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Blow-Up Phenomena and Decay for the Periodic Degasperis-Procesi Equation with Weak Dissipation

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Abstract

In the paper, several problems on the periodic Degasperis-Procesi equation with weak dissipation are investigated. At first, the local well-posedness of the equation is established by Kato's theorem and a precise blow-up scenario of the solutions is given. Then, several criteria guaranteeing the blow-up of the solutions are presented. Moreover, the blow-up rate and blow-up set of the blowing-up solutions are discussed. Furthermore, it is proved that the equation has global solutions and these global solutions decay to zero as time goes to infinite provided the potentials associated to their initial date are of one sign.

1 Introduction

Recently, Degasperis and Procesi [23] studied the following family of third order dispersive conservation laws,

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x, \tag{1.1}$$

where α, c_0, c_1, c_2, and c_3 are real constants. In [23] the authors found that there are only three equations that satisfy the asymptotic integrability condition within this family: the KdV equation, the Camassa-Holm equation and the Degasperis-Procesi equation.

If $\alpha = c_2 = c_3 = 0$, then Eq.(1.1) becomes the well-known KdV equation, describing the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity. In this model $u(t, x)$ represents the wave's height above a flat bottom, x is proportional to distance in the direction of propagation and t is proportional to the elapsed time. The KdV equation is completely integrable and its solitary waves are solitons [24]. The Cauchy problem of the KdV equation has been studied widely. It is known that the KdV equation is globally well-posed for $u_0 \in L^2(\mathbb{S})$, see [35, 47]. It is also observed that the KdV equation does not accomodate wave breaking (by wave breaking we mean the phenomenon that a wave remains bounded but its slope becomes unbounded in finite time, see [49]).

For $c_1 = -\frac{3}{2}c_3/\alpha^2$ and $c_2 = c_3/2$, Eq.(1.1) becomes the Camassa-Holm equation, modeling the unidirectional propagation of shallow water waves over a flat bottom. $u(t, x)$ stands for the fluid velocity at time t in the spatial x direction and c_0 is a nonnegative parameter related to the critical shallow water speed [3, 25, 34]. The Camassa-Holm equation is also a model for the propagation of axially symmetric waves in hyperelastic rods [20, 21]. It has a bi-Hamiltonian structure [30, 37] and is completely integrable [3, 9]. In this situation, the solitary waves are smooth if $c_0 > 0$ and peaked in the limiting case $c_0 = 0$, see [4]. The orbital stability of the peaked solitons is proved in [19]. The explicit interaction of the peaked solitons is given in [1]. Notice that the presence of a peak at the wave crest is analogous to the case of exact traveling wave solutions of the governing equations for water waves representing waves of greatest height – see the discussion in [11, 15].

The Cauchy problem of the Camassa-Holm equation has been studied extensively. It has been shown that this equation is locally well-posed [7, 13, 46] for initial data $u_0 \in H^s(\mathbb{S})$ with $s > \frac{3}{2}$. More interestingly, it has global strong solutions [8, 13, 17] and also blow-up solutions in finite time [7, 12, 13, 14, 17]. On the other hand, it has global weak solutions with initial data $u_0 \in H^1$, see [2, 18, 52]. The advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solitons and models wave breaking [4].

If $c_1 = -2c_3/\alpha^2$ and $c_2 = c_3$ in Eq.(1.1), then, after rescaling, shifting the dependent variable, and applying a Galilean boost [22], we find the Degasperis-Procesi equation of the form

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0,\ x \in \mathbb{R}. \tag{1.2}$$

Degasperis et al. [22] proved the formal integrability of Eq.(1.2) by constructing a Lax pair. They also showed that the equation has a bi-Hamiltonian structure and admits exact peakon solutions which are analogous to the Camassa-Holm peakons.

Dullin et al. [26] showed that the Degasperis-Procesi equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the Camassa-Holm shallow water equation. Lundmark and Szmigielski [41] presented an inverse scattering approach for computing n-peakon solutions to the Degasperis-Procesi equation. Whereas Vakhnenko and Parkes [48] investigated traveling wave solutions of the equation and Holm and Staley [33] studied stability of solitons and peakons numerically.

After the Degasperis-Procesi equation (1.2) was derived, many papers were devoted to its study, cf. [16, 32, 36, 38, 40, 42, 53, 54, 55, 56] and the citations therein. For example, Yin proved local well-posedness to Eq.(1.2) with initial data $u_0 \in H^s$, $s > \frac{3}{2}$ on the line [53] and on the circle [54]. In these two papers the precise blow-up scenario and a blow-up result were derived. The global existence of strong solutions and global weak solutions to Eq.(1.2) are also investigated in [55, 56]. Recently, Lenells [36] classified all weak traveling wave solutions. Matsuno [42] studied multisoliton solutions and their peakon limits. Analogous to the case of the Camassa-Holm equation [10], Henry [32] and Mustafa [44] showed that smooth solutions to Eq.(1.2) have infinite speed of propagation. Coclite and Karlsen [5] also obtained global existence results for entropy weak solutions in the class of $L^1(\mathbb{R}) \cap BV(\mathbb{R})$ and the class of $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$.

Despite the similarities to the Camassa-Holm equation, we would like to point out that these two equations are truly different. One of the important features of the Degasperis-Procesi equation is that it has not only peakon solitons [22] and periodic peakons [54], but also shock peakons [6, 40] and periodic shock waves [28]. On the other hand, the isospectral problem for the Degasperis-Procesi equation has the third-order equation in the Lax pair [22], while the isospectral problem for the Camassa-Holm equation is the second order equation [3]. Another indication of the fact that there is no simple transformation of the Degasperis-Procesi equation into the Camassa-Holm equation is the entirely different form of conservation laws for that two equations [3, 22]. Furthermore, the Camassa-Holm equation is a re-expression of geodesic flow on the diffeomorphism group [16] or on the Bott-Virasoro group [43]. Up to now, no geometric derivation of the Degasperis-Procesi equation is available.

Recently, several new global existence and blow-up results for strong solutions to the Degasperis-Procesi equation were presented in [38, 39]. It is proved that the first blow-up must occur as wave breaking and shock waves possibly appear afterwards [38]. Global weak solution and blow-up structure for this equation were investigated in [27, 28]. Initial boundary value problems for the Degasperis-Procesi equation were also discussed in [29].

In general, it is very difficult to avoid energy dissipation mechanisms in a real world. Ott and Sudan [45] ever investigated how KdV equation was modified by the presence of dissipation and the effect of such dissipation on the solitary solution of KdV equation. Ghidaglia [31] studied the long time behavior of solutions to the weakly dissipative KdV equation as a finite dimensional dynamical system. Recently, Wu and Yin discussed the blow-up, blow-up rate and decay of the solution of the weakly dissipative periodic Camassa-Holm equation [50], and the blow-up and decay of the solution of the weakly dissipative Degasperis-Procesi equation on the line [51].

In this paper, we would like to consider the dissipative periodic Degasperis-Procesi equation:

$$\begin{cases} u_t - u_{txx} + 4uu_x + L(u) = 3u_x u_{xx} + uu_{xxx}, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0,\ x \in \mathbb{R}, \end{cases}$$

where $L(u)$ is a dissipative term, L can be a differential operator or a quasi-differential operator according to different physical situations. We are interested in the effect of the weakly dissipative term on the periodic Degasperis-Procesi equation. In particular, we study the following periodic Degasperis-Procesi equation with weak dissipation:

$$\begin{cases} u_t - u_{txx} + 4uu_x + \lambda(u - u_{xx}) = 3u_x u_{xx} + uu_{xxx}, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geq 0,\ x \in \mathbb{R}, \end{cases} \tag{1.3}$$

where $L(u) = \lambda(1 - \partial_x^2)u$ is the weakly dissipative term, $\lambda > 0$ is a constant.

