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Remarks on the Waterbag Model of Dispersionless Toda Hierarchy

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Abstract

We construct the free energy associated with the waterbag model of dToda. Also, relations for conserved densities are investigated.

1 Introduction

The dispersionless Toda hierarchy (dToda) is one of the most important 2+1-dimensional integrable models. The dToda hierarchy appears in several areas of mathematics and physics, such as the integrable structure of interface dynamics [16, 28], the solution of Dirichlet boundary value problems [15], multi-slit conformal mappings of the unit disc (the radial Löwner equation [24, 25]), two-dimensional string theory [22], the large-N limit of normal matrix models [1, 27, 25] and the WDVV equation coming from topological field theory [4]. On the other hand, the dToda equation (see below) is also known as the Boyer-Finley equation, and can be used to generate a scalar-flat Kahler metric with a Killing field [5]. It also appears in the classification of self-dual Einstein metrics [3, 5] and in the twistor construction of Einstein-Weyl spaces [11, 26].

We quickly review some facts about the dToda hierarchy. It is defined by [23]

$$\begin{aligned} \frac{\partial\lambda}{\partial t_n} &= \{B_n(p),\lambda\}, \qquad \frac{\partial\lambda}{\partial\hat{t}_n} = \{\hat{B}_n(p),\lambda\}, \\ \frac{\partial\hat{\lambda}}{\partial t_n} &= \{B_n(p),\hat{\lambda}\}, \qquad \frac{\partial\hat{\lambda}}{\partial\hat{t}_n} = \{\hat{B}_n(p),\hat{\lambda}\}, \quad n=1,2,3,\cdots, \end{aligned} \tag{1.1}$$

where the Lax operators λ and $\hat{\lambda}$ are

$$\begin{aligned} \lambda &= e^p + \sum_{n=0}^{\infty} u_{n+1}e^{-np}, \\ \hat{\lambda}^{-1} &= \hat{u}_0 e^{-p} + \sum_{n=0}^{\infty} \hat{u}_{n+1}e^{np}, \end{aligned}$$

and

$$B_n(p) = [\lambda^n]_{\geq 0}, \quad \hat{B}_n(p) = [\hat{\lambda}^{-n}]_{\leq -1}.$$

Here $[\cdots]_{\geq 0}$ and $[\cdots]_{\leq -1}$ denote the non-negative part and negative part of λ^n and $\hat{\lambda}^{-n}$ respectively when expressed as Laurent series in e^p. For example,

$$B_1(p) = e^p + u_1, \quad \hat{B}_1(p) = \hat{u}_0 e^{-p}.$$

Finally, the Poisson Bracket in (1.1) is

$$\{f(t_0,p), g(t_0,p)\} = \frac{\partial f}{\partial t_0}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial t_0}.$$

One can view the complex-valued quantity λ as a local coordinate near ∞ and $\hat{\lambda}$ as a local coordinate near 0 [13].

According to dToda theory [23], there exist wave functions S, $\hat{S}$ and the dispersionless τ function F (or free energy), with

$$\begin{aligned} S(\lambda) &= \sum_{n=1}^{\infty} t_n \lambda^n + t_0 \ln\lambda - \sum_{n=1}^{\infty} \frac{\partial_{t_n} F}{n}\lambda^{-n}, \\ \hat{S}(\lambda) &= \sum_{n=1}^{\infty} \hat{t}_n \hat{\lambda}^{-n} + t_0 \ln\hat{\lambda} + \frac{\partial F}{\partial t_0} - \sum_{n=1}^{\infty} \frac{\partial_{\hat{t}_n} F}{n}\hat{\lambda}^{n}, \end{aligned}$$

such that

$$\begin{aligned} B_n(\lambda) &= \partial_{t_n} S(\lambda) = \lambda^n - \sum_{m=1}^{\infty} \frac{\partial^2_{t_n t_m} F}{m}\lambda^{-m}, \\ B_n(\hat{\lambda}) &= \partial_{t_n} \hat{S}(\hat{\lambda}) = \partial^2_{t_0 t_n} F - \sum_{m=1}^{\infty} \frac{\partial^2_{t_n \hat{t}_m} F}{m}\hat{\lambda}^{m}, \\ \hat{B}_n(\lambda) &= \partial_{\hat{t}_n} S(\lambda) = -\sum_{m=1}^{\infty} \frac{\partial^2_{\hat{t}_n t_m} F}{m}\lambda^{-m}, \\ \hat{B}_n(\hat{\lambda}) &= \partial_{\hat{t}_n} \hat{S}(\hat{\lambda}) = \hat{\lambda}^{-n} + \partial^2_{t_0 \hat{t}_n} F - \sum_{m=1}^{\infty} \frac{\partial^2_{\hat{t}_n \hat{t}_m} F}{m}\hat{\lambda}^{m}. \end{aligned}$$

