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Complex Lie Symmetries for Variational Problems

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Abstract

We present the use of complex Lie symmetries in variational problems by defining a complex Lagrangian and considering its Euler-Lagrange ordinary differential equation. This Lagrangian results in two real "Lagrangians" for the corresponding system of partial differential equations, which satisfy Euler-Lagrange like equations. Those complex Lie symmetries that are also Noether symmetries (i.e. symmetries of the complex Lagrangian) result in two real Noether symmetries of the real "Lagrangians". Also, a complex Noether symmetry of a second order complex ordinary differential equation results in a double reduction of the complex ordinary differential equation. This implies a double reduction in the corresponding system of partial differential equations.

1 Introduction

Euler-Lagrange (EL) equations, Lie-Bäcklund (LB) and Noether Symmetries (NSs) play a vital role in the study of invariances in the calculus of variations and differential equations. LB symmetries are used to reduce the order of an ordinary differential equation (ODE) or the number of independent variables in a partial differential equation (PDE) [6, 14]. NSs on the other hand are used to construct conservation laws for systems of EL equations by using Noether's Theorem [13, 9]. In this paper these results are extended to the complex domain by defining complex Lagrangians (c-Lagrangians), complex conservation laws and complex EL differential equations. Noether's theorem in the complex domain can be stated as follows: *for a complex EL differential equation, to each complex Noether symmetry associated with a c-Lagrangian there corresponds a complex conservation law, which can be determined explicitly by a formula.* A complex NS associated with a c-Lagrangian can be constructed in the same way as in the real case.

The use of complex Lie symmetries (CLSs) for complex ordinary differential equations (CODEs) and the system of PDEs was developed earlier [1]. A CLS can also be used to reduce the order of certain systems of PDEs that correspond to a CODE. The system of

PDEs corresponding to a CODE admits two real Lagrangian like quantities, which we call r-Lagrangians, that result from a c-Lagrangian and satisfy EL-like equations which come from splitting the complex EL equations. The use of a NS in reducing the order of DEs has been discussed in [9, 10, 11, 12] for example. For real scalar ODEs, a Noether symmetry that corresponds to a Lagrangian of the scalar ODE results in a double reduction of order. This result also holds for complex ODEs. It is seen that the order for a system of PDEs corresponding to a CODE can also be reduced.

The layout of the paper is as follows. In the next section we introduce the notion of c-Lagrangian and its complex EL equation. Various theorems concerning complex Noether point symmetries are given in the same section. The third section gives the two r-Lagrangians and the EL-like equations corresponding to the c-Lagrangian and complex EL equation. It also states the forms of the conditions on the r-Lagrangians and Noether point symmetries corresponding to the known results of section 2. In the same section we give applications of our approach with certain examples. Finally, we summarize the whole approach in the fourth section.

2 Lagrangians and EL Equations in the Complex Domain

Consider a second-order CODE of a complex-valued function $u(z)$ of the form

$$u'' = w\ (z, u, u'), \tag{2.1}$$

where w is a complex analytic function of its arguments. The CLSs admitted by the above CODE and the corresponding RLSs of the respective system of PDEs corresponding to the CODE have been discussed in [1]. Assume the above CODE arises from a c-Lagrangian $L(z, u, u')$, i.e. the above equation is equivalent to the complex EL equation

$$\frac{\partial L}{\partial u} - \frac{d}{dz}(\frac{\partial L}{\partial u'}) = 0. \tag{2.2}$$

Definition 1. $\mathbf{Z}$ is said to be a complex Noether point symmetry corresponding to a c-Lagrangian $L(z, u, u')$ of (2.1) if there exists a complex function $A(z, u)$ such that

$$\mathbf{Z}L + (\frac{d\varsigma}{dz})L = \frac{dA}{dz}, \tag{2.3}$$

where

$$\mathbf{Z} = \varsigma\frac{\partial}{\partial z} + \chi\frac{\partial}{\partial u} + \chi^{(1)}\frac{\partial}{\partial u'}, \qquad \frac{d}{dz} = \frac{\partial}{\partial z} + u'\frac{\partial}{\partial u} + ...\ . \tag{2.4}$$