We find that the behaviors of Eq.(1.3) are similar to the periodic Degasperis-Procesi equation [28] in a finite interval of time, such as, the local well-posedness and the blow-up phenomena. But there are considerable differences between Eq.(1.3) and the periodic Degasperis-Procesi equation in their long time behaviors. Global solution of Eq.(1.3) decays to zero as time goes to infinite provided the potential $y_0 = (1 - \partial_x^2)u_0$ is of one sign. This long time behavior is an important feature that the periodic Degasperis-Procesi equation does not possess. It is well-known that the periodic Degasperis-Procesi equation has periodic traveling wave peakons. Theorem 5.1 in the sequel shows that any global solution decays in the H^3-norm. This means that there are no periodic traveling wave solutions of Eq.(1.3). This is also another considerable difference between Eq.(1.3) and the periodic Degasperis-Procesi equation in their long time behaviors.

It is very interesting that Eq.(1.3) has the same blow-up rate and blow-up set as the periodic Degasperis-Procesi equation [28] does when the blow-up occurs. This fact shows that the blow-up rate and blow-up set of the Degasperis-Procesi equation are not affected by the weakly dissipative term. But the occurrence of blow-up of Eq.(1.3) is affected by the dissipative parameter.

It should be noticed that the weakly dissipative term breaks the conservation laws of the periodic Degasperis-Procesi equation [28, 54, 55]:

$$E_1(u) = \int_{\mathbb{S}} y dx, \quad E_2(u) = \int_{\mathbb{S}} yv dx, \quad E_3(u) = \int_{\mathbb{S}} u^3 dx,$$

where $y = (1 - \partial_x^2)u$ and $v = (4 - \partial_x^2)^{-1}u$, which play an important role in the study of the periodic Degasperis-Procesi equation.

Notation. In the following, we denote by $*$ the spatial convolution on $\mathbb{R}$. $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ is the circle of unit length. We write $(\hat{f}_n)$ for the Fourier series of $f \in L^2(\mathbb{S})$. We also use $(\cdot\, , \cdot)$ to represent the standard inner product in $L^2(\mathbb{S})$. For $1 \le p \le \infty$, the norm in the Lebesgue space L^p will be denoted by $\|\cdot\|_{L^p}$, while $\|\cdot\|_s$ will stand for the norm in the classical Sobolev spaces $H^s(\mathbb{S})$ for $s \ge 0$.

2 Local well-posedness and blow-up scenario

In this section, we prove the local well-posedness of the Cauchy problem of Eq.(1.3) by Kato's theorem and give a precise blow-up scenario of strong solutions to Eq.(1.3).

With $y := u - u_{xx}$, Eq.(1.3) takes the form:

$$\begin{cases} y_t + uy_x + 3u_x y + \lambda y = 0, & t > 0,\ x \in \mathbb{R}, \\ y(0,x) = u_0(x) - u_{0,xx}(x), & x \in \mathbb{R}, \\ y(t,x) = y(t,x+1), & t \ge 0,\ x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Note that if $G(x) := \frac{\cosh(x-[x]-\frac{1}{2})}{2\sinh(\frac{1}{2})}$, where $[x]$ stands for the integer part of $x \in \mathbb{R}$, then $(1-\partial_x^2)^{-1}f = G*f$ for all $f \in L^2(\mathbb{S})$ so that $G*y = u$. Using this identity, we can rewrite Eq.(2.1) as a quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} u_t + uu_x + \partial_x G * (\frac{3}{2}u^2) + \lambda u = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x) = u(t,x+1), & t \ge 0,\ x \in \mathbb{R}, \end{cases} \tag{2.2}$$

The local well-posedness of the Cauchy problem of Eq.(2.2) with initial data $u_0 \in H^s(\mathbb{S}),\ s > \frac{3}{2}$ can be obtained by applying the Kato's theorem. More precisely, we have the following local well-posedness result.

Theorem 2.1. Given $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, there exist a maximal $T = T(u_0) > 0$ and a unique strong solution u to Eq.(2.2), such that

$$u = u(\cdot, u_0) \in C([0,T); H^s(\mathbb{S})) \cap C^1([0,T); H^{s-1}(\mathbb{S})).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \mapsto u(\cdot, u_0) : H^s(\mathbb{S}) \to C([0,T); H^s(\mathbb{S})) \cap C^1([0,T); H^{s-1}(\mathbb{S}))$ is continuous and the maximal time of existence $T > 0$ is independent of s.

Proof. Set $A(u) = u\partial_x$, $f(u) = -\partial_x(1-\partial_x^2)^{-1}(\frac{3}{2}u^2) - \lambda u$, $Y = H^s(\mathbb{S})$, $X = H^{s-1}(\mathbb{S})$, $s > \frac{3}{2}$, and $Q = (1-\partial_x^2)^{\frac{1}{2}}$. Analogous to the proofs of Theorem 2.1 in [54], we can prove Theorem 2.1 by applying the Kato's theorem. ■

By the above local well-posedness result and energy estimates, one can prove the following precise blow-up scenario.

Theorem 2.2. Given $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, blow up of the strong solution $u = u(\cdot, u_0)$ in finite time T $< +\infty$ occurs if and only if

$$\lim_{t \to T} \inf\{\inf_{x \in \mathbb{S}} u_x(t,x)\} = -\infty.$$

Proof. By Theorem 2.1 and a simple density argument, we only need to show that the above theorem holds for $s = 3$. Let $T > 0$ be the maximal time of existence of the solution u to Eq.(2.1) (or Eq.(2.2)) with initial data $u_0 \in H^3(\mathbb{S})$. We have by Eq.(2.1)

$$\frac{d}{dt}\int_{\mathbb{S}} y^2 dx = 2\int_{\mathbb{S}} y y_t dx = -5\int_{\mathbb{S}} u_x y^2 dx - 2\lambda \int_{\mathbb{S}} y^2 dx. \tag{2.3}$$

If $u_0 \in H^4(\mathbb{S})$, we can obtain by Eq.(2.1)

$$\frac{d}{dt}\int_{\mathbb{S}} y_x^2 dx = 2\int_{\mathbb{S}} y_x y_{xt} dx = -7\int_{\mathbb{S}} u_x y_x^2 dx + 3\int_{\mathbb{S}} u_x y^2 dx - 2\lambda \int_{\mathbb{S}} y_x^2 dx. \tag{2.4}$$

As for $u_0 \in H^3(\mathbb{S})$, we will show that (2.4) still holds. In fact, we can approximate u_0 in $H^3(\mathbb{S})$ by function $u_0^n \in H^4(\mathbb{S})$. Moreover, we write $u^n = u^n(\cdot, u_0^n)$ for the solution of Eq.(2.1) with initial data u_0^n. By Lemma 2.1, we know that

$$u^n \in C([0,T_n); H^4(\mathbb{S})) \cap C^1([0,T_n); H^3(\mathbb{S})), \quad n \geq 1,$$

$$y^n = u^n - u^n_{xx} \in C([0,T_n); H^2(\mathbb{S})) \cap C^1([0,T_n); H^1(\mathbb{S})), \quad n \geq 1,$$

$u^n \to u$ in $H^3(\mathbb{S})$ and $T_n \to T$ as $n \to \infty$. Due to $u_0^n \in H^4(\mathbb{S})$, we have by (2.4)

$$\frac{d}{dt}\int_{\mathbb{S}} (y_x^n)^2 dx = -7\int_{\mathbb{S}} u_x^n (y_x^n)^2 dx + 3\int_{\mathbb{S}} u_x^n (y^n)^2 dx - 2\lambda \int_{\mathbb{S}} (y_x^n)^2 dx.$$

Since $u^n \to u$ in $H^3(\mathbb{S})$ as $n \to \infty$, it follows that $u_x^n \to u_x$ in $L^\infty(\mathbb{S})$ as $n \to \infty$. Note also that $y^n \to y$ in $H^1(\mathbb{S})$ and $y_x^n \to y_x$ in $L^2(\mathbb{S})$ as $n \to \infty$. Letting n goes to infinity in the above equation, we can easily deduce that (2.4) holds for $u_0 \in H^3(\mathbb{S})$.

