Journal of Nonlinear Mathematical Physics Volume 15, Supplement 1 (2008), 105–111 ARTICLE

On the Exact Solutions of the Nonlinear Wave and ϕ4-Model Equations

A H KARA [a], *A H BOKHARI* [b] *and F D ZAMAN* [b]

[a] *School of Mathematics and Centre for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Johannesburg, Wits 2050, Johannesburg, South Africa*
E-mail: Abdul.Kara@wits.ac.za

[b] *Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia*

Abstract

The nonlinear wave equation with variable long wave velocity and the Gordon-type equations (in particular, the ϕ4-model equation) display a range of symmetry generators, inter alia, translations, Lorentz rotations and scaling - all of which are related to conservation laws. We do a study of the symmetries of a large class with a view to reduction and solution of these equations which has been analysed, to some extent, using other techniques giving rise to a different class of solutions.

1 Introduction

The method of 'invariants' to reduce differential equations (des) is now well known via the Lie symmetry method (e.g., see [1] or [2]). This is especially useful when trying to solve nonlinear partial des (pdes).

The nonlinear pde

$$\beta u_{tt} + \alpha u_t = (f(u)u_x)_x + \lambda u(1 - u^n) \tag{1.1}$$

has a lengthy history of analysis, both analytically and numerically, for various combinations of the parameters n, α, β and λ. We have the following examples.
(i) When $n = 1$, $\alpha = 1$, $\beta = 0$ and $f = 1$, we have the Fisher equation which arises in the study of reaction-diffusion waves in biology. A detailed symmetry study with a variational bias on its reduced form had been done in [3]. (See also [3] for details of the methods adopted and analysis of the Fitzhugh-Nagumo equation).
(ii) The case $n = 2$, $\alpha = 0$, $\beta = 1$ and $f = 1$ gives the $\phi - 4$ model equation.
(iii) With $\lambda = 0$, $\alpha = 1$ and $\beta = 0$, we have the nonlinear diffusion equation.
(iv) When $\lambda = 0$, $\alpha = 0$ and $\beta = 1$, we have the nonlinear (1-1) wave equation whose long wave speed is given by $f(u)$. In some studies, the speed is assumed to be a function of u_x,

i.e., f is replaced by $g(u_x)$, say.
(v) In (iv) above, if α is assumed to be nonzero but 'small', then the wave equation is construed as a wave equation with a 'damping' term.
(vi) The Telegraph equation is also obtainable with $n = 1$ and $\alpha, \beta \neq 0$.
In (iii)-(v), $\lambda \neq 0$ implies a model with a 'source' term. In our studies below, we will consider various classes of (ii), (iv) and (v). A symmetry analysis of the respective equation will be done leading to a reduction and solution of the pde or it may better that the reduced form can be pursued using alternative techniques.

Some preliminaries and notation are briefly presented here. Consider an rth-order system of partial differential equations of n independent and m dependent variables, viz.,

$$E^{\beta}(x, u, u_{(1)}, \dots, u_{(r)}) = 0\,, \qquad \beta = 1, \dots, \tilde{m}\,. \tag{1.2}$$

A conservation law of (1.2) is the equation $D_i T^i = 0$, on the solutions of (1.2). Here the *total differentiation operator* is

$$D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha} \frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha} \frac{\partial}{\partial u_j^{\alpha}} + \cdots\,, \quad i = 1, \dots, n. \tag{1.3}$$

The tuple $T = (T^1, \dots, T^n)$ is called a *conserved vector/flow* of (1.2). A generalized Lie symmetry operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \zeta_i^{\alpha} \frac{\partial}{\partial u_i^{\alpha}} + \zeta_{i_1 i_2}^{\alpha} \frac{\partial}{\partial u_{i_1 i_2}^{\alpha}} + \cdots\,, \tag{1.4}$$

where ξ^i, η^{α} are dependent only on (x, u) in the case of point symmetries and the additional coefficients are

$$\begin{aligned} \zeta_i^{\alpha} &= D_i(W^{\alpha}) + \xi^j u_{ij}^{\alpha}\,, \\ \zeta_{i_1 i_2}^{\alpha} &= D_{i_1} D_{i_2}(W^{\alpha}) + \xi^j u_{j i_1 i_2}^{\alpha}, \\ &\vdots \end{aligned} \tag{1.5}$$

and W^{α} is the Lie characteristic function defined by $W^{\alpha} = \eta^{\alpha} - \xi^j u_j^{\alpha}$. The conserved flow T is easily obtainable from Noether's Theorem if (1.2) is variational and the corresponding Lagrangian generates Noether symmetries which are the generators that leave invariant the respective variational functional. In such situations, one may twice reduce the de using the symmetry as the reduced equation inherits the symmetry via the conservation law (see [4, 5]).