Now we restate Noether's theorem for complex NSs, and theorems for the first integral and solvability by quadratures, without proof as there is no difference made by complexity.
Theorem 1. *If* $\mathbf{Z}$ *is a Noether point symmetry for a c-Lagrangian* $L(z, u, u')$ *of (2.1) then*

$$I = \varsigma L + (\chi - u'\varsigma)L_{u'} - A, \tag{2.5}$$

is a complex first integral of (2.1) associated with $\mathbf{Z}$, i.e., $dI/dz = 0$ on solutions of (2.1).
Theorem 2. *The first integral* I *associated with the complex Noether point symmetry* $\mathbf{Z}$ *satisfies the relation*

$$\mathbf{Z}I = 0. \tag{2.6}$$

Theorem 3. *If for a c-Lagrangian* $L(z, u, u')$ *of (2.1) there corresponds a Noether point complex symmetry, then (2.1) is solvable by quadrature.*

In the subsequent section, EL-like equations, two conserved quantities and conditions corresponding to (2.5) and (2.6) are obtained for the system of PDEs.

3 Euler-Lagrange-Like Equations

One obtains the following system of PDEs corresponding to CODE (2.1)

$$f_{xx} - f_{yy} + 2g_{xy} = 4G(x, y, f, g, h, l), \quad f_x = g_y, \tag{3.1}$$

$$g_{xx} - g_{yy} - 2f_{xy} = 4H(x, y, f, g, h, l), \quad f_y = -g_x, \tag{3.2}$$

where

$$u = f + ig, \quad z = x + iy, \quad w = G + iH, \tag{3.3}$$

and

$$h = f_x + g_y, l = g_x - f_y. \tag{3.4}$$

Write L as

$$L = L_1(x, y, f, g, h, l) + iL_2(x, y, f, g, h, l). \tag{3.5}$$

The system of EL-like equations corresponding to (2.2) is

$$\frac{\partial L_1}{\partial f} + \frac{\partial L_2}{\partial g} - d_x\left(\frac{\partial L_1}{\partial h} + \frac{\partial L_2}{\partial l}\right) - d_y\left(\frac{\partial L_2}{\partial h} - \frac{\partial L_1}{\partial l}\right) = 0, \tag{3.6}$$

$$\frac{\partial L_2}{\partial f} - \frac{\partial L_1}{\partial g} - d_x\left(\frac{\partial L_2}{\partial h} - \frac{\partial L_1}{\partial l}\right) + d_y\left(\frac{\partial L_1}{\partial h} + \frac{\partial L_2}{\partial l}\right) = 0, \tag{3.7}$$

where

$$\begin{aligned} d_x &= \partial_x + f_x\partial_f + g_x\partial_g + h_x\partial_h + l_x\partial_l, \\ d_y &= \partial_y + f_y\partial_f + g_y\partial_g + h_y\partial_h + l_y\partial_l. \end{aligned} \tag{3.8}$$

The above system of DEs (3.6) and (3.7) couples the two r-Lagrangians, L_1 and L_2. Note that they are not separately Lagrangians as they do not satisfy EL equations separately but only the coupled system (3.6) and (3.7). The corresponding system of conditions relative to (2.3) in the real domain is given by

$$\begin{aligned} &2\mathbf{X}L_1 - 2\mathbf{Y}L_2 + (d_x\varsigma_1 + d_y\varsigma_2)L_1 - (d_x\varsigma_2 - d_y\varsigma_1)L_2 = d_xA_1 + d_yA_2, \\ &2\mathbf{X}L_2 + 2\mathbf{Y}L_1 + (d_x\varsigma_1 + d_y\varsigma_2)L_2 + (d_x\varsigma_2 - d_y\varsigma_1)L_1 = d_xA_2 - d_yA_1, \end{aligned} \tag{3.9}$$

where we have set

$$\begin{aligned} \varsigma &= \varsigma_1 + i\varsigma_2, \quad A = A_1 + iA_2, \\ \mathbf{Z} &= \mathbf{X} + i\mathbf{Y}. \end{aligned} \tag{3.10}$$