Adding (2.3) and (2.4), we get

$$\begin{aligned}\frac{d}{dt}\left(\int_{\mathbb{S}} y^2dx+\int_{\mathbb{S}} y_x^2dx\right)=&-7\int_{\mathbb{S}} u_xy_x^2dx-2\int_{\mathbb{S}} u_xy^2dx\\ &-2\lambda\left(\int_{\mathbb{S}} y^2dx+\int_{\mathbb{S}} y_x^2dx\right).\end{aligned} \tag{2.5}$$

If u_x is bounded from below on $[0,T)$, for example $u_x\geq -c$, c is a positive constant, then we get by (2.5) and Gronwall's inequality

$$\|y\|_1^2\leq \exp\{(7c-2\lambda)t\}\|y(0)\|_1^2.$$

This implies that the H^3-norm of the solution u of Eq.(2.1) does not blow up in finite time. ■

Consider now the following ordinary differential equation

$$\begin{cases} q_t=u(t,q), & t\in[0,T),\\ q(0,x)=x, & x\in\mathbb{R}.\end{cases} \tag{2.6}$$

Applying classical results in the theory of ordinary differential equations, one can obtain the following useful result which will be used in the sequel.

Lemma 2.1. [51] Let $u_0\in H^s(\mathbb{R}),\ s\geq 3$, and let $T>0$ be the maximal existence time of the corresponding strong solution u to Eq.(2.2). Then the Eq.(2.6) has a unique solution $q\in C^1([0,T)\times\mathbb{R},\mathbb{R})$. Moreover, the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$q_x(t,x)=\exp\left(\int_0^t u_x(s,q(s,x))ds\right)>0,\ \ \forall (t,x)\in[0,T)\times\mathbb{R}.$$

Furthermore, setting $y:=u-u_{xx}$, we have

$$y(t,q(t,x))q_x^3(t,x)=y_0(x)\exp(-\lambda t),\quad \forall (t,x)\in[0,T)\times\mathbb{R}.$$

The following lemma will be used in the proof of the blow-up rate of blowing-up solutions.

Lemma 2.2. [12] Let $T>0$ and $v\in C^1([0,T);H^2(\mathbb{S}))$. Then for every $t\in[0,T)$, there exists at least one point $\xi(t)\in\mathbb{S}$ with

$$m(t):=\inf_{x\in\mathbb{S}}[v_x(t,x)]=v_x(t,\xi(t)).$$

The function m(t) is absolutely continuous on $(0,T)$ with

$$\frac{dm}{dt}=v_{tx}(t,\xi(t))\qquad a.e.\quad on\ (0,T).$$

3 Blow-up

In this section, we shall derive a priori estimate for the $L^\infty-$norm of the strong solutions. This enables us to establish several blow-up results for Eq.(2.2) with certain initial profiles.

As above and henceforth, the results are only proved with regard to $s = 3$, since we can obtain the same conclusion for general case $s > \frac{3}{2}$ by using Theorem 2.1 and a simple density argument.

Lemma 3.1. If $u_0 \in H^s(\mathbb{S}),\ s > \frac{3}{2}$, then as long as the solution $u(t,x)$ given by Theorem 2.1 exists, we have

$$\int_{\mathbb{S}} y(t,x)v(t,x)dx = \exp(-2\lambda t)\int_{\mathbb{S}} y_0(x)v_0(x)dx,$$

where $y(t,x) = u(t,x) - u_{xx}(t,x)$ and $v(t,x) = (4-\partial_x^2)^{-1}u$. Moreover, we have

$$\frac{1}{4}\exp(-2\lambda t)\|u_0\|_{L^2}^2 \le \|u(t)\|_{L^2}^2 \le 4\exp(-2\lambda t)\|u_0\|_{L^2}^2.$$

Proof. Let $T > 0$ be the maximal time of existence of the solution u to Eq.(2.1) (or Eq.(2.2)) with initial data $u_0 \in H^3(\mathbb{S})$ such that

$$u \in C([0,T); H^3(\mathbb{S})) \cap C^1([0,T); H^2(\mathbb{S})),$$

which is guaranteed by Theorem 2.1. By Eq.(2.1), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} yv\,dx &= \frac{1}{2}\int_{\mathbb{S}} y_t v\,dx + \frac{1}{2}\int_{\mathbb{S}} yv_t dx = \int_{\mathbb{S}} y_t\,v\,dx \\ &= -\int_{\mathbb{S}} v(yu)_x\,dx - 2\int_{\mathbb{S}} vyu_x dx - \lambda\int_{\mathbb{S}} yv\,dx.\end{aligned}$$

Using the relations $y = u - u_{xx}$ and $4v - v_{xx} = u$, it yields that

$$\begin{aligned}\int_{\mathbb{S}} v(yu)_x\,dx &= -\int_{\mathbb{S}} v_x yu\,dx = -\int_{\mathbb{S}} v_x u^2\,dx + \int_{\mathbb{S}} v_x u u_{xx} dx \\ &= \int_{\mathbb{S}} v_x u^2\,dx - \int_{\mathbb{S}} v_x u_x^2 dx.\end{aligned}$$

On the other hand,

$$2\int_{\mathbb{S}} v\,y\,u_x\,dx = -\int_{\mathbb{S}} v_x\,u^2\,dx + \int_{\mathbb{S}} v_x u_x^2\,dx.$$

Combining the above three relations, we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} yv\,dx = -\lambda\int_{\mathbb{S}} yvdx.$$

Consequently, this implies the first desired result.

In view of the proved equality, Parseval's equality yields

$$\begin{aligned}\|u(t)\|_{L^2}^2 &= \sum_{n=-\infty}^{\infty} |\hat{u}_n|^2 \le 4 \sum_{n=-\infty}^{\infty} \frac{1+4\pi^2 n^2}{4+4\pi^2 n^2}|\hat{u}_n|^2\\ &= 4(y(t)\,,\, v(t)) = 4\exp(-2\lambda t)(y_0\,,\, v_0)\\ &= 4\exp(-2\lambda t)\sum_{n=-\infty}^{\infty} \frac{1+4\pi^2 n^2}{4+4\pi^2 n^2}|(\widehat{u_0})_n|^2\\ &\le 4\exp(-2\lambda t)\sum_{n=-\infty}^{\infty} |(\widehat{u_0})_n|^2 = 4\exp(-2\lambda t)\|u_0\|_{L^2}^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|u(t)\|_{L^2}^2 &= \sum_{n=-\infty}^{\infty} |\hat{u}_n|^2 \ge \sum_{n=-\infty}^{\infty} \frac{1+4\pi^2 n^2}{4+4\pi^2 n^2}|\hat{u}_n|^2\\ &= (y(t),\ v(t)) = \exp(-2\lambda t)(y_0,\ v_0)\\ &= \exp(-2\lambda t)\sum_{n=-\infty}^{\infty} \frac{1+4\pi^2 n^2}{4+4\pi^2 n^2}|(\widehat{u_0})_n|^2\\ &\ge \frac{1}{4}\exp(-2\lambda t)\sum_{n=-\infty}^{\infty} |(\widehat{u_0})_n|^2 = \frac{1}{4}\exp(-2\lambda t)\|u_0\|_{L^2}^2.\end{aligned}$$

This completes the proof of Lemma 3.1. ■

In order to have a neat notation, let

$$k = \coth\left(\frac{1}{2}\right) = \frac{\cosh(\frac{1}{2})}{\sinh(\frac{1}{2})}.$$

The following important estimate can be obtained by Lemma 3.1.

