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Filtration of a Visco-Elastic Liquid with Relaxation: a Note on Lie Point Symmetries and Reductions

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Abstract

We present the Lie point symmetries admitted by third order partial differential equations (PDEs) which model the pressure of a visco-elastic liquid with relaxation which filtrates through a porous medium. The symmetries are used to construct reductions of the PDEs to ordinary differential equations (ODEs). Some boundary value problems are also discussed.

1 Introduction

The method of Lie point symmetries [1, 2, 3, 4] is one of the few algorithmic methods at our disposal to find analytic solutions to partial differential equations (PDEs). In recent years it has been widely studied and applied to various models that originate in a wide variety of fields like physics, engineering, population dynamics and finance. A summary of some of the results is published in [5]. The solutions obtained from a Lie point symmetry analysis are usually invariant solutions. This is a small subset of the general solution of a PDE. That said, the solutions obtained are often of physical relevance as they contain the symmetries inherent in the model and if nothing else, can be used to test numerical simulations.

The flows of viscoelastic fluids through a porous medium have received a wide coverage in engineering applications. Such applications include enhanced oil recovery, paper and textile coating, insulation systems, geothermal engineering etc. Although viscoelastic flows through porous media are quite prevalent in nature, the literature on such flows with modified Darcy's law is scant yet. Some attempts which involve modified Darcy's law for viscoelastic flows may be mentioned in the following works [6, 7, 8, 9, 10, 11, 12].

We consider the equation

$$p_t = \left[|p_x|^{n-1} p_x + f \left(|p_x|^{n-1} p_x \right)_t \right]_x \tag{1.1}$$

that describes the filtration of a visco-elastic liquid (oil) with relaxation through a porous medium. Here p is the pressure of the oil and subscripts t and x denote partial derivatives. This model was mentioned in [13].

In this article we give a Lie point symmetry classification of (1.1). Some group invariant solutions to this equation are also presented.

In order to deal with the sign of p_x, and more specifically its derivative, comfortably we consider $z = (\mathrm{sign}(p_x))^{n-1}\, p_x^n = kp_x^n$. The function k is a piecewise continuous function which has the value 1 or -1 at each value of x and t. Furthermore, it only changes sign when $p_x = 0$ (i.e. when p_x^{n-1} changes sign). Thus,

$$\frac{\partial z}{\partial x} = \frac{\partial k}{\partial x} p_x^n + knp_x^{n-1}p_{xx} = \sum_i \delta(x - x_i)\delta(t - t_i)p_x^n + knp_x^{n-1}p_{xx}$$

where (x_i, t_i) are values of x and t for which p_x^{n-1} changes sign and $\delta(x - x_i)$ is the dirac delta function. Therefore we have shown $\frac{\partial z}{\partial x} = knp_x^{n-1}p_{xx}$. Similarly $\frac{\partial z}{\partial t} = knp_x^{n-1}p_{xt}$.

We can therefore consider $\mathrm{sign}(p_x)^{n-1}$ as a constant with respect to differentiation. In this article we will denote $\mathrm{sign}(p_x)^{n-1} = k$.

Eq. (1.1) is therefore written as

$$p_t = [kp_x^n + f(kp_x^n)_t]_x\,. \tag{1.2}$$

We consider first the PDE

$$p_t = [kp_x^n + F(p_x)_t]_x \tag{1.3}$$

which is a slight generalization of (1.2) in that we replace $f(kp_x^n)$ with $F(p_x)$. The group classification of Lie point symmetries of (1.2) and (1.3) are given in Section 2. Some invariant solutions of (1.2) and (1.3) are presented in Sections 4 and 3 respectively. Lastly, some related boundary value problems are discussed in Section 5.

2 Lie point symmetries of (1.2) and (1.3)

Here we first calculate the Lie point symmetries of Eq. (1.3). One can then show that the symmetries of Eq. (1.3) are also symmetries of Eq. (1.2). For the sake of compactness we present the symmetries of the two equations together.