In particular, we have

$$\begin{aligned} p(\lambda) &= \partial_{t_0} S(\lambda) = \ln\lambda - \sum_{m=1}^{\infty} \frac{\partial^2_{t_0 t_m} F}{m}\lambda^{-m}, \\ p(\hat{\lambda}) &= \partial_{t_0} \hat{S}(\hat{\lambda}) = \ln\hat{\lambda} + \partial^2_{t_0 t_0} F - \sum_{m=1}^{\infty} \frac{\partial^2_{t_0 \hat{t}_m} F}{m}\hat{\lambda}^{m}. \end{aligned} \tag{1.2}$$

Also,

$$\begin{aligned} H_m^+ &= \partial^2_{t_0 t_m} F = \frac{1}{m}\oint_{\infty} \lambda^m dp = \frac{1}{m}\oint_{\infty} \lambda^m \frac{d\xi}{\xi}, \\ H_m^- &= \partial^2_{t_0 \hat{t}_m} F = \frac{1}{m}\oint_{0} \hat{\lambda}^{-m} dp = \frac{1}{m}\oint_{0} \hat{\lambda}^{-m} \frac{d\xi}{\xi}, \quad m \geq 1 \end{aligned} \tag{1.3}$$

are the conserved densities of the dToda hierarchy, where $p = \ln\xi$. Then the dToda hierarchy (1.1) can be expressed as

$$\frac{\partial p(\lambda)}{\partial t_n} = \frac{\partial B_n(p(\lambda))}{\partial t_0}, \quad \frac{\partial p(\lambda)}{\partial \hat{t}_n} = \frac{\partial \hat{B}_n(p(\lambda))}{\partial t_0},$$
$$\frac{\partial p(\hat{\lambda})}{\partial t_n} = \frac{\partial B_n(p(\hat{\lambda}))}{\partial t_0}, \quad \frac{\partial p(\hat{\lambda})}{\partial \hat{t}_n} = \frac{\partial \hat{B}_n(p(\hat{\lambda}))}{\partial t_0}, \tag{1.4}$$

λ, $\hat{\lambda}$ being fixed. The systems (1.4) for all n are the conservation laws for the dToda hierarchy.

From (1.2),one knows that

$$B_1(p) = e^p + u_1 = e^p + \partial^2_{t_0 t_1} F, \quad \hat{B}_1(p) = \hat{u}_0 e^{-p} = e^{\partial^2_{t_0 t_0} F} e^{-p}.$$

Then from (1.4), one has

$$p_{\hat{t}_1} = \partial_{t_0}[e^{\partial^2_{t_0 t_0} F} e^{-p}], \quad p_{t_1} = \partial_{t_0}[e^p + \partial^2_{t_0 t_1} F].$$

Then $p_{t_1 \hat{t}_1} = p_{\hat{t}_1 t_1}$ will imply

$$\partial^2_{t_1 \hat{t}_1} F = -e^{\partial^2_{t_0 t_0} F}.$$

The latter is the dToda equation, also known as the Boyer-Finley equation.

This paper is organized as follows. In the next section, we construct the waterbag model of dToda type from the Hirota equation. Section 3 is devoted to finding the free energy associated with the waterbag model from the Landau-Ginzburg formulation in topological field theory. Also, equations for the conserved densities are obtained. In the final section, we discuss some further problems to be investigated.

2 Dispersionless Hirota Equation and Symmetry Constraints

The dToda hierarchy (1.1) (or (1.4)) is equivalent to the dispersionless Hirota equation [10]:

$$\begin{aligned} D_\mu p(\lambda) &= -\partial_{t_0} \ln[e^{p(\lambda)} - e^{p(\mu)}], \quad \hat{D}_{\hat{\mu}} p(\lambda) = -\partial_{t_0} \ln[1 - e^{p(\hat{\mu}) - p(\lambda)}], \\ D_\mu p(\hat{\lambda}) &= -\partial_{t_0} \ln[1 - e^{p(\mu) - p(\hat{\lambda})}], \quad D_{\hat{\mu}} p(\hat{\lambda}) = -\partial_{t_0} \ln[e^{p(\hat{\lambda})} - e^{p(\hat{\mu})}], \end{aligned}$$

where

$$D_\mu = \sum_{m=1}^{\infty} \frac{\mu^{-m}}{m} \partial_{t_m}, \quad \hat{D}_{\hat{\mu}} = \sum_{m=1}^{\infty} \frac{\hat{\mu}^m}{m} \partial_{\hat{t}_m}.$$