Lemma 3.2. Assume that $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$. Let T be the maximal existence time of the solution u to the Eq.(2.2) guaranteed by Theorem 2.1. Then we have

$$\|u(t)\|_{L^\infty} \le \exp(-\lambda t)\left(\frac{3k}{\lambda}\|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty}\right), \quad \forall t \in [0, T].$$

Proof. Let $T > 0$ be the maximal time of existence of the solution u to Eq.(2.2) with the initial data $u_0 \in H^3(\mathbb{S})$ such that $u \in C([0,T); H^3(\mathbb{S})) \cap C^1([0,T); H^2(\mathbb{S}))$, which is guaranteed by Theorem 2.1. By (2.2) we have

$$u_t + uu_x = -3G * (uu_x) - \lambda u. \tag{3.1}$$

Note that

$$\begin{aligned}
-3G*(uu_x) &= -\frac{3}{2\sinh(\frac{1}{2})}\int_0^1 \cosh(x-\xi-[x-\xi]-\frac{1}{2})uu_\xi d\xi \\
&= -\frac{3e^{x-\frac{1}{2}}}{4\sinh(\frac{1}{2})}\int_0^x e^{-\xi}uu_\xi d\xi - \frac{3e^{-x+\frac{1}{2}}}{4\sinh(\frac{1}{2})}\int_0^x e^{\xi}uu_\xi d\xi \\
&\quad - \frac{3e^{x+\frac{1}{2}}}{4\sinh(\frac{1}{2})}\int_x^1 e^{-\xi}uu_\xi d\xi - \frac{3e^{-x-\frac{1}{2}}}{4\sinh(\frac{1}{2})}\int_x^1 e^{\xi}uu_\xi d\xi \qquad (3.2)\\
&= -\frac{3e^{x-\frac{1}{2}}}{8\sinh(\frac{1}{2})}\int_0^x e^{-\xi}u^2 d\xi + \frac{3e^{-x+\frac{1}{2}}}{8\sinh(\frac{1}{2})}\int_0^x e^{\xi}u^2 d\xi \\
&\quad - \frac{3e^{x+\frac{1}{2}}}{8\sinh(\frac{1}{2})}\int_x^1 e^{-\xi}u^2 d\xi + \frac{3e^{-x-\frac{1}{2}}}{8\sinh(\frac{1}{2})}\int_x^1 e^{\xi}u^2 d\xi.
\end{aligned}$$

In view of (2.6), we have

$$\begin{aligned}
\frac{du(t,q(t,x))}{dt} &= u_t(t,q(t,x)) + u_x(t,q(t,x))\frac{dq(t,x)}{dt} \\
&= (u_t + uu_x)(t,q(t,x)).
\end{aligned}$$

It then follows from (3.1) and (3.2) that

$$\begin{aligned}
-&\frac{3e^{q(t,x)-\frac{1}{2}}}{8\sinh(\frac{1}{2})}\int_0^{q(t,x)} e^{-\xi}u^2 d\xi - \frac{3e^{q(t,x)+\frac{1}{2}}}{8\sinh(\frac{1}{2})}\int_{q(t,x)}^1 e^{-\xi}u^2 d\xi \\
&\le \frac{du(t,q(t,x))}{dt} + \lambda u(t,q(t,x)) \\
&\le \frac{3e^{-q(t,x)+\frac{1}{2}}}{8\sinh(\frac{1}{2})}\int_0^{q(t,x)} e^{\xi}u^2 d\xi + \frac{3e^{-q(t,x)-\frac{1}{2}}}{8\sinh(\frac{1}{2})}\int_{q(t,x)}^1 e^{\xi}u^2 d\xi.
\end{aligned}$$

It thus transpires that

$$\begin{aligned}
&\left|\frac{du(t,q(t,x))}{dt} + \lambda u(t,q(t,x))\right| \\
&\le \frac{3}{4\sinh(\frac{1}{2})}\int_0^1 \cosh(q(t,x)-\xi-[q(t,x)-\xi]-\frac{1}{2})u^2 d\xi \\
&\le \frac{3\cosh(\frac{1}{2})}{4\sinh(\frac{1}{2})}\int_{\mathbb{S}} u^2(t,\xi)d\xi = \frac{3k}{4}\|u(t,x)\|_{L^2}^2.
\end{aligned}$$

In view of Lemma 3.1, we have

$$\left|\frac{du(t,q(t,x))}{dt} + \lambda u(t,q(t,x))\right| \le 3k\exp(-2\lambda t)\|u_0\|_{L^2}^2.$$

Integrating the above inequality with respect to $t < T$ on $[0,t]$ yields

$$|\exp(\lambda t)u(t,q(t,x)) - u_0(x)| \le \frac{3k}{\lambda}\|u_0\|_{L^2}^2.$$

Thus,

$$\begin{aligned}|u(t,q(t,x))| &\leq \|u(t,q(t,x))\|_{L^\infty}\\ &\leq \exp(-\lambda t)\left(\frac{3k}{\lambda}\|u_0\|_{L^2}^2+\|u_0\|_{L^\infty}\right).\end{aligned} \tag{3.3}$$

Using the Sobolev embedding (and the periodicity in the spatial variable) to ensure the uniform boundedness of $u_x(r,\eta)$ for $(r,\eta)\in[0,t]\times\mathbb{R}$ with $t\in[0,T)$, in view of Lemma 2.1, we get for every $t\in[0,T)$ a constant $C(t)>0$ such that

$$e^{-C(t)} \leq q_x(t,x) \leq e^{C(t)}, \qquad x\in\mathbb{R}.$$

We now deduce from the above equation that the function $q(t,\cdot)$ is strictly increasing on $\mathbb{R}$ with $\lim_{x\to\pm\infty} q(t,x)=\pm\infty$ as long as $t\in[0,T)$. Thus, by (3.3) we can obtain

$$\|u(t,x)\|_{L^\infty} = \|u(t,q(t,x))\|_{L^\infty} \leq \exp(-\lambda t)\left(\frac{3k}{\lambda}\|u_0\|_{L^2}^2+\|u_0\|_{L^\infty}\right).$$

This completes the proof of Lemma 3.2. ∎

We now present the first blow-up result.

Theorem 3.1. Let $u_0\in H^s(\mathbb{S}),\ s>\frac{3}{2}$. Assume that there exists $x_0\in\mathbb{S}$ such that

$$u_0'(x_0) < -\frac{1}{2}\lambda - \frac{1}{2}\sqrt{\lambda^2+6\left(\frac{3k}{\lambda}\|u_0\|_{L^2}^2+\|u_0\|_{L^\infty}\right)^2}.$$

Then the corresponding solution of Eq.(2.2) blows up in finite time.