The conservation laws of an equation lead, inter alia, to a system of reduced equations corresponding to the pde by the introduction of an additional dependent variable called the *potential variable.* For example, in the case of two independent variables x and t, the conservation law

$$D_t T^1 + D_x T^2 = 0$$

lead to $v_x = T^1$ and $v_t = -T^2$, where v is the potential variable. In certain special cases, the Lie point symmetries of this system lead to nonlocal (potential) symmetries of the original pde which would provide new exact solutions to the pde (see [2]). There are other advantages to studying the Lie symmetries of the potential system as discussed in [3].

The principle Lie algebra of point symmetry generators of (1.1) is the translations $\{\partial_x, \partial_t\}$. Separately, these would lead to *linear momentum* and *energy* conservation solutions, respectively. The latter is usually regarded as steady state solutions. The combination $c\partial_x + \partial_t$ leads to travelling wave or soliton type solutions, where c represents the speed of the wave. In this case, the combined symmetry leads to the invariants $y = x - ct$ and $w(y) = u$ so that (1.1) becomes

$$(\beta c^2 - f)w'' - \alpha c w' - f'(w')^2 - \lambda w(1 - w^n) = 0. \tag{1.6}$$

2 The ϕ4-model equation

In this section, we consider the well known ϕ4-model equation of mathematical physics

$$u_{tt} = u_{xx} + \lambda u(1 - u^2). \tag{2.1}$$

2.1 *Exact and approximate analytical solutions*

There are numerous but specific studies of the equation as far as the travelling wave type solutions are concerned leading to kink-antikink analyses [6] and some discrete models [7]. After doing a Lie point symmetry analysis, we reduce the equation by successive reduction and obtain exact and approximate analytical solutions of travelling wave type and those invariant under Lorentz rotations. To our knowledge, the latter, which is of physical significance as it involves *Lorentz spin* conservation, have not been considered by any exact or analytical methods. We also briefly look at a variational form and conservation laws of (2.1).
Firstly, from (1.6), equation (2.1) leads to the nonlinear ode

$$(1 - c^2)w'' + \lambda w(1 - w^2) = 0 \tag{2.2}$$

which has a Lagrangian

$$L = \frac{1}{2}w'^2 + \frac{\lambda}{1 - c^2}(-\frac{1}{2}w^2 - \frac{1}{4}w^4).$$

The Noether symmetries corresponding to L would lead to a double reduction of (2.2) and, hence, to a solution. Alternatively, we note that (2.2) reduces to the first-order ode

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\mu} = \frac{\lambda}{1 - c^2}\frac{\mu(1 - \mu^2)}{\gamma},$$

where $\mu = w$, $\gamma = w'$, so that

$$\gamma = \sqrt{\frac{2\lambda}{1 - c^2}(\frac{\mu^2}{2} - \frac{\mu^4}{4}) + k},$$

where k is a constant. Substituting back, the first-order ode in w and y is integrable. The simplest case $k = 0$ leads to the solution of (2.1)

$$u = \pm\frac{\sqrt{2tanh^2(c_1 - s\sqrt{\lambda/(1 - c^2)}) - 2}}{tanh(c_1 - s\sqrt{\lambda/(1 - c^2)})}, \tag{2.3}$$