In addition if we set $\chi = \chi_1 + i\chi_2$ in (2.4) with $\chi^{(1)} = \chi_1^{(1)} + i\chi_2^{(1)}$, then

$$\begin{aligned} 2X &= \varsigma_1\partial_x + \varsigma_2\partial_y + \chi_1\partial_f + \chi_2\partial_g + \chi_1^{(1)}\partial_h + \chi_2^{(1)}\partial_l, \\ 2Y &= \varsigma_2\partial_x - \varsigma_1\partial_y + \chi_2\partial_f - \chi_1\partial_g + \chi_2^{(1)}\partial_h - \chi_1^{(1)}\partial_l, \end{aligned} \tag{3.11}$$

with

$$\begin{aligned} \chi_1^{(1)} &= d_x\chi_1 + d_y\chi_2 - \frac{h}{2}(d_x\varsigma_1 + d_y\varsigma_2) + \frac{l}{2}(d_x\varsigma_2 - d_y\varsigma_1), \\ \chi_2^{(2)} &= d_x\chi_2 - d_y\chi_1 - \frac{h}{2}(d_x\varsigma_2 - d_y\varsigma_1) - \frac{l}{2}(d_x\varsigma_1 + d_y\varsigma_2). \end{aligned} \tag{3.12}$$

The first integral (2.5), $I = I_1 + iI_2$, results in two real conserved quantities

$$\begin{aligned} I_1 &= \varsigma_1 L_1 - \varsigma_2 L_2 + (\chi_1 - \frac{h}{2}\varsigma_1 + \frac{l}{2}\varsigma_2)(\partial_h L_1 + \partial_l L_2) - (\chi_2 - \frac{h}{2}\varsigma_2 - \frac{l}{2}\varsigma_1)(\partial_h L_2 - \partial_l L_1) - A_1, \\ I_2 &= \varsigma_1 L_2 + \varsigma_2 L_1 + (\chi_1 - \frac{h}{2}\varsigma_1 + \frac{l}{2}\varsigma_2)(\partial_h L_2 - \partial_l L_1) + (\chi_2 - \frac{h}{2}\varsigma_2 - \frac{l}{2}\varsigma_1)(\partial_h L_1 + \partial_l L_2) - A_2 \end{aligned} \tag{3.13}$$

which satisfy

$$\begin{aligned} d_x I_1 + d_y I_2 &= 0, \\ d_x I_2 - d_y I_1 &= 0 \end{aligned} \tag{3.14}$$

on solutions of (3.1) and (3.2) as well as the coupled equations

$$\begin{aligned} &\mathbf{X}I_1 - \mathbf{Y}I_2 = 0, \\ &\mathbf{X}I_2 + \mathbf{Y}I_1 = 0. \end{aligned} \tag{3.15}$$

We now present some illustrative examples.

Example 1. Consider the complexified free particle equation

$$u'' = 0, \tag{3.16}$$

which admits a c-Lagrangian of the form

$$L = \frac{1}{2}u'^{\,2}. \tag{3.17}$$

The CODE (3.16) admits an 8−dimensional complex Lie algebra [1]. It has 5 CLSs which are also the NSs with respect to the c-Lagrangian (3.17) (for simplicity we do not write the first extension)

$$\mathbf{Z}_1 = \frac{\partial}{\partial z}, \quad \mathbf{Z}_2 = \frac{\partial}{\partial u}, \quad \mathbf{Z}_3 = 2z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}, \tag{3.18}$$

$$\mathbf{Z}_4 = z^2\frac{\partial}{\partial z} + zu\frac{\partial}{\partial u}, \quad \mathbf{Z}_5 = z\frac{\partial}{\partial u}. \tag{3.19}$$