Proof. Let $T>0$ be the existence time of the solution $u(t,\cdot)$ of Eq.(2.2) with the initial data $u_0\in H^3(\mathbb{S})$. Differentiating Eq.(2.2) with respect to x, in view of $\partial_x^2 G*f = G*f-f$, we get

$$u_{tx} = -u_x^2 - uu_{xx} + \frac{3}{2}u^2 - G*\left(\frac{3}{2}u^2\right) - \lambda u_x. \tag{3.4}$$

Note that

$$\begin{aligned}\frac{du_x(t,q(t,x))}{dt} &= u_{xt}(t,q(t,x)) + u_{xx}(t,q(t,x))\frac{dq(t,x)}{dt}\\ &= u_{tx}(t,q(t,x)) + u(t,q(t,x))u_{xx}(t,q(t,x)).\end{aligned}$$

Thus, we have

$$\begin{aligned}\frac{du_x(t,q(t,x))}{dt} =& -u_x^2(t,q(t,x)) + \frac{3}{2}u^2(t,q(t,x))\\ &- G*\left(\frac{3}{2}u^2(t,q(t,x))\right) - \lambda u_x(t,q(t,x)).\end{aligned} \tag{3.5}$$

In view of $G * \left(\frac{3}{2}u^2\right)(t, q(t,x)) \geq 0$, we infer from (3.5) and Lemma 3.2 that

$$\begin{aligned}\frac{du_x(t,q(t,x))}{dt} &\leq -u_x^2(t,q(t,x)) - \lambda u_x(t,q(t,x)) + \frac{3}{2}u^2(t,q(t,x)) \\ &\leq -u_x^2(t,q(t,x)) - \lambda u_x(t,q(t,x)) + \frac{3}{2}\left(\frac{3k}{\lambda}\|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty}\right)^2.\end{aligned}$$

Set $m(t) = u_x(t, q(t, x_0))$ and

$$\alpha^2 = \frac{3}{2}\left(\frac{3k}{\lambda}\|u_0\|_{L^2}^2 + \|u_0\|_{L^\infty}\right)^2.$$

Then, we obtain

$$\begin{aligned}\frac{dm(t)}{dt} &\leq -m^2(t) - \lambda m(t) + \alpha^2 \\ &= -\frac{1}{4}\left(2m(t) + \lambda - \sqrt{\lambda^2 + 4\alpha^2}\right)\left(2m(t) + \lambda + \sqrt{\lambda^2 + 4\alpha^2}\right).\end{aligned}$$

From the hypothesis, we have $m(0) < -\frac{1}{2}\lambda - \frac{1}{2}\sqrt{\lambda^2 + 4\alpha^2}$. Thus $\frac{dm}{dt}|_{t=0} < 0$. By continuity with respect to t of $m(t)$, we have $\frac{dm}{dt} < 0$, $\forall t \in [0, T)$. Therefore, $m(t) < -\frac{1}{2}\lambda - \frac{1}{2}\sqrt{\lambda^2 + 4\alpha^2}$, $\forall t \in [0, T)$. Solving the above inequality yields

$$\frac{2m(0) + \lambda + \sqrt{\lambda^2 + 4\alpha^2}}{2m(0) + \lambda - \sqrt{\lambda^2 + 4\alpha^2}} \exp\left(\sqrt{\lambda^2 + 4\alpha^2}\, t\right) - 1 \leq \frac{2\sqrt{\lambda^2 + 4\alpha^2}}{2m(t) + \lambda - \sqrt{\lambda^2 + 4\alpha^2}} \leq 0.$$

Since

$$0 < \frac{2m(0) + \lambda + \sqrt{\lambda^2 + 4\alpha^2}}{2m(0) + \lambda - \sqrt{\lambda^2 + 4\alpha^2}} < 1,$$

there exists T,

$$T \leq \frac{1}{\sqrt{\lambda^2 + 4\alpha^2}} \ln\left(\frac{2m(0) + \lambda - \sqrt{\lambda^2 + 4\alpha^2}}{2m(0) + \lambda + \sqrt{\lambda^2 + 4\alpha^2}}\right),$$

such that $\lim_{t \uparrow T} m(t) = -\infty$. Hence, the theorem is proved according to Theorem 2.2. ■

We now present the second blow-up result.

Theorem 3.2. Let $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, $u_0 \not\equiv 0$. Assume that the corresponding solution $u(t,x)$ of Eq. (2.2) has a zero for any time $t \geq 0$, and that

$$0 < \lambda \leq \left(\frac{7}{24}\right)^{\frac{3}{4}}\left(\frac{9(1-\varepsilon)}{16\sinh\left(\frac{1}{2}\right)}\right)^{\frac{1}{4}}\|u_0\|_{L^2}, \quad 0 < \varepsilon < 1.$$

Then the corresponding solution of Eq.(2.2) blows up in finite time.

Proof. Let $T > 0$ be the maximal time of existence of the solution u to Eq.(2.2) with the initial data $u_0 \in H^3(\mathbb{S})$.

By assumption, for each $t \in [0, T)$ there is a $\xi_t \in [0, 1]$ such that $u(t, \xi_t) = 0$. Then for $x \in \mathbb{S}$ we have

$$u^2(t,x) = \left(\int_{\xi_t}^{x} u_x\, dx\right)^2 \leq (x - \xi_t)\int_{\xi_t}^{x} u_x^2\, dx, \quad x \in [\xi_t, \xi_t + \frac{1}{2}]. \tag{3.6}$$

Thus, the above relation and an integration by parts yield

$$\begin{aligned}\int_{\xi_t}^{\xi_t+\frac{1}{2}} u^2u_x^2\,dx &\le \int_{\xi_t}^{\xi_t+\frac{1}{2}} (x-\xi_t)u_x^2\left(\int_{\xi_t}^{x} u_x^2\right)dx\\ &\le \frac{1}{4}\left(\int_{\xi_t}^{\xi_t+\frac{1}{2}} u_x^2\,dx\right)^2.\end{aligned}$$

Combining this with a similar estimate on $[\xi_t+\frac{1}{2},\xi_t+1]$, we obtain

$$\int_{\mathbb{S}} u^2u_x^2\,dx \le \frac{1}{4}\left(\int_{\mathbb{S}} u_x^2\,dx\right)^2. \tag{3.7}$$

By (3.6), we also have

$$\sup_{x\in\mathbb{S}} u^2(t,x) \le \frac{1}{2}\int_{\mathbb{S}} u_x^2\,dx. \tag{3.8}$$

Let us assume that the solution $u(t,x)$ exists globally in time. Note that $G(x)\ge \frac{1}{2\sinh(\frac{1}{2})}$ for all $x\in\mathbb{S}$ and

$$-3\int_{\mathbb{S}} u\,u_x^2u_{xx}\,dx = \int_{\mathbb{S}} u_x^4\,dx.$$

Then, by (3.4) and (3.7)-(3.8), we get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx + 3\lambda\int_{\mathbb{S}} u_x^3\,dx\\ &= -3\int_{\mathbb{S}} u_x^4\,dx - 3\int_{\mathbb{S}} u_x^2uu_{xx}\,dx + \frac{9}{2}\int_{\mathbb{S}} u_x^2u^2\,dx - \frac{9}{2}\int_{\mathbb{S}} u_x^2G*\left(u^2\right)\,dx\\ &\le -2\int_{\mathbb{S}} u_x^4\,dx + \frac{9}{8}\left(\int_{\mathbb{S}} u_x^2\,dx\right)^2 - \frac{9}{2\sinh(\frac{1}{2})}\int_{\mathbb{S}} u_x^2dx\int_{\mathbb{S}} u^2\,dx.\end{aligned} \tag{3.9}$$