$(s = x - ct)$. Equation (2.1) also admits the Lorentz rotation symmetry $x\partial_t + t\partial_x$ whose invariants are $y = x^2 - t^2$ and $w = u$. With this choice, (2.1) leads to an ode in $w = w(y)$ given by

$$4yw'' + 4w' + \lambda w(1 - w^2) = 0, \tag{2.4}$$

which has a straightforward/standard Lagrangian

$$L = 2yw'^2 - \lambda(\frac{1}{2}w^2 - \frac{1}{4}w^4).$$

However, it can be shown that L does not generate any Noether symmetry so that a double reduction cannot be done as suggested above in the travelling wave case. Moreover, (2.4) does not admit any Lie point symmetries. We, thus, need to resort to other analytical methods. In particular, we may use the well known perturbation technique (see [8, 9]) on $yw'' + w' + \epsilon w(1 - w^2) = 0$ with the assumption that $\epsilon = \lambda/4$ is relatively small. Substitution back into u and (x, t) would give an explicit analytical solution to the $\phi 4$-model equation.

2.2 *Conservation laws*

One can determine the components of the conserved flow (T^1, T^2) by the definition of the conservation law. This route is cumbersome. We resort to the straightforward/standard Lagrangian of (2.1) given by

$$L = \frac{1}{2}u_x^2 - \frac{1}{2}u_t^2 - \frac{\lambda}{4}(2u^2 - u^4).$$

Its Noether symmetries $X = \xi\partial_x + \tau\partial_t + \eta\partial_u$ are given by

$$XL + L(D_t\tau + D_x\xi) = Df + Dg, \tag{2.5}$$

where X is prolonged accordingly and (f, g) is a gauge vector dependent on (x, t, u). It turns out that the complete algebra of Lie points symmetries form the algebra of Noether symmetries which implies that three conserved vectors can be obtained by Noether's theorem

$$\begin{aligned} T^1 &= L\tau + (\eta - \xi u_x - \tau u_t)L_{u_t} - f, \\ T^2 &= L\xi + (\eta - \xi u_x - \tau u_t)L_{u_x} - g. \end{aligned} \tag{2.6}$$

That is, conservation of linear momentum and energy are given by $(u_x u_t, L - u_x^2)$ and $(L + u_t^2, -u_x u_t)$, respectively. Lorentz spin is given by

$$(\frac{1}{2}xu_x^2 + \frac{1}{2}xu_t^2 + tu_xu_t - \frac{\lambda}{2}x(\frac{1}{2}u^2 - \frac{1}{4}u^4), -\frac{1}{2}tu_x^2 - \frac{1}{2}tu_t^2 - xu_xu_t - \frac{\lambda}{2}t(\frac{1}{2}u^2 - \frac{1}{4}u^4)).$$

3 The nonlinear wave equation

3.1 *Long wave speed dependencies on* u

3.1.1 A (1-1) nonlinear wave equation with long wave speed $f(u) = mu^{m-1}$ is given by $u_{tt} = (mu^{m-1}u_x)_x$ $(m \neq 0)$ which is

$$u_{tt} - mu^{m-1}u_{xx} + (1-m)mu^{m-2}u_x^2 = 0. \tag{3.1}$$

It can be shown that its algebra of Lie point symmetry generators is spanned by

$$\{\partial_x, \partial_t, x\partial_x + t\partial_t, \frac{1}{2}(m-1)t\partial_t + u\partial_u\}.$$

The travelling wave solution (with wave speed c) is obtained by combining ∂_x and ∂_t so that $y = x - ct$ and $w = u$ leading to the nonlinear ode $c^2w'' - mw^{m-1}w'' + (1-m)mw^{m-2}w'^2 = 0$ which after one integration leads to the variables separable ode $(c^2 - mw^{m-1})\mathrm{d}w = k\mathrm{d}y$ so that an implicit solution is

$$c^2u - u^m + k_1 = k(x - ct),$$

where k and k_1 are constants. The steady state solution is obtained when $c = 0$, viz.,

$$u = (k_1 - kx)^{\frac{1}{m}}.$$

The dilation symmetry $x\partial_x + t\partial_t$ leads to invariants $y = \frac{x}{t}$ and $w = u$ by which (3.1) becomes the ode, for $w = w(y)$,

$$y^2w''' + 2yw' - mw^{m-1}w'' + (1-m)mw'^2 = 0$$

which after one integration leads to the first-order ode $(y^2 - mw^{m-1})w' = k$.
The conserved form in (3.1) leads to the potential system

$$-mu^{m-1}u_x + v_t = 0, \quad v_x - u_t = 0 \tag{3.2}$$

which generates a number of symmetries equivalent to the symmetries above and the additional point symmetry