One can reduce the order of (3.16) by two using any one of the above symmetries. For example, for the NS $\mathbf{Z}_2$, the first integral is

$$I = u' = a, \tag{3.20}$$

by using (2.5). From (2.6), $\mathbf{Z}_2$ is also a symmetry of the c-Lagrangian (3.17). The above equation (3.20) yields the solution

$$u = az + b, \tag{3.21}$$

where a and b are complex constants. Similarly, if we use the NS $\mathbf{Z}_3$ we obtain

$$I = u\ u' - zu'^2 = c. \tag{3.22}$$

Since $\mathbf{Z}_3$ satisfies (2.6) it is also a symmetry of the above equation (3.22). It can be transformed into separable form by introducing

$$w = uz^{-\frac{1}{2}}, \tag{3.23}$$

which is

$$w^2 - 4z^2 w'^2 = C. \tag{3.24}$$

Thus, a NS reduces a second order CODE to quadratures. The solution in new coordinates (z, w) is

$$w(z) = \alpha z^{\frac{1}{2}} + \beta z^{-\frac{1}{2}}, \tag{3.25}$$

where α and β are complex constants. Inserting (3.23) into (3.25), we easily get the form of the solution (3.21). The system of PDEs corresponding to the CODE (3.16) is

$$f_{xx} - f_{yy} + 2g_{xy} = 0,\ f_x = g_y, \tag{3.26}$$

$$g_{xx} - g_{yy} - 2f_{xy} = 0,\ f_y = -g_x. \tag{3.27}$$

The respective r-Lagrangians L_1 and L_2 are

$$L_1 = \frac{1}{8}(h^2 - l^2), \tag{3.28}$$

$$L_2 = \frac{1}{4}hl. \tag{3.29}$$

One obtains the respective system of PDEs (3.26) and (3.27) by replacing the above r-Lagrangians in (3.6) and (3.7). In order to reduce the order we decompose $\mathbf{Z}_2$ into its real parts

$$2\mathbf{X}_2 = \frac{\partial}{\partial f},\ \ 2\mathbf{Y}_2 = -\frac{\partial}{\partial g}, \tag{3.30}$$

which satisfy (3.9). From (3.13), setting $I_1 = c_1$ and $I_2 = c_2$, the two conserved quantities are $(A_1 = A_2 = 0)$

$$I_1 = h = 2c_1, I_2 = l = 2c_2. \tag{3.31}$$

Using the CR equations one obtains the solution of the system (3.26) and (3.27). Similarly, $\mathbf{Z}_3$ yields $(A_1 = A_2 = 0)$

$$2\mathbf{X}_3 = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + f\frac{\partial}{\partial f} + g\frac{\partial}{\partial g},\ 2\mathbf{Y}_3 = 2y\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y} + g\frac{\partial}{\partial f} - f\frac{\partial}{\partial g}, \tag{3.32}$$

which satisfy (3.9). The conserved quantities from (3.13), setting $I_1 = c_1$ and $I_2 = c_2$, are ($A_1 = A_2 = 0$)

$$I_1 = fh - gl - \frac{1}{2}x(h^2 - l^2) + yhl = 2c_1, \tag{3.33}$$

$$I_2 = fl + gh - \frac{1}{2}y(h^2 - l^2) - xhl = 2c_2. \tag{3.34}$$

By introducing new coordinates using (3.23) we have

$$F = r^{-\frac{1}{2}}(f\cos\theta/2 + g\sin\theta/2), \tag{3.35}$$

$$G = r^{-\frac{1}{2}}(g\cos\theta/2 - f\sin\theta/2), \tag{3.36}$$

where

$$r^2 = x^2 + y^2,\ \theta = \tan^{-1}(y/x), \tag{3.37}$$

we obtain the following system

$$F^2 - G^2 - 4r^2[(H^2 - J^2)\cos 2\theta - 2HJ\sin 2\theta] = C_1, \tag{3.38}$$