By the Cauchy-Schwartz inequality, we have

$$\frac{9}{8}\left(\int_{\mathbb{S}} u_x^2\,dx\right)^2 \le \frac{9}{8}\int_{\mathbb{S}} u_x^4\,dx \tag{3.10}$$

Furthermore, Lemma 3.1 and (3.8) imply that

$$\begin{aligned}-\int_{\mathbb{S}} u_x^2dx\int_{\mathbb{S}} u^2\,dx &\le -2\left(\int_{\mathbb{S}} u^2dx\right)^2\\ &\le -\frac{1}{8}\exp(-4\lambda t)\|u_0\|_{L^2}^4.\end{aligned} \tag{3.11}$$

Combing (3.9) with (3.10)-(3.11), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx + 3\lambda\int_{\mathbb{S}} u_x^3\,dx\\ &\le -\frac{7}{8}\int_{\mathbb{S}} u_x^4\,dx - \frac{9}{16\sinh(\frac{1}{2})}\exp(-4\lambda t)\|u_0\|_{L^2}^4, \quad t\ge 0.\end{aligned}$$

An application of Hölder's inequality yields

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx + 3\lambda\int_{\mathbb{S}} u_x^3\,dx\\&\le -\frac{7}{8}\left(\int_{\mathbb{S}} u_x^3\,dx\right)^{\frac{4}{3}} - \frac{9}{16\sinh(\frac{1}{2})}\exp(-4\lambda t)\|u_0\|_{L^2}^4, \quad t\ge 0.\end{aligned} \tag{3.12}$$

Define $V(t) := \int_{\mathbb{S}} u_x^3(t,x)\,dx$ for all $t\ge 0$. By (3.12), we have

$$\frac{d}{dt}V(t) + 3\lambda V(t) \le -\frac{9}{16\sinh(\frac{1}{2})}\exp(-4\lambda t)\|u_0\|_{L^2}^4, \quad t\ge 0.$$

Then

$$\begin{aligned}V(t) \le \exp(-3\lambda t)&\left(V(0) - \frac{9}{16\lambda\sinh(\frac{1}{2})}\|u_0\|_{L^2}^4\right)\\&- \frac{9}{16\lambda\sinh(\frac{1}{2})}\|u_0\|_{L^2}^4, \quad t\ge 0.\end{aligned}$$

Thus, we can find that there exists some $t_0\ge 0$ such that

$$V(t) \le -\frac{9(1-\varepsilon)}{16\lambda\sinh(\frac{1}{2})}\|u_0\|_{L^2}^4 < 0 \tag{3.13}$$

for all $t\ge t_0$, where $0<\varepsilon<1$. Then, again by (3.12) we obtain

$$\frac{d}{dt}V(t) + 3\lambda V(t) \le -\frac{7}{8}\left(V(t)\right)^{\frac{4}{3}}, \quad t > t_0.$$

Solving this differential inequality, we get

$$\left[\left(\frac{1}{(V(t_0))^{\frac{1}{3}}} + \frac{7}{24\lambda}\right)\exp(\lambda(t-t_0)) - \frac{7}{24\lambda}\right]^3 \le \frac{1}{V(t)} < 0, \quad t\ge t_0. \tag{3.14}$$

By assumption and (3.13), we know

$$\frac{1}{(V(t_0))^{\frac{1}{3}}} + \frac{7}{24\lambda} > 0.$$

The inequality (3.14) will lead to a contradiction as $t\ge t_0$ is big enough. This shows $T<\infty$ and completes the proof of the theorem. ■

By Theorem 3.2, we deduce the following useful corollaries.

Corollary 3.1. If $u_0\in H^3(\mathbb{S})$, $u_0\not\equiv 0$ and $\int_{\mathbb{S}} u_0\,dx = 0$ or $\int_{\mathbb{S}} y_0\,dx = 0$, and assume that

$$0<\lambda\le\left(\frac{7}{24}\right)^{\frac{3}{4}}\left(\frac{9(1-\varepsilon)}{16\sinh\left(\frac{1}{2}\right)}\right)^{\frac{1}{4}}\|u_0\|_{L^2},\ 0<\varepsilon<1,$$

then the corresponding solution $u(t,x)$ blows up in finite time.

Proof. Note the periodicity of u and G. Integrating Eq.(2.2) by parts, we get

$$\frac{d}{dt}\int_{\mathbb{S}} u(t,x)\,dx + \lambda\int_{\mathbb{S}} u(t,x)\,dx = -\int_{\mathbb{S}} uu_x\,dx - \int_{\mathbb{S}} \partial_x\left(G * \frac{3}{2}u^2\right) = 0.$$

It thus follows that

$$\int_{\mathbb{S}} u(t,x)\,dx = \exp(-\lambda t)\int_{\mathbb{S}} u_0(x)\,dx = 0.$$

The above relation shows that $u(t,x)$ has a zero for all $t \in \mathbb{S}$. Thus, Theorem 3.2 ensures that the solution $u(t,x)$ blows up in finite time.

The proof of the case $\int_{\mathbb{S}} y_0\,dx = 0$ is similar to that of the case $\int_{\mathbb{S}} u_0\,dx = 0$. ■

Corollary 3.2. If $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, $u_0 \not\equiv 0$ and $\int_{\mathbb{S}} u_0^3\,dx = 0$, and assume that

$$0 < \lambda \leq \left(\frac{7}{24}\right)^{\frac{3}{4}}\left(\frac{9(1-\varepsilon)}{16\sinh\left(\frac{1}{2}\right)}\right)^{\frac{1}{4}}\|u_0\|_{L^2},\ 0<\varepsilon<1,$$

then the corresponding solution $u(t,x)$ to Eq.(2.2) blows up in finite time.

Proof. Multiplying Eq.(2.2) by u^2 and integrating by parts, we obtain

$$\frac{1}{3}\frac{d}{dt}\int_{\mathbb{S}} u^3dx + \lambda\int_{\mathbb{S}} u^3dx = -\int_{\mathbb{S}} u^3u_xdx - \int_{\mathbb{S}} u^2\partial_x\left(G * \frac{3}{2}u^2\right)dx = 0.$$

It thus follows that

$$\int_{\mathbb{S}} u^3(t,x)\,dx = \exp(-3\lambda t)\int_{\mathbb{S}} u_0^3(x)\,dx = 0.$$

The above equality implies that $u(t,x)$ has a zero for all $t \in \mathbb{S}$. Thus, Theorem 3.2 ensures that the solution $u(t,x)$ blows up in finite time. ■

Corollary 3.3. If $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, $u_0 \not\equiv 0$ and $u_0(x)$ or $y_0(x)$ is odd, and assume that

$$0 < \lambda \leq \left(\frac{7}{24}\right)^{\frac{3}{4}}\left(\frac{9(1-\varepsilon)}{16\sinh\left(\frac{1}{2}\right)}\right)^{\frac{1}{4}}\|u_0\|_{L^2},\ 0<\varepsilon<1,$$

then the corresponding solution $u(t,x)$ to Eq.(2.2) blows up in finite time.

Proof. If $u_0(x)$ or $y_0(x)$ is odd, then the solution $u(t,x)$ is odd for all $t \geq 0$. This also shows that $u(t,x)$ has a zero for all $t \in \mathbb{S}$. Thus, Theorem 3.2 ensures that the solution $u(t,x)$ blows up in finite time. ■

Finally we give another criterion that guarantees the blow-up of the solutions of Eq.(2.2).