We can also express them in terms of the S-function

$$\begin{aligned} D_\mu S(\lambda) &= -\ln[\frac{e^{p(\lambda)} - e^{p(\mu)}}{\mu}], \quad \hat{D}_{\hat{\mu}} S(\lambda) = -\ln[1 - e^{p(\hat{\mu}) - p(\lambda)}], \\ D_\mu \hat{S}(\hat{\lambda}) &= -\ln[1 - e^{p(\mu) - p(\hat{\lambda})}], \quad \hat{D}_{\hat{\mu}} \hat{S}(\hat{\lambda}) = -\ln[\frac{e^{p(\hat{\lambda})} - e^{p(\hat{\mu})}}{\mu}]. \end{aligned} \tag{2.1}$$

Next, we consider the symmetry constraints. These symmetry constraints relate the “non-isospectral symmetry” , to the “isospectral symmetry” using the wave function S and the dispersionless τ function F, similar to the case with dispersion [14]. The point is that such a symmetry constraint can reduce a 2+1-dimensional dispersionless equation to a set of 1+1-dimensional systems of integrable hydrodynamic type with a finite number of dependent variables. The details can be found in [2]. From this integrable hydrodynamic system, one can find the exact solutions using the generalized hodograph method [18, 21].

Given the infinitesimal symmetries δF, the symmetry constraints can be written as [2]

$$\delta F = \sum_{i=1}^{N} \varepsilon_i S_i, \quad \text{where} \quad S_i = S(\lambda_i).$$

- Case(I): $F_{t_0} = \sum_{i=1}^{N} \varepsilon_i S_i$.
  Near ∞ we have, by (2.1),

$$\begin{aligned} p &= \ln\lambda - D_\lambda F_{t_0} = \ln\lambda - D_\lambda \sum_{i=1}^{N} \varepsilon_i S_i \\ &= \ln\lambda - \sum_{i=1}^{N} \varepsilon_i D_\lambda S_i \\ &= \ln\lambda - \sum_{i=1}^{N} \varepsilon_i \ln[\frac{e^{p(\lambda)} - e^{h^i}}{\lambda}], \quad h^i = p(\lambda_i) \\ &= \ln\lambda - \sum_{i=1}^{N} \varepsilon_i \ln[e^p - e^{h^i}] + (\sum_{i=1}^{N} \varepsilon_i) \ln\lambda. \end{aligned}$$

Let $(\sum_{i=1}^{N} \varepsilon_i) = 0$. Then one gets

$$\lambda = e^p \prod_{i=1}^{N} (e^p - e^{h^i})^{-\varepsilon_i}.$$

Moreover, near 0 we also have $\partial^2_{t_0 t_0} F = \sum_{i=1}^{N} \varepsilon_i h^i$. Then

$$\begin{aligned} p(\hat{\lambda}) &= \ln\hat{\lambda} + \partial^2_{t_0 t_0} F - \hat{D}_{\hat{\lambda}} F_{t_0} \\ &= \ln\hat{\lambda} + \sum_{i=1}^{N} \varepsilon_i h^i + \sum_{i=1}^{N} \varepsilon_i \ln[1 - e^{p(\hat{\lambda}) - h^i}] \\ &= \ln\hat{\lambda} + \sum_{i=1}^{N} \varepsilon_i h^i + \sum_{i=1}^{N} \varepsilon_i \ln[e^{-p(\hat{\lambda})} - e^{-h^i}] + (\sum_{i=1}^{N} \varepsilon_i) p(\hat{\lambda}). \end{aligned}$$

Then one obtains

$$\hat{\lambda} = e^{p - \sum_{i=1}^{N} \varepsilon_i h^i} \prod_{i=1}^{N} (e^{-p} - e^{-h^i})^{-\varepsilon_i}.$$

Actually, we can see that

$$\begin{aligned} \hat{\lambda} &= e^p e^{-(\sum_{i=1}^{N} \varepsilon_i) p} e^{-\sum_{i=1}^{N} \varepsilon_i h^i} \prod_{i=1}^{N} (e^{-p} - e^{-h^i})^{-\varepsilon_i} \\ &= e^p \prod_{i=1}^{N} (e^{h^i} - e^p)^{-\varepsilon_i} = e^p \prod_{i=1}^{N} (e^p - e^{h^i})^{-\varepsilon_i} = \lambda. \end{aligned} \tag{2.2}$$

Also,

$$H_1^+ = \sum_{i=1}^N \varepsilon_i e^{h^i}, \quad H_1^- = -e^{\sum_{i=1}^N \varepsilon_i h^i}(\sum_{i=1}^N \varepsilon_i e^{-h^i}).$$

The t_1 and $\hat{t}_1$ evolutions are

$$\begin{aligned} \partial_{t_1} h^i &= \partial_{t_0}[e^{h^i} + \sum_{i=1}^N \varepsilon_i e^{h^i}], \\ \partial_{\hat{t}_1} h^i &= \partial_{t_0}[e^{-h^i + \sum_{i=1}^N \varepsilon_i h^i}]. \end{aligned} \tag{2.3}$$