$$2FG - 4r^2[(H^2 - J^2)\sin 2\theta + 2HJ\cos 2\theta] = C_2, \tag{3.39}$$

where

$$H = \frac{1}{2}(F_x + G_y),\ J = \frac{1}{2}(G_x - F_y) \tag{3.40}$$

and

$$\partial_x = \cos\theta\partial_r - \frac{1}{r}\sin\theta\partial_\theta,\ \partial_y = \sin\theta\partial_r + \frac{1}{r}\partial_\theta\,.$$

so that $F_x = \cos\theta F_r - (1/r)\sin\theta F_\theta$ etc in (3.40). The solution of the system (3.38) and (3.39), with $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$, is

$$F = r^{\frac{1}{2}}(\alpha_1\cos\theta/2 - \alpha_2\sin\theta/2) + r^{-\frac{1}{2}}(\beta_1\cos\theta/2 + \beta_2\sin\theta/2), \tag{3.41}$$

$$G = r^{\frac{1}{2}}(\alpha_1\sin\theta/2 + \alpha_2\cos\theta/2) + r^{-\frac{1}{2}}(-\beta_1\sin\theta/2 + \beta_2\cos\theta/2). \tag{3.42}$$

Notice that the above solution cannot be obtained directly without using complex transformations. It also demonstrates the fact that a simple solution may look complicated in some coordinates. Substituting (3.41) and (3.42) into (3.35) and (3.36), one obtains the solution of the system (3.26) and (3.27).

Example 2. The complexified oscillator equation is given by

$$u'' = -u. \tag{3.43}$$

The CLS of this CODE have been studied in [1]. An obvious c-Lagrangian admitted by such an equation is

$$L = \frac{1}{2}u'^2 - \frac{1}{2}u^2. \tag{3.44}$$

A simple NS of (3.43) with respect to the c-Lagrangian (3.44) is

$$\mathbf{Z} = \frac{\partial}{\partial z}, \tag{3.45}$$

which gives the first integral

$$I = \frac{1}{2}u^2 + \frac{1}{2}u'^2 = c. \tag{3.46}$$

The solution of the above CODE (3.43) is

$$u = \alpha \cos z + \beta \sin z, \tag{3.47}$$

where α and β are complex constants. The system of PDEs corresponding to (3.43) is

$$f_{xx} - f_{yy} + 2g_{xy} = -4f,\ f_x = g_y \tag{3.48}$$

$$g_{xx} - g_{yy} - 2f_{xy} = -4g,\ f_y = -g_x. \tag{3.49}$$

The respective r-Lagrangians, L_1 and L_2, are

$$L_1 = \frac{1}{8}(h^2 - l^2) - \frac{1}{2}(f^2 - g^2), \tag{3.50}$$

$$L_2 = \frac{1}{4}hl - fg. \tag{3.51}$$

The obvious NSs of system (3.48) and (3.49) for the r-Lagrangians (3.50) and (3.51) are

$$2\mathbf{X} = \frac{\partial}{\partial x},\ 2\mathbf{Y} = -\frac{\partial}{\partial y} \tag{3.52}$$

which satisfy (3.9). From (3.13) two conserved quantities, $I_1 = c_1$ and $I_2 = c_2$, are $(A_1 = A_2 = 0)$

$$I_1 = f^2 - g^2 + \frac{1}{4}(h^2 - l^2) = -2c_1, \tag{3.53}$$

$$I_2 = fg + \frac{1}{4}hl = -c_2, \tag{3.54}$$

which, on using the CR equations, yields the solution of (3.48) and (3.49) as

$$f = \alpha_1 \cos x \cosh y + \alpha_2 \sin x \sinh y + \beta_1 \sin x \cosh y - \beta_2 \cos x \sinh y, \tag{3.55}$$

$$g = \alpha_2 \cos x \cosh y - \alpha_1 \sin x \sinh y + \beta_1 \cos x \sinh y + \beta_2 \sin x \cosh y. \tag{3.56}$$

This is easier to obtain by utilising (3.47).

