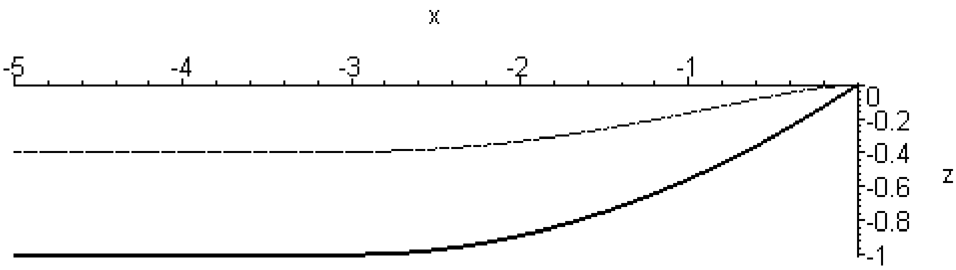

Figure 2b

Figure 2: Two examples of the free surface with variable vorticity, as described by equations (3.7)-(3.9) with: (a) $\alpha = \ell = 1$; (b) $\alpha = 3$, $\ell = 1$. The bottom profile is the same as that used for figure 1.

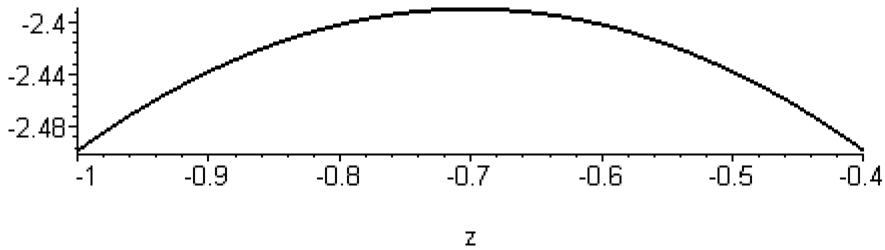

Figure 3: An example of the variable vorticity, given in (3.9), for $\alpha = 3$, $\ell = 1$, in the region of constant depth ($B(X) = 1$; see figure 2b).