Lie point symmetries of (1.3)

$$X = \xi^t(t,x,p)\frac{\partial}{\partial t} + \xi^x(t,x,p)\frac{\partial}{\partial x} + \eta(t,x,p)\frac{\partial}{\partial p}$$

are determined by the equation

$$X^{(3)}\left[p_t - knp_x^{n-1}p_{xx} - F'p_{xxt} - F''p_{xx}p_{xt}\right] = 0 \tag{2.1}$$

on the solution space of Eq. (1.3). Here $X^{(3)}$ is the appropriate third order extension of X namely

$$\begin{aligned} X^{(3)} \;\; = \;\; & X + \zeta^t(t,x,p,p_{(1)})\frac{\partial}{\partial p_t} + \zeta^x(t,x,p,p_{(1)})\frac{\partial}{\partial p_x} + \zeta^{xx}(t,x,p,p_{(1)},p_{(2)})\frac{\partial}{\partial p_{xx}} \\ & + \zeta^{xt}(t,x,p,p_{(1)},p_{(2)})\frac{\partial}{\partial_{xt}} + \zeta^{xxt}(t,x,p,p_{(1)},p_{(2)},p_{(3)})p\frac{\partial}{\partial p_{xxt}} \end{aligned}$$

with all ith order partial derivatives of p denoted by $p_{(i)}$ and

$$\zeta^i = D_i(\eta) - p_l D_i(\xi^l), \quad \zeta^{ij} = D_i(\zeta^j) - p_{jl} D_i(\xi^l), \quad \zeta^{ijk} = D_i(\zeta^{jk}) - p_{jkl} D_i(\xi^l),$$

are the standard prolongation formulae where i, j, k and l can be x or t and D_i is the total derivative operator to the ith independent variable. The convention of summation over repeated indeces are followed.

Eq. (2.1) yields the following group classification of Lie point symmetries for F in Eq. (1.3) not a constant. Eq. (1.3) with $F(p_x)$ and n arbitrary admits the three translation symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x} \quad \text{and} \quad X_3 = \frac{\partial}{\partial p}.$$

These symmetries are also admitted by Eq. (1.2) with $f(z)$ and n arbitrary.

The Lie algebra is extended in the following cases:

Case 1: $F = Ap_x + B$ or $f = Az^{1/n} + B$, $\quad n$ arbitrary,

$$X_4 = p\frac{\partial}{\partial p} + (1-n)t\frac{\partial}{\partial t}.$$

Case 1.1: $n = 1$,

$$X_g = g(t,x)\frac{\partial}{\partial p},$$

where $g(x,t)$ is a solution of $g_t = g_{xx} + Ag_{xxt}$ (which is Eq. (1.3) with $F = Ap_x + B$ and $n = 1$ or Eq. (1.2) with $f = Az + B$ and $n = 1$).

Case 2: $F = \frac{B}{p_x} + D$ or $f = \frac{B}{z^{1/n}} + D$, $\quad n$ arbitrary,

$$X_4 = x\frac{\partial}{\partial x} + (n+1)t\frac{\partial}{\partial t}.$$

Case 3: $F = B(p_x + C)^A + D$ or $f = B(z + C)^A$, $\quad n = 1$,

$$X_4 = 2(A-1)t\frac{\partial}{\partial t} + (A-1)x\frac{\partial}{\partial x} + [p(A+1) + 2Cx]\frac{\partial}{\partial p}.$$

Case 4: $F = A\ln|p_x| + B$ or $f = A\ln|z^{1/n}| + B$, $\quad n$ arbitrary,

$$X_4 = x\frac{\partial}{\partial x} + 2nt\frac{\partial}{\partial t} - p\frac{\partial}{\partial p}.$$