Example 3. Consider the complexified Emden-Fowler equation

$$u'' + \frac{2}{z}u' = 3u^5. \tag{3.57}$$

It admits a c-Lagrangian of the form

$$L = \frac{1}{2}z^2u'^2 + \frac{1}{2}z^2u^6. \tag{3.58}$$

The complex NS of (3.57) with respect to the c-Lagrangian (3.58) is

$$\mathbf{Z} = 2z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}. \tag{3.59}$$

The first integral is

$$I = -z^3 u'^2 - z^2 u\, u' + z^3 u^6, \tag{3.60}$$

by using (2.5). Hence, the reduced equation is

$$z^3 u'^2 + z^2 u\, u' - z^3 u^6 = c, \tag{3.61}$$

which can be put into a separable form on introducing

$$w = uz^{1/2}, \tag{3.62}$$

which is

$$z^2 w'^2 - \frac{1}{4}w^2 - w^6 = C. \tag{3.63}$$

This first-order equation (3.63) is variables separable and thus can be reduced to quadrature. The corresponding system of PDEs of the CODE (3.57) is

$$f_{xx} - f_{yy} + 2g_{xy} + \frac{4x}{x^2+y^2}h + \frac{4y}{x^2+y^2}l = 12(f^5 - 8f^3g^2 + 5fg^4), f_x = g_y, \tag{3.64}$$

$$g_{xx} - g_{yy} - 2f_{xy} + \frac{4x}{x^2+y^2}l - \frac{4y}{x^2+y^2}h = 12(g^5 - 8f^2g^3 + 5f^4g), f_y = -g_x. \tag{3.65}$$

The c-Lagrangian L (3.58) yields the two r-Lagrangians

$$\begin{aligned} L_1 &= \frac{1}{8}(x^2-y^2)(h^2-l^2) - \frac{1}{2}xyhl + \frac{1}{2}(x^2-y^2)(f^2-g^2)(f^4+g^4-14f^2g^2) \\ &- 2xyfg(3f^4+3g^4-10f^2g^2), \end{aligned} \tag{3.66}$$

$$\begin{aligned} L_2 &= \frac{1}{4}(x^2-y^2)hl + \frac{1}{4}xy(h^2-l^2) + (x^2-y^2)fg(3f^4+3g^4-10f^2g^2) \\ &+ xy(f^2-g^2)(f^4+g^4-14f^2g^2). \end{aligned} \tag{3.67}$$

It turns out that the system of PDEs (3.64) and (3.65) is obtained on inserting L_1 and L_2 in the EL-like equations (3.6) and (3.7). The two real NSs corresponding to the system of PDEs (3.64) and (3.65) for the r-Lagrangians L_1 and L_2, are

$$2\mathbf{X} = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g}, \tag{3.68}$$

$$2\mathbf{Y} = 2y\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y} + f\frac{\partial}{\partial g} - g\frac{\partial}{\partial f}. \tag{3.69}$$

The conserved quantities are given by ($A_1 = A_2 = 0$)