We can also express them in Hamiltonian form as

$$\begin{bmatrix} h^1 \\ h^2 \\ \vdots \\ h^N \end{bmatrix}_{t_1} = \eta^{ij}\partial_{t_0} \begin{bmatrix} \frac{\partial H_1^+}{\partial h^1} \\ \frac{\partial H_1^+}{\delta h^2} \\ \vdots \\ \frac{\partial H_1^+}{\delta h^N} \end{bmatrix}, \quad \begin{bmatrix} h^1 \\ h^2 \\ \vdots \\ h^N \end{bmatrix}_{\hat{t}_1} = \eta^{ij}\partial_{t_0} \begin{bmatrix} \frac{\partial H_1^-}{\partial h^1} \\ \frac{\partial H_1^-}{\delta h^2} \\ \vdots \\ \frac{\partial H_1^-}{\delta h^N} \end{bmatrix},$$

where

$$\eta^{ij} = \begin{bmatrix} 1+\frac{1}{\varepsilon_1} & 1 & \dots & \dots & 1 \\ 1 & 1+\frac{1}{\varepsilon_2} & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & 1 \\ 1 & 1 & \dots & 1 & 1+\frac{1}{\varepsilon_N} \end{bmatrix}.$$

As in the case of dispersionless discrete KP hierarchy [30], case(I) is called the waterbag model.
Once (I) has been done, we similarly have the following cases.

- Case(II): $F_{t_m} = \sum_{i=1}^N \varepsilon_i S_i$.
  In that case

$$\begin{aligned} B_m(\lambda) &= \lambda^m - D_\lambda F_{t_m} = \lambda^m - \sum_{i=1}^N \varepsilon_i D_\lambda S_i \\ &= \lambda^m - \sum_{i=1}^N \varepsilon_i \ln(e^p - e^{h^i}). \end{aligned}$$

Hence the Lax operator is

$$\begin{aligned} \lambda^m &= e^{mp} + u_{m-1}e^{(m-1)p} + u_{m-2}e^{(m-2)p} + \dots + u_1 e^p \\ &\quad + u_0 + \sum_{i=1}^N \varepsilon_i \ln(e^p - e^{h^i}). \end{aligned} \tag{2.4}$$

- Case (III): $F_{\hat{t}_m} = \sum_{i=1}^N \varepsilon_i S_i$.
  In that case

$$\begin{aligned}\hat{B}_m(\hat{\lambda}) &= \hat{\lambda}^{-m} + \partial^2_{t_0\hat{t}_m} F - \hat{D}_{\hat{\lambda}} F_{\hat{t}_m} = \hat{\lambda}^{-m} + \partial^2_{t_0\hat{t}_m} F - \sum_{i=1}^N \varepsilon_i \hat{D}_{\hat{\lambda}} S_i \\ &= \hat{\lambda}^{-m} + \partial^2_{t_0\hat{t}_m} F - \sum_{i=1}^N \varepsilon_i \ln(e^{-p} - e^{-h^i}).\end{aligned}$$

  Hence the Lax operator is

$$\begin{aligned}\hat{\lambda}^{-m} &= \hat{u}_m e^{-mp} + \hat{u}_{m-1} e^{-(m-1)p} + \hat{u}_{m-2} e^{-(m-2)p} + \cdots \\ &\quad + \hat{u}_1 e^{-p} + \hat{u}_0 + \sum_{i=1}^N \varepsilon_i \ln(e^{-p} - e^{-h^i}).\end{aligned} \tag{2.5}$$

3 Residue formula and free energy

In this section, we compute the free energy associated with the waterbag model of case (I) in the last section, given by equation (2.2). Also, relations for the conserved densities are investigated.

The free energy is a function $\mathbb{F}(t^1, t^2, \cdots, t^n)$ such that the associated functions

$$c_{ijk} = \frac{\partial^3 \mathbb{F}}{\partial t^i \partial t^j \partial t^k}$$

satisfy the following conditions.

- The matrix $\eta_{ij} = c_{1ij}$ is constant and non-degenerate. This together with the inverse matrix η^{ij} are used to raise and lower indices.
- The functions $c^i_{jk} = \eta^{ir} c_{rjk}$ define an associative commutative algebra with a unity element (Frobenius algebra).

