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Partially Solvable Spin Chains and QES Spin Models

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Abstract

In this paper we prove an extension of the usual freezing trick argument which can be applied to a number of quasi-exactly solvable spin models of Calogero–Sutherland type. In order to illustrate the application of this method we analyze a partially solvable spin chain presenting near-neighbors interactions which was introduced and studied in J. Phys. A: Math. Theor. **40** (2007) 1857–1883; Nucl. Phys. **789** (2008) 452–482. Our discussion focuses on the existence of integer eigenvalues.

1 Introduction

In the last few years there has been a revival of interest in solvable spin chains, due in part to their remarkable connections with SUSY Yang–Mills and string theories [16, 2, 20, 1, 10, 11]. The first example of such chains is the spin $1/2$ Heisenberg chain [14], whose Hamiltonian is given by

$$\widetilde{\mathsf{H}}_{\mathrm{He}} = \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1} \,.$$

This chain describes a set of N spins on a lattice with short-range, position-independent interactions, and it is well known that it can be exactly solved using the Bethe ansatz [3]. In the above equation $\mathbf{S}_i$ denotes the spin operator of the i-th site, the sum runs from 1 to N (as always hereafter), and we define $\mathbf{S}_{N+1} = \mathbf{S}_1$. For future reference it is convenient to subtract the ground state energy from the above Hamiltonian and write it in the less conventional form

$$\mathsf{H}_{\mathrm{He}} = \sum_i (1 - S_{i,i+1}) \,, \tag{1.1}$$

where S_{ij} is the operator exchanging the i-th and j-th spins

A different kind of solvable spin chain, independently introduced by Haldane [13] and Shastry [21], describes an arrangement of N spins on a circle interacting pairwise with strength inversely proportional to the chord distance. The Hamiltonian of the Haldane–Shastry (HS) chain can be written as

$$\mathsf{H}_{\mathrm{HS}} = \frac{1}{2} \sum_{i<j} \sin(\xi_i - \xi_j)^{-2} (1 - S_{ij}) \,, \qquad \xi_i = \frac{i\pi}{N} \,, \tag{1.2}$$

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in terms of the aforementioned spin exchange operators S_{ij}. Although the particles' spin in the original HS chain was assumed to be $1/2$, one can more generally consider particles with n internal degrees of freedom transforming under the fundamental representation of $\mathrm{su}(n)$.

The ultimate reason for the solvability of the HS chain lies in its intimate connection with the spin Sutherland model of A_N type [23, 24]. Indeed, Polychronakos [17] noted that the complete integrability of the HS chain could be inferred from that of the spin Sutherland model by suitably taking a strong coupling limit (the so-called "freezing trick"). By applying the same technique to the A_N spin Calogero model [4], the latter author arrived at the celebrated spin chain that is nowadays referred to as the Polychronakos–Frahm (PF) chain, namely [18]

$$\mathsf{H}_{\mathrm{PF}} = \sum_{i<j} (\xi_i - \xi_j)^{-2} (1 - S_{ij}) . \tag{1.3}$$

Here the chain sites ξ_i are the coordinates of the unique maximum of the scalar part of the potential of the Calogero spin model of A_N type, i.e., the only solutions to the system

$$\xi_i = \sum_{j \neq i} \frac{1}{\xi_i - \xi_j} , \qquad i = 1, \dots, N$$

such that $\xi_1 < \cdots < \xi_N$. Contrary to what happens in the HS chains, in this case the sites are no longer equispaced. The partition function of this chain was successfully computed by Polychronakos as an appropriate limit of the quotient of the partitions functions of the spin and spinless Calogero models. Variations of this technique were developed to analyze spin chains with free parameters related to BC_N Calogero–Sutherland (CS) models [5, 9].

The bottom line of Polychronakos's freezing trick is that, provided that some technical conditions are satisfied, one can associate to every solvable spin model of CS type H a solvable spin chain H whose spectrum can be computed from the knowledge of the partition functions of H and its scalar counterpart. Given the influence that quasi-exactly solvable (QES) quantum systems [25, 22, 26] have exerted in various areas of Mathematical Physics, it is natural to wonder whether Polychronakos's freezing trick argument can be extended to systems for which only a proper subset of its spectrum is explicitly known. Since the explicit knowledge of the full spectrum of H is required in order to compute its partition function, it is clear that this extension is nontrivial.