Case 5: $F = A\ln|p_x + C| + B$ or $f = A\ln|z + C| + B$, $\quad n = 1$,

$$X_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - (p + 2Cx)\frac{\partial}{\partial p}.$$

3 Invariant solutions of Eq. (1.3)

We use linear combinations or superpositions of the symmetries of the PDE to find solutions which are invariant under these superpositions. This approach was for example followed in [14]. The method of finding invariant solutions is also refered to as a method of reduction. This is because of the fact that a solution invariant under a symmetry of a PDE of order q with m independent variables is a solution of a PDE of order q with $m-1$ independent variables, called the reduced equation. In our case PDEs (1.2) and (1.3) are third order PDEs with two independent variables. Thus the invariant solutions will solve a third order ODE. ODEs are much easier to handle numerically than PDEs, so though we do not have solutions in closed form, we have made some progress.

A summary of invariant solutions of Eq. (1.3) calculated in this way is given here (see also [15] for detailed calculations): For $F(p_x)$ and n arbitrary, the family of solutions invariant under the Lie point symmetry $X = \frac{\partial}{\partial t} + c_2\frac{\partial}{\partial x} + c_1\frac{\partial}{\partial p}$ is $p = c_1 t + h(x - c_2 t)$ where $h(s)$ satisfies

$$c_1 - c_2 h_s - knh_s^{n-1}h_{ss} + c_2 F'' h_{ss}^2 + c_2 F' h_{sss} = 0.$$

The space of invariant solutions is enlarged in the following cases.

Case 1: $F = Ap_x + B$ with symmetry operator $X = (c_3 + (1-n)t)\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial x} + (c_1 + p)\frac{\partial}{\partial p}$,

Case 1.1: $n \neq 1$,

$$p = (t(n-1) - c_3)^{\frac{1}{1-n}} h\left(\frac{x(n-1) + c_2 \ln|t(n-1) - c_3|}{n-1}\right) - c_1$$

where h satisfies

$$h - c_2 h' + kn(h')^{n-1}h'' - Ah'' + Ac_2 h''' = 0. \qquad (3.1)$$

Case 1.2: $n = 1$ and $c_3 \neq 0$,
$p = e^{t/c_3} h(\frac{c_3 x - c_2 t}{c_3}) - c_1$ where h satisfies the linear ODE with constant coefficients

$$h - c_2 h' - (c_3 + A)h'' + Ac_2 h''' = 0.$$

Case 1.3: $n = 1$, $c_3 = 0$ and $c_2 \neq 0$,
$p = k_1 e^{t/(c_2^2 - A)} e^{x/c_2} - c_1$.

Case 2: $F = \frac{A}{p_x} + B$, n arbitrary with $X = (c_3 + (1+n)t)\frac{\partial}{\partial t} + (c_2 + x)\frac{\partial}{\partial x} + c_1\frac{\partial}{\partial p}$,

Case 2.1: $n \neq -1$,

$$p = \frac{c_1 \ln|c_3 + (1+n)t|}{1+n} + h\left((c_3 + (1+n)t)^{-1/(1+n)}(x + c_2)\right)$$

where $h(s)$ satisfies

$$c_1(h')^3 - s(h')^4 - kn(h')^{n+2}h'' + 2As(h'')^2 - Ash'h''' = 0. \qquad (3.2)$$

Case 2.2 $n = -1$, $c_3 \neq 0$,

$$p = \frac{c_1 t}{c_3} + h\left((c_2 + x)e^{-t/c_3}\right)$$

where $h(s)$ satisfies

$$c_1(h')^3 - s(h')^4 + c_3h'h'' + 2As(h'')^2 - Ash'h''' = 0.$$

Case 2.3: $n = -1$, $c_3 = 0, c_1 \neq 0$,

$$p = c_1 \ln|c_2 + x| + \frac{t}{c_1} + k_1.$$

Case 3: $F = B(p_x + C)^A + D$, $n = 1$ with $X = (c_3 + 2(A-1)t)\frac{\partial}{\partial t} + (c_2 + (A-1)x)\frac{\partial}{\partial x} + (c_1 + 2Cx + (A+1)p)\frac{\partial}{\partial p}$