The equations of associativity give a system of non-linear PDEs for $\mathbb{F}(t)$

$$\frac{\partial^3 \mathbb{F}(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 \mathbb{F}(t)}{\partial t^\mu \partial t^\gamma \partial t^\sigma} = \frac{\partial^3 \mathbb{F}(t)}{\partial t^\alpha \partial t^\gamma \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 \mathbb{F}(t)}{\partial t^\mu \partial t^\beta \partial t^\sigma}.$$

These equations constitute the Witten-Dijkgraaf-Verlinde-Verlinde (or WDVV) equations. In general, two dimensional topological field theories (TFTs) can be classified by the solutions of the WDVV equations of associativity [7, 8, 29] in the sense that a particular solution of the WDVV equations provides the primary free energy of some topological model. In fact, various classes of solutions to the WDVV equations have been obtained (see [9, 12, 13] and references therein), which turn out to be the tau-functions of dispersionless integrable hierarchies. Accordingly, investigating the solution space of the WDVV equations will deepen our understanding of 2d TFT.

The geometrical setting in which to understand the free energy $\mathbb{F}(t)$ is the Frobenius manifold [8]. Given any solution of the WDVV equation, one can construct a Frobenius manifold M associated with it. One way to construct such a manifold is derived via the Landau-Ginzburg formalism as the structure on the parameter space M of the appropriate form

$$\lambda = \lambda(p; t^1, t^2, \cdots, t^n).$$

The Frobenius structure is given by the flat metric

$$\eta(\partial,\partial') = -\sum res_{d\lambda=0}\{\frac{\partial(\lambda dp)\partial'(\lambda dp)}{d\lambda(p)}\} \tag{3.1}$$

and the tensor

$$c(\partial,\partial',\partial'') = -\sum res_{d\lambda=0}\{\frac{\partial(\lambda dp)\partial'(\lambda dp)\partial''(\lambda dp)}{d\lambda(p)dp}\} \tag{3.2}$$

defines a totally symmetric $(3,0)$-tensor c_{ijk}.

Geometrically, a solution of WDVV equation defines a multiplication

$$\circ : TM \times TM \longrightarrow TM$$

of vector fields on the parameter space M, i.e.

$$\partial_{t^\alpha} \circ \partial_{t^\beta} = c^\gamma_{\alpha\beta}(t)\partial_{t^\gamma}.$$

From $c^\gamma_{\alpha\beta}(t)$, one can construct integrable hierarchies of hydrodynamic type whose corresponding Hamiltonian densities are defined recursively by the formula [8, 9]

$$\frac{\partial^2 \psi^{(l)}_\alpha}{\partial t^i \partial t^j} = c^k_{ij}\frac{\partial \psi^{(l-1)}_\alpha}{\partial t^k}, \tag{3.3}$$

where $l \geq 1, \alpha = 1,2,\cdots,n$, and $\psi^0_\alpha = \eta_{\alpha\varepsilon}t^\varepsilon$. The integrability conditions for this systems are automatically satisfied when the c^k_{ij} are defined as above.

In the proof of the following theorem, to make the computations more easily, **one uses** $\ln\lambda$ **to replace** λ.

Theorem 1. *Let the Lax operator be defined as in* (2.2). *Then*

$$\begin{array}{ll}
(I) & \eta(\partial_{h^i},\partial_{h^j}) = \eta_{ij} = -\varepsilon_i\varepsilon_j, \quad i \neq j, \\
(II) & \eta(\partial_{h^i},\partial_{h^i}) = \eta_{ii} = -\varepsilon_i^2 + \varepsilon_i, \\
(III) & c(\partial_{h^i},\partial_{h^j},\partial_{h^k}) = c_{ijk} = \varepsilon_i\varepsilon_j\varepsilon_k, \quad i \neq j \neq k, \\
(IV) & c(\partial_{h^i},\partial_{h^i},\partial_{h^k}) = c_{iik} = \varepsilon_i\varepsilon_k\left[\varepsilon_i + \dfrac{e^{h^k}}{e^{h^i}-e^{h^k}}\right], \quad i \neq k, \\
(V) & c(\partial_{h^i},\partial_{h^i},\partial_{h^i}) = c_{iii} = \varepsilon_i^3 + \varepsilon_i\left[1 - \varepsilon_i - \displaystyle\sum_{l=1,l\neq i}^{N}\dfrac{\varepsilon_l e^{h^l}}{e^{h^i}-e^{h^l}}\right].
\end{array}$$

Proof. We see that $\frac{\partial \ln\lambda}{\partial h^i} = \varepsilon_i\frac{e^{h^i}}{\xi - e^{h^i}}$, where $p = \ln\xi$. Also, we have

$$\frac{d\ln\lambda}{dp} = 1 - \sum_{k=1}^{N}\varepsilon_k\frac{e^{h^k}}{\xi - e^{h^k}} = \frac{\prod_{k=1}^{N}(\xi - \omega_k)}{\prod_{k=1}^{N}(\xi - e^{h^k})}. \tag{3.4}$$