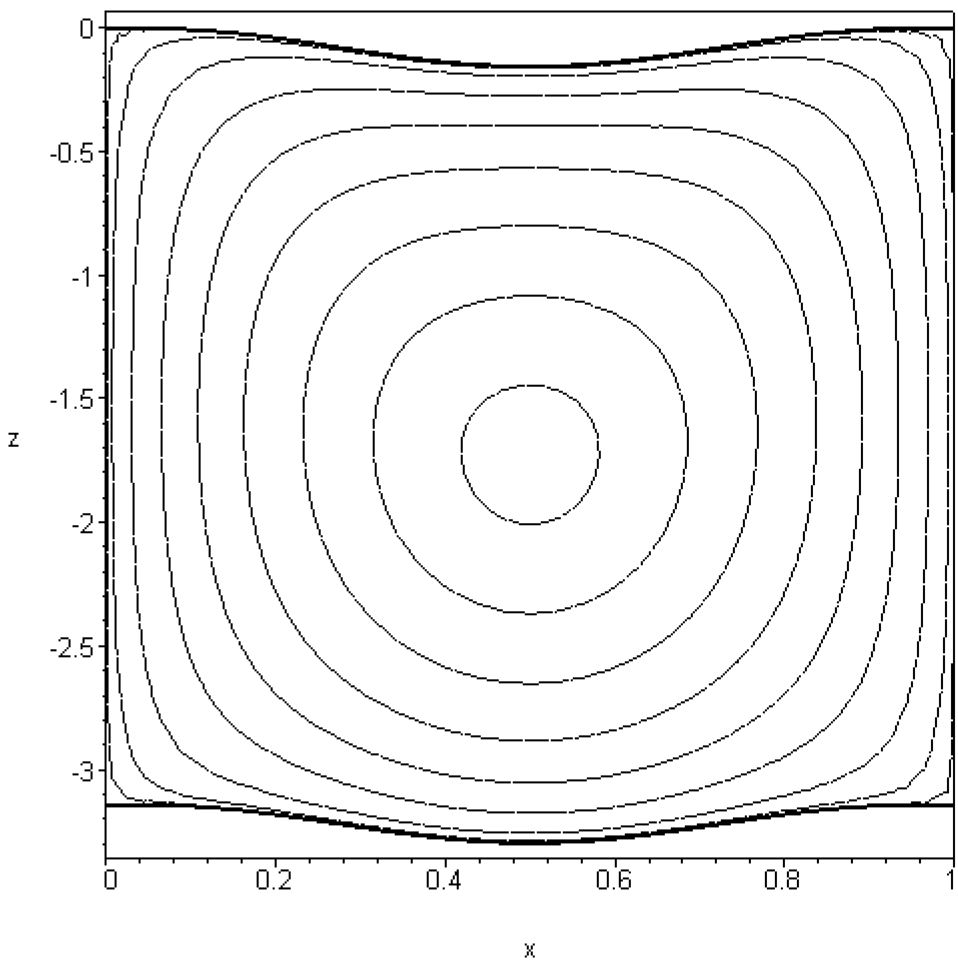

Figure 4: An example of an isolated flow field with vorticity, described by equation (3.10); the bottom profile is proportional to $X^3(1-X)^3$, for $0 \le X \le 1$. The heavy lines at the top and the bottom are the free surface and the bottom profile, respectively.

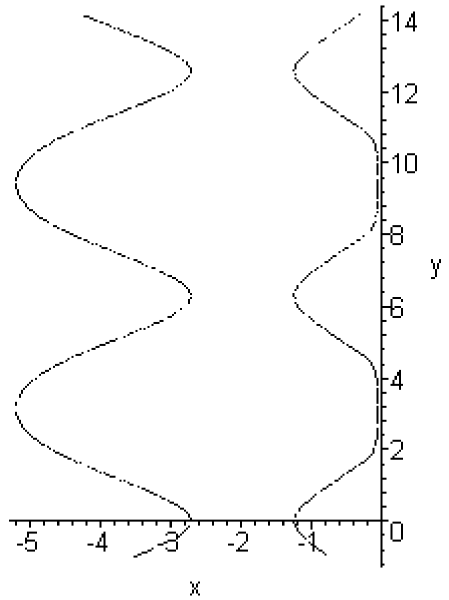

Figure 5: Solution of equation (6.11), for the run-up pattern, displaying the two viable solutions in the case $n = 2$, $\mu = 25$ (and $\nu = 0$).

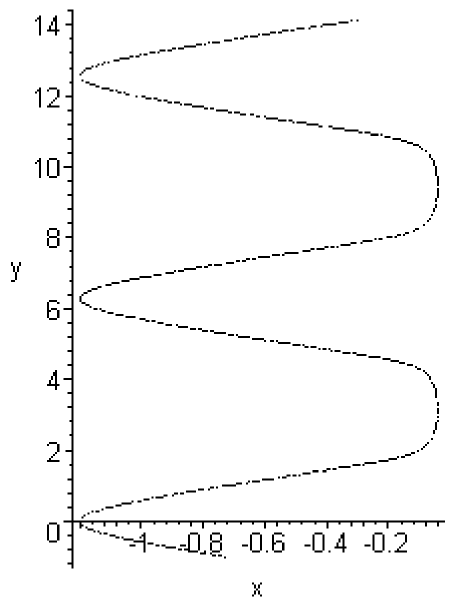

Figure 6: Solution of equation (6.11), for the run-up pattern in the presence of a background flow, in the case $n = 2$, $\mu = 25$, $\nu = 0 \cdot 2$; for these values of n and μ, the critical value of ν is approximately $0 \cdot 11855$.

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