$$p = -\frac{c_1 + 2Cx}{A+1} - \frac{C(c_2 + (A-1)x}{A+1} + (c2 + (A-1)x)^{\frac{A+1}{A-1}}\, h\left(\frac{(c_2 + (A-1)x)^2}{c_3 + 2(A-1)t}\right)$$

where $h(s)$ satisfies

$$\begin{aligned}
&(A+1)h + (A-1)(s + 3A + 1)sh' + 2(A-1)^2s^2h'' \\
&- 2(A-1)AB\left((A+1)h + 2(A-1)sh'\right)^{A-1}[(A+1)h \\
&+ 2(A-1)(5A-1)s^3h'' + 2(A-1)^2s^4h'''] \\
&- 2(A-1)^2AB\left((A+1)s^2h' + 2(A-1)s^3h''\right)\left((A+1)h + 2(A-1)sh'\right)^{A-2} \\
&\left((A+1)h + (A-1)(3A+1)sh' + 2(A-1)^2s^2h''\right) = 0.
\end{aligned}$$

Case 4: $F = A\ln|p_x| + B$ and n arbitrary with $X = (c_3 + 2nt)\frac{\partial}{\partial t} + (c_2 + x)\frac{\partial}{\partial x} + (c_1 - p)\frac{\partial}{\partial p}$,

$$p = c_1 + (c_3 + 2nt)^{-1/(2n)}h\left((c_2 + x)(c_3 + 2nt)^{-1/(2n)}\right)$$

where $h(s)$ satisfies

$$h(h')^2 + s(h')^3 + kn(h')^{n+1}h'' - Ah'h'' + As(h'')^2 - Ash'h''' = 0.$$

Case 5: $F = A\ln|p_x+C|+B$, $n = 1$ with $X = (c_3+2t)\frac{\partial}{\partial t}+(c_2+x)\frac{\partial}{\partial x}+(c_1-p-2Cx)\frac{\partial}{\partial p}$,

$$p = -\frac{2Ct(x + c_2)}{c_3 + 2t} + c_1 + 2Cc_2 + h\left(\frac{x + c_2}{\sqrt{c_3 + 2t}}\right)/\sqrt{c_3 + 2t}$$

where $h(s)$ satisfies

$$2Cc_3s + shh' + h'' - \frac{A(h'' + sh''')}{h' + Cc_3} + \frac{As(h'')^2}{(h' + Cc_3)^2} = 0.$$

4 Invariant solutions of Eq. (1.2)

In this section we present invariant solutions of Eq. (1.2) based on the group classification of Lie point symmetries given in Section 2 and the solutions of Eq. (1.3) given in the previous section.

We note that $F(p_x) = f(z)$ and thus $F' = knp_x^{n-1}f'$ and $F'' = kn(n-1)p_x^{n-2}f' + (knp_x^{n-1})^2 f''$. Thus for $f(kp_x^n)$ and n arbitrary, the family of solutions invariant under the Lie point symmetry $X = \frac{\partial}{\partial t} + c_2\frac{\partial}{\partial x} + c_1\frac{\partial}{\partial p}$ is $p = c_1 t + h(x - c_2 t)$ where $h(s)$ satisfies

$$c_1 - c_2 h_s - n|h_s|^{n-1}h_{ss} + c_2(n(n-1)|h_s|n - 3h_s f' + n^2|h_s|^{2n-2}f'')h_{ss}^2$$

$$+ c_2 n|h_s|^{n-1}f'h_{sss} = 0.$$

The invariant solutions of Eq. (1.2) differ from those of Eq. (1.3) only in two other cases where the value of k has an influence on the reduced equation, namely Case 1.1 and Case 2.1.