In the following proofs, we use the formula (3.4) and the fact that the residue at infinity is zero.
(I)

$$
\begin{aligned}
\eta(\partial_{h^i},\partial_{h^j}) &= \sum_{d\ln\lambda=0} Res \frac{\frac{\partial\ln\lambda}{\partial h^i}\frac{\partial\ln\lambda}{\partial h^j}}{\xi\frac{d\ln\lambda}{dp}} d\xi \\
&= \sum_{d\ln\lambda=0} Res \frac{\varepsilon_i \frac{e^{h^i}}{\xi-e^{h^i}} \varepsilon_j \frac{e^{h^j}}{\xi-e^{h^j}}}{\xi(1-\sum_{k=1}^N \varepsilon_k \frac{e^{h^k}}{\xi-e^{h^k}})} d\xi \\
&= \sum_{d\ln\lambda=0} Res \frac{\varepsilon_i\varepsilon_j e^{h^i} e^{h^j} \prod_{k=1}^N(\xi-e^{h^k})}{\xi(\xi-e^{h^i})(\xi-e^{h^k})\prod_{k=1}^N(\xi-\omega_k)} \\
&= -Res_{\xi=0} \frac{\varepsilon_i\varepsilon_j e^{h^i} e^{h^j} \prod_{k=1}^N(\xi-e^{h^k})}{\xi(\xi-e^{h^i})(\xi-e^{h^k})\prod_{k=1}^N(\xi-\omega_k)} = -\varepsilon_i\varepsilon_j, \quad i\neq j.
\end{aligned}
$$

(II)

$$
\begin{aligned}
\eta(\partial_{h^i},\partial_{h^i}) &= \sum_{d\ln\lambda=0} Res \frac{\varepsilon_i^2 e^{2h^i} \prod_{k=1}^N(\xi-e^{h^k})}{\xi(\xi-e^{h^i})^2\prod_{k=1}^N(\xi-\omega_k)} \\
&= -(Res_{\xi=0}+Res_{\xi=e^{h^i}}) \sum_{d\ln\lambda=0} Res \frac{\varepsilon_i^2 e^{2h^i} \prod_{k=1}^N(\xi-e^{h^k})}{\xi(\xi-e^{h^i})^2\prod_{k=1}^N(\xi-\omega_k)} \\
&= -\varepsilon_i^2 - \frac{\varepsilon_i^2 e^{2h^i} \prod_{k=1,k\neq i}^N(e^{h^i}-e^{h^k})}{e^{h^i}\prod_{k=1}^N(e^{h^i}-\omega_k)} \\
&= -\varepsilon_i^2 - \varepsilon_i^2 e^{h^i}\frac{1}{-\varepsilon_i e^{h^i}} = -\varepsilon_i^2+\varepsilon_i.
\end{aligned}
$$

(III)

$$
\begin{aligned}
c(\partial_{h^i},\partial_{h^j},\partial_{h^k}) &= \sum_{d\ln\lambda=0} Res \frac{\varepsilon_i e^{h^i}\varepsilon_j e^{h^j}\varepsilon_k e^{h^k}\prod_{l=1}^N(\xi-e^{h^l})}{\xi(\xi-e^{h^i})(\xi-e^{h^j})(\xi-e^{h^j})\prod_{l=1}^N(\xi-\omega_l)} d\xi \\
&= -Res_{\xi=0} \frac{\varepsilon_i e^{h^i}\varepsilon_j e^{h^j}\varepsilon_k e^{h^k}\prod_{l=1}^N(\xi-e^{h^l})}{\xi(\xi-e^{h^i})(\xi-e^{h^j})(\xi-e^{h^k})\prod_{l=1}^N(\xi-\omega_l)} d\xi \\
&= \varepsilon_i\varepsilon_j\varepsilon_k, \quad i\neq j\neq k.
\end{aligned}
$$

(IV)

$$
\begin{aligned}
c(\partial_{h^i},\partial_{h^i},\partial_{h^k}) &= \sum_{d\ln\lambda=0} Res \frac{\varepsilon_i^2 e^{2h^i}\varepsilon_k e^{h^k}\prod_{l=1}^N(\xi-e^{h^l})}{\xi(\xi-e^{h^i})^2(\xi-e^{h^k})\prod_{l=1}^N(\xi-\omega_l)} d\xi \\
&= -[Res_{\xi=0}+Res_{\xi=e^{h^i}}] \frac{\varepsilon_i^2 e^{2h^i}\varepsilon_k e^{h^k}\prod_{l=1}^N(\xi-e^{h^l})}{\xi(\xi-e^{h^i})^2(\xi-e^{h^k})\prod_{l=1}^N(\xi-\omega_l)} d\xi \\
&= -[-\varepsilon_i^2\varepsilon_k + \frac{\varepsilon_i^2 e^{2h^i}\varepsilon_k e^{h^k}\prod_{l=1,l\neq i}^N(e^{h^i}-e^{h^l})}{e^{h^i}(e^{h^i}-e^{h^k})\prod_{l=1}^N(e^{h^i}-\omega_l)}] \\
&= -[-\varepsilon_i^2\varepsilon_k - \frac{\varepsilon_i\varepsilon_k e^{h^k}}{e^{h^i}-e^{h^k}}] = \varepsilon_i\varepsilon_k[\varepsilon_i + \frac{e^{h^k}}{e^{h^i}-e^{h^k}}].
\end{aligned}
$$