$$I_1 = \frac{1}{4}x(x^2-3y^2)(h^2-l^2) - \frac{1}{2}y(3x^2-y^2)hl + (-f-hx+ly)(\frac{1}{2}(x^2-y^2)h - xyl)$$
$$+(g+hy+lx)(\frac{1}{2}(x^2-y^2)l + xyh) + x(x^2-3y^2)(f^2-g^2)(f^4+g^4-14f^2g^2)$$
$$-2yfg(3x^2-y^2)(3f^4+3g^4-10f^2g^2), \quad (3.70)$$
$$I_2 = \frac{1}{4}y(3x^2-y^2)(h^2-l^2) + \frac{1}{2}xhl(x^2-3y^2) + (-f-hx+ly)(\frac{1}{2}(x^2-y^2)l + xyh)$$
$$-(g+hy+lx)(\frac{1}{2}(x^2-y^2)h - xyl) + 2x(x^2-3y^2)fg(3f^4+3g^4-10f^2g^2)$$
$$+y(3x^2-y^2)(f^2-g^2)(f^4+g^4-14f^2g^2), \quad (3.71)$$

as these satisfy the coupled system (3.13). We introduce

$$F = r^{1/2}(f\cos\theta/2 - g\sin\theta/2), \quad (3.72)$$

$$G = r^{1/2}(g\cos\theta/2 + f\sin\theta/2), \quad (3.73)$$

where r and θ are defined by (3.37). One obtains the system

$$r^2[(H^2-J^2)\cos 2\theta - 2HJ\sin 2\theta] + (12F^2G^2 - \tfrac{1}{4})(F^2-G^2) - (F^2-G^2)^3 = C_1$$
$$r^2[(H^2-J^2)\sin 2\theta + 2HJ\cos 2\theta] - \tfrac{1}{2}FG + 8F^3G^3 - 6FG(F^2-G^2)^2 = C_2, \quad (3.74)$$

where H and J are given by (3.40). The solution of (3.74) and (3.75) can easily be deduced from the solution of (3.63) by using separation of variables. Then one can revert to the original variables via the transformation (3.72) and (3.73).

4 Summary and Discussion

An inverse problem [4, 5, 7, 8, 9, 15] in variational calculus is to find a Lagrangian whose EL equation is that DE. The Lagrangian of a DE is not unique and certain DEs, like scalar-evolution equations, do not admit Lagrangians [5, 8]. There are several techniques developed to construct Lagrangians for special classes of ODEs, PDEs and their systems by various authors [4, 5, 7, 8, 9, 15]. The question of finding a Lagrangian is an open problem as most of the approaches apply to some special classes of DEs (both ODEs and PDEs) and their systems. Complex NSs depend on the choice of a c-Lagrangian admitted by the CODE, as a CODE can admit several c-Lagrangians. Thus, conserved quantities associated with r-Lagrangians of some system of PDEs are also not unique.

We have presented certain systems of PDEs that admit two r-Lagrangians that satisfy EL-like equations. It thus provides us with an explicit derivation of Lagrangians, conservation laws and NSs of systems of PDEs. Equations considered here are very important in this regard as they give us a system of those PDEs that can be handled with CLS analysis. It may be pointed out that if one takes y (the imaginary part of the independent variable z) and g (the imaginary part of the dependent variable u) to be zero, one gets the usual symmetry analysis of real DEs and PDEs.

It would be very interesting to use this approach to find the c-Lagrangians and conservation laws associated with NSs for a system of CODEs. It is worth pointing out that in our CLS analysis one always gets systems of an even number of PDEs. Thus, if we have a system of two CODEs we will get a system of four PDEs.

CLS analysis can also be used to reduce a system of ODEs by taking a complex valued function of one real variable. It is expected that certain symmetries will be preserved under this projection, which can then be used for reduction.

The CLS analysis has also been used for linearization of CODEs [2, 3]. Lie's results on necessary and sufficient conditions for linearization of a real scalar second-order ODE are extended to the complex domain. It is found that the linearization of CODEs yield the linearization of certain systems of non-linear PDEs. An important implication of such a construction is that the real mappings can be obtained in a nontrivial way from complex mappings to linearize systems of non-linear PDEs. Also, the linearization of complex differential equations of a complex function of one real variable can be used to linearize systems of ODEs and the linearizing mappings are obtained nontrivially from complex mappings. It is the case that the process of analytic continuation can be used in a non-trivial way.

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