In Case 1.1 the reduced Eq. (3.1) is replaced by the equation

$$h - c_2 h' + kn(h')^{n-1}h'' - Ak^{1/n}h'' + Ak^{1/n}c_2 h''' = 0.$$

Similarly, the equation to be solved for h in Case 2.1 for an invariant solution of Eq. (1.2),

$$k^{1/n}c_1(h')^3 - k^{1/n}s(h')^4 - k^{(1+n)/n}n(h')^{n+2}h'' + 2As(h'')^2 - Ash'h''' = 0,$$

replaces (3.2).

In all other cases the invariant solutions of Eq. (1.2) are identical to those of Eq. (1.3) as $F(p_x) = F(z^{1/n}) = f(z)$ in these cases.

5 Boundary value problems

In this section we consider boundary value problems for Case 1.1 and 2.1 of Eq. (1.2). A boundary value problem has a solution invariant under a symmetry if not only the PDE but also the boundary and the boundary conditions are invariant under the symmetry (see for example [3] or Volume 2 of [5]).

The boundary value problem consisting of Eq.(1.2) with $f = Az^{1/n} + B, \quad n > 1$, i.e. Case 1.1, and boundary conditions

$$p(x, t_0) = p_0, \quad p(x_0, t) = p_0 + p_1(t - t_0)^{1/(1-n)}, \quad p(x_1, t) = p_0 + p_2(t - t_0)^{1/(1-n)}, \tag{5.1}$$

admits the symmetry operator

$$X = (1-n)(t - t_0)\frac{\partial}{\partial t} + (p - p_0)\frac{\partial}{\partial p}.$$

The solution to the boundary value problem is of the form

$$p(x, t) = ((n-1)(t - t_0))^{1/(1-n)}\, h(x - t_0) + p_0,$$

where $h(x-t_0)$ solves the boundary value problem

$$h + kn(h')^{n-1}h'' - Ak^{1/n}h'' = 0,$$
$$h(x_0-t_0) = (n-1)^{1/(n-1)}p_1, \quad h(x_1-t_0) = (n-1)^{1/(n-1)}p_2$$

which can be solved numerically. Note that $k = sign(p_x)^{n-1} = sign(h')^{n-1}$.

We consider Eq. (1.2) with $f = \frac{B}{z^{1/n}} + D, \quad n > -1$, that is Case 2.1, with the boundary conditions

$$p(x,t_0) = p_0, \quad p(x_0,t) = p_1, \quad p_x(x_0,t) = p_3(t-t_0)^{\frac{-1}{1+n}}.$$

This boundary value problem is invariant under the symmetry with operator

$$X = (1+n)(t-t_0)\frac{\partial}{\partial t} + (x-x_0)\frac{\partial}{\partial x}.$$

The solution to this boundary value problem is given by

$$p = h\left(((1+n)(t-t_0))^{-1/(1+n)}(x-x_0)\right)$$

where $h(s)$ satisfies

$$k^{1/n}s(h')^4 + k^{(1+n)/n}n(h')^{n+2}h'' - 2As(h'')^2 + Ash'h''' = 0$$

subject to the boundary conditions

$$h(\infty) = p_0, \quad h(0) = p_1, \quad h'(0) = p_3(1+n)^{\frac{1}{1+n}}.$$

Again, $k = sign(h')^{n-1}$.

6 Conclusion

We have applied a group classification of Lie point symmetries to a third order non linear PDE (1.3) and used the results to determine a group classification of Lie point symmetries of Eq. (1.2) which describes the filtration of a visco-elastic liquid with relaxation through a porous medium. The results of the two group classifications were used to determine families of invariant solutions for (1.2) and (1.3). In some cases the invariant solutions depend on functions which need to be determined by solving a third order ODE, called the reduced equation.

Two initial/boundary value problems were presented and reduced to boundary value problems for a second order ODE and a third order ODE which can be solved numerically.

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