(V)

$$\begin{aligned}
c(\partial_{h^i},\partial_{h^i},\partial_{h^i}) &= \sum_{d\ln\lambda=0} Res \frac{\varepsilon_i^3 e^{3h^i}\prod_{l=1}^N(\xi-e^{h^l})}{\xi(\xi-e^{h^i})^3\prod_{l=1}^N(\xi-\omega_l)}d\xi \\
&= -[Res_{\xi=0}+Res_{\xi=e^{h^i}}]\frac{\varepsilon_i^3 e^{3h^i}\prod_{l=1}^N(\xi-e^{h^l})}{\xi(\xi-e^{h^i})^3\prod_{l=1}^N(\xi-\omega_l)}d\xi \\
&= -\{-\varepsilon_i^3+\varepsilon_i^3 e^{3h^i}\frac{d}{d\xi}[\frac{\prod_{l=1,l\neq i}^N(\xi-e^{h^l})}{\xi\prod_{l=1}^N(\xi-\omega_l)}]\,|_{\xi=e^{h^i}}\} \\
&= \varepsilon_i^3+\varepsilon_i^3 e^{3h^i}\frac{1-\varepsilon_i-\sum_{l=1,l\neq i}^N\frac{\varepsilon_l e^{h^l}}{e^{h^i}-e^{h^l}}}{\varepsilon_i^2 e^{3h^i}} \\
&= \varepsilon_i^3+\varepsilon_i(1-\varepsilon_i-\sum_{l=1,l\neq i}^N\frac{\varepsilon_l e^{h^l}}{e^{h^i}-e^{h^l}}).
\end{aligned}$$

■

Let's define $\Omega=\sum_{i=1}^N\frac{\partial}{\partial h^i}$. Then we can verify directly that

$$\eta(\partial_{h^i},\partial_{h^j})=c(\partial_{h^i},\partial_{h^j},\Omega)=\sum_{k=1}^N c(\partial_{h^i},\partial_{h^j},\partial_{h^k}). \tag{3.5}$$

Also, from the Theorem, it's not difficult to check directly the compatibility (or Egorov's) condition

$$\partial_{h^i}c_{lmn}=\partial_{h^l}c_{imn},\quad i,l,m,n=1\cdots N.$$

Hence one can get the free energy associated with (2.2) as

$$\begin{aligned}
F(\vec{h}) &= \sum_{1\le i<j<k\le N}\varepsilon_i\varepsilon_j\varepsilon_k h^i h^j h^k+\frac{1}{6}\sum_{i=1}^N(\varepsilon_i-\varepsilon_i^2+\varepsilon_i^3)(h^i)^3 \\
&+ \frac{1}{2}\sum_{i\neq k}^N\varepsilon_i^2\varepsilon_k(h^i)^2h^k \\
&+ \frac{1}{2}\sum_{1\le i<k\le N}\varepsilon_i\varepsilon_k[Li_3(e^{h^i-h^k})+Li_3(e^{h^k-h^i})],
\end{aligned} \tag{3.6}$$

where $Li_3(e^x)=\sum_{k=1}^\infty\frac{e^{kx}}{k^3}$ is the poly-logarithmic function. Moreover, from (3.5), one knows that $t^1=\sum_{i=1}^N h^i$.

This solution looks like (but is different to) the formula related to classical Lie algebras in [19]. In [6], the solution of the WDVV equation associated with the waterbag reduction for dispersionless KP(dKP) is found and, using this solution, one can construct the recursive operator of the conserved densities in (3.3). Hence the bi-Hamiltonian structure of the waterbag reduction of the dKP hierarchy can be found. For more solutions of the WDVV equation associated with integrable hydrodynamic systems, one should refer to [17]. We remark that the free energy (3.6) is invariant under any permutation of $(h^1,h^2,\cdots h^N)$, which is different from the formula in [8].