Theorem 3.3. Assume that $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, is odd and that $u_0'(0) < -\lambda$. Then the corresponding solution of Eq.(2.2) blows up in finite time.

Proof. The proof of the theorem is similar to that of Theorem 2.5 in [51], we omit it here. ■

4 Blow-up rate and blow-up set

In this section we will investigate the blow-up rate and the blow-up set of strong solutions to Eq.(2.2).

Theorem 4.1. Let $T < \infty$ be the blow-up time of the corresponding solution u to Eq.(2.2) with initial data $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$. Then we have

$$\lim_{t\to T}\left(\inf_{x\in\mathbb{S}}\{u_x(t,x)\}(T-t)\right) = -1,$$

while the solution u remains bounded.

Proof. Let $T > 0$ be the maximal time of existence of the solution u to Eq.(2.2) with initial data $u_0 \in H^3(\mathbb{S})$. By the assumption $T < \infty$ and Theorem 2.2, we know that

$$\liminf_{t\to T} m(t) = -\infty, \tag{4.1}$$

where $m(t) := \inf_{x\in\mathbb{S}}\{u_x(t,x)\}$ for $t \in [0,T)$. Lemma 2.2 implies that m is locally Lipschitz with

$$m(t) = u_x(t,\xi(t)), \quad t \in [0,T).$$

Note that $u_{xx}(t,\xi(t)) = 0$ for *a.e.* $t \in (0,T)$. Then, from (3.4) we deduce that

$$\frac{d}{dt}m(t) + m^2(t) + \lambda m(t) = \frac{3}{2}u^2(t,\xi(t)) - \frac{3}{2}G*(u^2)(t,\xi(t)). \tag{4.2}$$

By Young's inequality, in view of Lemma 3.1, we have

$$\|G*u^2(t,\cdot)\|_{L^\infty} \le \|G\|_{L^\infty}\|u^2\|_{L^1} \le \frac{\cosh(\frac{1}{2})}{2\sinh(\frac{1}{2})}\|u(t,\cdot)\|^2_{L^2} \le 2k\|u_0\|^2_{L^2}.$$

Now Lemma 3.2 and the above inequality imply that for all $t \in [0,T)$,

$$\begin{aligned}
&\left|\frac{3}{2}u^2(t,\xi(t)) - G*\left(\frac{3}{2}u^2(t,\xi(t))\right)\right| \\
&\le \frac{3}{2}\left(\|u(t,\cdot)\|^2_{L^\infty} + \|G*u^2(t,\cdot)\|_{L^\infty}\right) \\
&\le \frac{3}{2}\left[\left(\frac{3k}{\lambda}\|u_0\|^2_{L^2} + \|u_0\|_{L^\infty}\right)^2 + 2k\|u_0\|^2_{L^2}\right].
\end{aligned} \tag{4.3}$$

Set

$$\beta = \frac{3}{2}\left[\left(\frac{3k}{\lambda}\|u_0\|^2_{L^2} + \|u_0\|_{L^\infty}\right)^2 + 2k\|u_0\|^2_{L^2}\right].$$

Combining (4.2) with (4.3), we deduce that

$$\left|\frac{d}{dt}m(t) + m^2(t) + \lambda m(t)\right| \le \beta, \quad a.e. \quad on\ (0,T).$$

Hence,

$$-\beta - \frac{1}{4}\lambda^2 \le \frac{dm}{dt} + \left(m(t) + \frac{1}{2}\lambda\right)^2 \le \beta + \frac{1}{4}\lambda^2,\ t \in [0,T). \tag{4.4}$$

Let $\varepsilon \in (0,1)$. Since $\lim_{t\to T}\inf(m(t)+\frac{1}{2}\lambda) = -\infty$(by (4.1)), there is some $t_0 \in (0,T)$ with $m(t_0) + \frac{1}{2}\lambda < 0$ and

$$\left(m(t_0) + \frac{1}{2}\lambda\right)^2 > \frac{1}{\varepsilon}\left(\beta + \frac{1}{4}\lambda^2\right).$$

We can prove that

$$\left(m(t) + \frac{1}{2}\lambda\right)^2 > \frac{1}{\varepsilon}\left(\beta + \frac{1}{4}\lambda^2\right), \quad t \in [t_0, T). \tag{4.5}$$

In fact, since $m(t)$ is locally Lipschitz (it belongs to $W^{1,\infty}_{loc}(\mathbb{R})$ by Theorem 2.1), there is some $\delta > 0$ such that

$$\left(m(t) + \frac{1}{2}\lambda\right)^2 > \frac{1}{\varepsilon}\left(\beta + \frac{1}{4}\lambda^2\right),\ \ t \in (t_0, t_0+\delta).$$

From (4.4), we have

$$\frac{dm}{dt} < (\varepsilon - 1)\left(m(t) + \frac{1}{2}\lambda\right)^2 < 0, \quad a.e. \quad on\ (t_0, t_0+\delta).$$

Being locally Lipschitz, the function $m(t)$ is absolutely continuous. Therefore, by integrating the above relation on $[t_0, t_0+\delta]$, we obtain that $m(t_0+\delta) \le m(t_0)$. Thus,

$$m(t_0+\delta) + \frac{1}{2}\lambda \le m(t_0) + \frac{1}{2}\lambda < 0.$$

By the above inequality, we have

$$\left(m(t_0+\delta) + \frac{1}{2}\lambda\right)^2 \ge \left(m(t_0) + \frac{1}{2}\lambda\right)^2 > \frac{1}{\varepsilon}\left(\beta + \frac{1}{4}\lambda^2\right).$$

The relation (4.5) is proved by a continuous extension.

A combination of (4.4) with (4.5) enables us to infer

$$-1 - \varepsilon < \frac{\frac{dm}{dt}}{\left(m(t) + \frac{1}{2}\lambda\right)^2} < -1 + \varepsilon, \quad a.e. \quad on\ (t_0, T). \tag{4.6}$$

For $t \in (t_0, T)$, integrating (4.6) on (t, T), we obtain

$$-1 - \varepsilon < \frac{1}{\left(m(t) + \frac{1}{2}\lambda\right)(T-t)} < -1 + \varepsilon,\ \ t \in (t_0, T).$$

Letting ε goes to zero, we obtain

$$\lim_{t\to T}\left[\left(m(t) + \frac{1}{2}\lambda\right)(T-t)\right] = -1,$$

that is

$$\lim_{t\to T}(T-t)m(t) = -1.$$

This completes the proof of the theorem. ■

We now present the exact blow-up set for the corresponding breaking-wave solution to Eq.(2.2) for a large class of initial data.

Theorem 4.2. Assume that $u_0 \in H^s(\mathbb{S})$ $s > \frac{3}{2}$, $u_0'(0) < -\lambda$ or

$$0 < \lambda \leq \left(\frac{7}{24}\right)^{\frac{3}{4}} \left(\frac{9(1-\varepsilon)}{16\sinh\left(\frac{1}{2}\right)}\right)^{\frac{1}{4}} \|u_0\|_{L^2}, \ 0 < \varepsilon < 1,$$

and that the corresponding associated potential $y_0 := u_0 - u_{0,xx} \not\equiv 0$ is odd.
(a) Suppose that $(x - \frac{1}{2})y_0(x) \leq 0$ on $\mathbb{S}$. Then the solution to Eq.(2.2) with initial data u_0 blows up in finite time only at the point $x = \frac{1}{2}$ and

$$\lim_{t \to T} u_x(t, \frac{1}{2}) = -\infty.$$

(b) Suppose that $(x - \frac{1}{2})y_0(x) \geq 0$ on $\mathbb{S}$. Then the solution to Eq.(2.2) with initial data u_0 blows up in finite time only at two points $x = 0$ and $x = 1$, and

$$\lim_{t \to T} u_x(t, 0) = \lim_{t \to T} u_x(t, 1) = -\infty.$$

Proof. The proof is similar to that of Theorem 4.3 in [28], so we omit it. ■

5 Global solution and its decay

In this section, we will show that there exist global strong solutions to Eq.(2.1) and these global solutions decay to zero at time goes to infinite provided the initial data u_0 satisfy certain sign conditions.