$$(tu^m + (m-1)xv)\partial_x + (tv + \frac{1}{m}xu)\partial_t - \frac{2uv}{m}\partial_u - (\frac{2}{m+1}u^{m+1} + \frac{1+m}{2m}v^2)\partial_v$$

which is a potential symmetry of (3.1). An exact solution of (3.1) would lead to a solution of (3.1) different from the above, i.e., a new solution.
3.1.2 Equation (3.1) with a damping term ϵu_t leads to the potential form

$$-mu^{m-1}u_x + v_t = 0, \quad v_x - u_t = \epsilon u. \tag{3.3}$$

Without showing the details, we note that for $m \neq 1$, the travelling wave similarity variables lead to the solution $c^2 \ln u - \frac{m}{m-1}u^{m-1} = \epsilon c(x - ct) + k$, k constant. Here, it is clear that one cannot construe ϵ as small.
Now we note that (3.3) admits the scaling symmetry

$$\frac{m-1}{m+1}x\partial_x + \frac{2}{m+1}u\partial_u + v\partial_v$$

reducing (3.3) to the system (with $y = t$, $u = \alpha x^{\frac{2}{m-1}}$ and $v = \beta x^{\frac{m+1}{m-1}}$ where $\alpha = \alpha(y)$ and $\beta = \beta(y)$)

$$\frac{m+1}{m-1}\beta = \alpha' + \epsilon\alpha, \quad \beta' = \frac{2m}{m-1}\alpha^m, \tag{3.4}$$

so that

$$\alpha'' + \epsilon\alpha' - \frac{2m(m+1)}{(m-1)^2}\alpha^m = 0. \tag{3.5}$$

This equation can be analysed using inverse variational methods as it admits a Lagrangian after multiplication by $e^{\epsilon y}$. Alternatively, as a sample case, we choose $m = 2$ to show the use of the *approximate Lie symmetry method* in analysing (3.5) assuming ϵ small (see [10]). Equation (3.5) admits the approximate symmetry

$$y(1 + \frac{\epsilon}{10}y)\partial_y - 2\alpha(1 + \frac{\epsilon}{5}y)\partial_\alpha.$$

The zero and first invariants can then be used to obtain a reduction in the usual way. Furthermore, a series or numerical study of (3.5) may be made. All of these routes require a detailed attention and can be pursued as a separate study.

3.2 *Long wave speed dependencies on* u_x

A potential form of the wave equation

$$u_{tt} = (au_x + \frac{b}{2}u_x^2)_x \tag{3.6}$$

is

$$u_t = v_x, \quad v_t = au_x + \frac{b}{2}u_x^2.$$

Its Lie point symmetry generators are the obvious translations and

$$x\partial_x + t\partial_t + u\partial_u + v\partial_v, \quad \frac{b}{2a^2}x\partial_x + (\frac{x}{a} + \frac{3b}{2a^2})\partial_u + (t + \frac{2b}{a^2}v)\partial_v, \quad t\partial_u + x\partial_v.$$

Reducing (3.6) or the potential form by the travelling wave similarity variables $y = x - ct$ and $w = u$ leads, on integration, to the ode $\frac{b}{2}w'^2 + aw' - c^2w' = k$ (where k is a constant) whose solution leads to

$$u = (\frac{c^2 - a \pm \sqrt{c^2 - 2c^2a + a^2 - 2kb}}{b})(x - ct) + k_2,$$

k_2 a constant.

The scaling transformation leads to the invariants $y = x/t$, $\alpha = \frac{u}{x}$ and $\beta = \frac{v}{t}$, where $\alpha = \alpha(y)$ and $\beta = \beta(y)$. The potential form leads to the system of odes

$$-y^2\alpha' = \beta', \quad \beta - y\beta' = a(\alpha + y\alpha') + \frac{\beta}{2}(\alpha + y\alpha')^2$$

whose analytical solution can be somewhat cumbersome to obtain. A numerical scheme may be the best here.

4 Conclusion

We have presented the role of invariants in obtaining reduction and exact and approximate analytical solutions to some classes of nonlinear wave equations with a source term in the case of the ϕ-4 model equation.

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