Furthermore, we have

$$
\begin{aligned}
c^{\gamma}_{\alpha\beta} &= 0, \quad \alpha\neq\beta\neq\gamma; \quad c^{\beta}_{\alpha\alpha}=\varepsilon_\alpha\frac{e^{h^\alpha}}{e^{h^\alpha}-e^{h^\beta}}, \quad \alpha\neq\beta;\\
c^{\beta}_{\alpha\beta} &= \varepsilon_\beta\frac{e^{h^\beta}}{e^{h^\alpha}-e^{h^\beta}}, \quad \alpha\neq\beta; \quad c^{\alpha}_{\alpha\alpha}=1-\sum_{\gamma\neq\alpha}\varepsilon_\gamma\frac{e^{h^\gamma}}{e^{h^\alpha}-e^{h^\gamma}}.
\end{aligned}
\tag{3.7}
$$

If we define $\phi_i=\frac{\partial\ln\lambda}{\partial h^i}=\varepsilon_i\frac{e^{h^i}}{e^p-e^{h^i}}$, then we have

$$
\phi_i\phi_j=c^l_{ij}\phi_l+Q_{ij}\frac{\partial\ln\lambda}{\partial h^i},
\tag{3.8}
$$

where

$$
Q_{ij}=\begin{cases} -\phi_i, & i=j,\\ 0, & i\neq j.\end{cases}
$$

From (3.5), one knows that Ω is the unit element of the associative algebra (3.8).

Now, we have the following

Theorem 2. *Let H^+_n and H^-_n be the conserved densities defined in* (1.3). *Then one has*

$$
(I)\,\frac{\partial^2H^+_n}{\partial h^i\partial h^j}=c^k_{ij}\frac{\partial H^+_n}{\partial h^k}, \quad (II)\,\frac{\partial^2H^-_n}{\partial h^i\partial h^j}=c^k_{ij}\frac{\partial H^-_n}{\partial h^k}.
$$

Proof. (I)

$$
\begin{aligned}
\frac{\partial^2H^+_n}{\partial h^i\partial h^j} &= \frac{\partial}{\partial h^i}\oint_\infty\lambda(\xi)^{n-1}\frac{\partial\lambda}{\partial h^j}\xi^{-1}d\xi=\frac{\partial}{\partial h^i}\oint_\infty\lambda(\xi)^{n-1}\lambda\frac{\partial\ln\lambda}{\partial h^j}\xi^{-1}d\xi\\
&= \frac{\partial}{\partial h^i}\oint_\infty\lambda(\xi)^n\varepsilon_j\frac{e^{h^j}}{e^p-e^{h^j}}\xi^{-1}d\xi\\
&= n\oint_\infty\lambda(\xi)^n\varepsilon_i\frac{e^{h^i}}{e^p-e^{h^i}}\varepsilon_j\frac{e^{h^j}}{e^p-e^{h^j}}\xi^{-1}d\xi+\oint_\infty\lambda(\xi)^n\varepsilon_j\frac{\partial}{\partial h^i}(\frac{e^{h^j}}{e^p-e^{h^j}})\xi^{-1}d\xi\\
&= n\oint_\infty\lambda(\xi)^n(c^k_{ij}\phi_k\xi^{-1}d\xi+Q_{ij}\frac{d\lambda}{\lambda})-\oint_\infty\lambda(\xi)^n\varepsilon_j\frac{\partial}{\partial p}(\frac{e^{h^j}}{e^p-e^{h^j}})\xi^{-1}d\xi\\
&= nc^k_{ij}\oint_\infty\lambda(\xi)^n\phi_k\xi^{-1}d\xi+n\oint_\infty\lambda(\xi)^{n-1}Q_{ij}d\lambda\\
&- \oint_\infty\lambda(\xi)^n[\frac{\partial}{\partial p}(\varepsilon_j\frac{e^{h^j}}{e^p-e^{h^j}})]\xi^{-1}d\xi\\
&= c^k_{ij}\frac{\partial}{\partial h^k}\oint_\infty\lambda^n\xi^{-1}d\xi-\oint_\infty\frac{\partial}{\partial p}[\lambda(\xi)^n\varepsilon_j\frac{e^{h^j}}{e^p-e^{h^j}}]dp\\
&= c^k_{ij}\frac{\partial H^+_n}{\partial h^k}, \quad n\geq 1.
\end{aligned}
$$

(II) The calculation is similar. ■

4 Concluding remarks

We find the free energy associated with the waterbag model (2.2) using the Landau-Ginzburg formulation. From the free energy, one can establish the equations for the conserved densities H_n^+ and H_n^-. Unlike the waterbag model of dKP [6], here one can't construct the recursive operator for H_n^+ or H_n^- from Theorem 3.2. Therefore, the bi-Hamiltonian structure of (2.3) is still unknown. On the other hand, we can construct an integrable hierarchy via (3.3) and (3.7); however, it won't be the dToda hierarchy. Finally, finding the free energies associated with (2.4) and (2.5) would also be very interesting.

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