Theorem 5.1. Assume $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$. If $y_0 = u_0 - u_{0,xx}$ does not change sign on $\mathbb{S}$, then Eq.(2.1) (or Eq.(2.2)) has a global strong solution

$$u = u(\cdot, u_0) \in C([0, \infty); H^s(\mathbb{S})) \cap C^1([0, \infty); H^{s-1}(\mathbb{S})).$$

Moreover, the global solution decays to 0 in the H^1-norm and the H^3-norm as time goes to infinite.

Proof. Let $T > 0$ be the maximal time of existence of the solution u to Eq.(2.1) (or Eq.(2.2)) with initial data $u_0 \in H^3(\mathbb{S})$. We first consider the case where $y_0 \geq 0$ on $\mathbb{S}$. If $y_0 \geq 0$, then Lemma 2.1 ensure that $y \geq 0$ for all $t \in [0, T)$. Using $u = G * y$ and the positivity of G, we infer that $u(t, \cdot) \geq 0$ for all $t \in [0, T)$. We know that

$$u(t, x) = \int_{\mathbb{S}} G(x - \eta) y(t, \eta) d\eta, \ \ (t, x) \in [0, T) \times \mathbb{S},$$

where $G(x) := \frac{\cosh(x - [x] - \frac{1}{2})}{2\sinh(\frac{1}{2})}$, $x \in \mathbb{S}$. Fix $(t, x) \in [0, T) \times \mathbb{S}$ and let $\sigma := \frac{1}{4\sinh(\frac{1}{2})}$. We have

$$\begin{aligned} u(t, x) = &\sigma e^x \int_0^x e^{-\eta - \frac{1}{2}} y(t, \eta) d\eta + \sigma e^{-x} \int_0^x e^{\eta + \frac{1}{2}} y(t, \eta) d\eta \\ &+ \sigma e^x \int_x^1 e^{-\eta + \frac{1}{2}} y(t, \eta) d\eta + \sigma e^{-x} \int_x^1 e^{\eta - \frac{1}{2}} y(t, \eta) d\eta. \end{aligned}$$

Differentiation with respect to x yields for $(t,x)\in[0,T)\times\mathbb{S}$

$$\begin{aligned}u_x(t,x)=&\sigma e^x\int_0^x e^{-\eta-\frac12}y(t,\eta)d\eta-\sigma e^{-x}\int_0^x e^{\eta+\frac12}y(t,\eta)d\eta\\&+\sigma e^x\int_x^1 e^{-\eta+\frac12}y(t,\eta)d\eta-\sigma e^{-x}\int_x^1 e^{\eta-\frac12}y(t,\eta)d\eta.\end{aligned}$$

From the above two equations, we deduce that

$$\begin{aligned}u(t,x)+u_x(t,x)&=2\sigma e^x\left(\int_0^x e^{-\eta-\frac12}y(t,\eta)d\eta+\int_x^1 e^{-\eta+\frac12}y(t,\eta)d\eta\right),\\u(t,x)-u_x(t,x)&=2\sigma e^{-x}\left(\int_0^x e^{\eta+\frac12}y(t,\eta)d\eta+\int_x^1 e^{\eta-\frac12}y(t,\eta)d\eta\right).\end{aligned}$$

Since $y\ge 0$ for all $t\in[0,T)$, it follows that for $t\in[0,T)$,

$$|u_x(t,x)|\le u(t,x),\ \forall(t,x)\in[0,T)\times\mathbb{S}. \tag{5.1}$$

By Lemma 3.2, we have

$$\begin{aligned}|u_x(t,x)|&\le u(t,x)\\&\le\exp\{-\lambda t\}\left(\frac{3k}{\lambda}\|u_0\|_{L^2}^2+\|u_0\|_{L^\infty}\right),\ \forall(t,x)\in[0,T)\times\mathbb{S}.\end{aligned} \tag{5.2}$$

The above inequality and Lemma 2.2 imply $T=\infty$. This proves that the solution u exists globally in time.

Multiplying Eq.(2.2) by u and integrating by parts, in view of (5.1) and Lemmas 3.1-3.2, we get

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{\mathbb{S}}\left(u^2+u_x^2\right)dx+\lambda\int_{\mathbb{S}}\left(u^2+u_x^2\right)dx\\=&-4\int_{\mathbb{S}}u^2u_xdx+3\int_{\mathbb{S}}uu_xu_{xx}dx+\int_{\mathbb{S}}u^2u_{xxx}dx\\=&-\frac12\int_{\mathbb{S}}u_x^3dx\le\frac12\int_{\mathbb{S}}u^3dx\le\frac12\|u\|_{L^\infty}\int_{\mathbb{S}}u^2dx\\\le&2\exp\{-3\lambda t\}\left(\frac{3k}{\lambda}\|u_0\|_{L^2}^2+\|u_0\|_{L^\infty}\right)\|u_0\|_{L^2}^2.\end{aligned}$$

Integrating the above inequality with respect to t, we have

$$\|u(t)\|_1^2\le\exp\{-2\lambda t\}\left(\frac{12k}{\lambda^2}\|u_0\|_{L^2}^4+\frac{4}{\lambda}\|u_0\|_{L^2}^2\|u_0\|_{L^\infty}+\|u_0\|_1^2\right).$$

This shows that the corresponding global solution with $y_0\ge 0$ decays to 0 in the H^1-norm.

By (2.5), we obtain that $-u_x\le\frac{\lambda}{7}$ for sufficiently large t. This yields in combination with (2.5) and Gronwall's inequality,

$$\|y\|_1^2\le e^{-\lambda t}\|y_0\|_1^2$$

for large t. The above inequality implies that the corresponding global solution with $y_0\ge 0$ decays to 0 in the H^3-norm. This completes the proof of the theorem with the assumption $y_0\ge 0$ on $\mathbb{S}$. In the case when $y_0(x)\le 0$ on $\mathbb{S}$, one can repeat the above proof to get the desired result. ■

Remark 5.1. Note that the global solution to the periodic Degasperis-Procesi equation does not generally decay to zero as time goes to infinite [28, 54]. Theorem 5.1 shows that there is a considerable difference between Eq.(1.3) and the periodic Degasperis-Procesi equation in their long time behaviors. More precisely, the energy dissipation will affect the long time behavior of global solutions to the periodic Degasperis-Procesi equation.

Remark 5.2. It is known the periodic Degasperis-Procesi equation has periodic peaked traveling wave solutions [28, 54]. Theorem 5.1 shows that global H^3-solutions with y_0 of one sign decay in the H^1-norm and the H^3-norm. Lemma 3.2 shows that any global solution decays in the $L^\infty-$norm. This means that there are no periodic traveling wave solutions of the dissipative equation (1.3). This is also another considerable difference between Eq.(1.3) and the periodic Degasperis-Procesi equation in their long time behaviors.

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