An interesting type of QES spin models with near-neighbors interactions has been recently introduced in Refs. [6, 7]. The associated spin chain reads

$$\mathsf{H}_{\mathrm{NN}} = \sum_{i} (\xi_i - \xi_{i+1})^{-2} (1 - S_{i,i+1}) , \tag{1.4}$$

where the sites' coordinates $\xi_1 < \cdots < \xi_N$ are given by the only solution to the algebraic equations

$$\xi_i = \frac{1}{\xi_i - \xi_{i-1}} + \frac{1}{\xi_i - \xi_{i+1}} , \qquad i = 1, \dots, N \tag{1.5}$$

compatible with the above ordering. A novel feature of this chain is that it is somehow intermediate between the Heisenberg chain, presenting short-range, position independent interactions, and the spin chains of HS type, where the interactions are long ranged and depend on the sites' positions. As a matter of fact, one would recover (1.1) if the spins were equispaced, whereas the equation of the sites is obtained from that of the PF chain by keeping only the summands involving nearest neighbors. The analysis of the spectrum of the operator (1.4), carried out in Ref. [8], is thus naturally related to the general problem mentioned in the preceding paragraph.

The paper is organized as follows. In Section 2 we prove a general convergence result relating QES spin systems of CS type and partially solvable spin chains. The hypotheses of the lemma have been written in such a way that can be readily verified for a wide panoply of such models. In Section 3 we make use of the freezing lemma to study the algebraic energies of the spin chain (1.4). This section is mainly based on Ref. [8]. The paper concludes with a digression on the existence of integer energy levels in this and some other related models with short-range interactions.

2 A freezing lemma

In this section we shall state and prove a lemma which allows to obtain a (partially) solvable spin chain starting with a QES spin model. In some sense, this result can be understood as a refinement of Polychronakos's freezing trick [18]. The applicability conditions of the lemma have been stated in such a way that can be readily verified for most models of CS type.

Let us introduce the necessary definitions. Let Σ be a finite-dimensional Hilbert space and consider a spin Hamiltonian

$$H = -\sum_i \partial_{x_i}^2 + 2a\left[U_1(\mathbf{x}) + \mathsf{h}(\mathbf{x})\right] + a^2 U_2(\mathbf{x}) \tag{2.1}$$

acting on (a dense subset of) $L^2(C)\otimes\Sigma$, C being a domain in $\mathbb{R}^N$. Here $a>0$ is a coupling parameter and we denote by $U_i:\mathbb{R}^N\to\mathbb{R}$ and $\mathsf{h}:\mathbb{R}^N\to\operatorname{End}(\Sigma)$ sufficiently regular scalar and spin multiplication operators. We need also consider an associated scalar Hamiltonian

$$H^{\mathrm{sc}} = -\sum_i \partial_{x_i}^2 + 2aU_1(\mathbf{x}) + a^2 U_2(\mathbf{x})\,, \tag{2.2}$$

defined on $L^2(\mathbb{R}^N)$. Both H and H^{sc} are assumed to be lower bounded, and we assume that H^{sc} has an eigenvalue at the bottom of the spectrum. Let $\hat\mu$ denote the normalized ground state function of H^{sc}, which can be chosen to be strictly positive [19]. There is no loss of generality in assuming that the ground state energy of H^{sc} is 0, and we have introduced a factor 2 above in the above equations for convenience.

We shall assume that $\hat\mu$ is of the form

$$\hat\mu(\mathbf{x},a) = C(a)\,\mathrm{e}^{a\lambda(\mathbf{x})}\,,$$

where the function λ has a unique maximum ξ in C, which must also coincide with a global minimum of U_2. For the sake of simplicity, we also assume that this maximum is hyperbolic. We define the spin chain Hamiltonian $\mathsf{H}\in\operatorname{End}(\Sigma)$ as

$$\mathsf{H} = \mathsf{h}(\xi)\,. \tag{2.3}$$

Lemma 1. *Let Ψ be an eigenfunction of H with energy E. Assume that there exists a Σ-valued polynomial $F\in\mathbb{C}[\mathbf{x},a^{-1}]\otimes\Sigma$ such that*

$$\hat\mu^{-1}\Psi - F = \mathscr{O}(a^{-1})\,, \tag{2.4}$$

and suppose, moreover, that this eigenfunction can be normalized so that

$$\hat{\mu}^{-1}\Psi \in \mathbb{C}[\mathbf{x}, a^{-1}] \otimes \Sigma. \tag{2.5}$$

Then the limits

$$\chi = \lim_{a\to\infty} F(\xi), \qquad \mathsf{E} = \lim_{a\to\infty} \frac{E}{2a}$$

exist, and

$$\mathsf{H}\chi = \mathsf{E}\chi. \tag{2.6}$$

Proof. A first observation is that $\hat{\mu}^2$ converges to a Dirac delta distribution supported at ξ as a tends to infinity, i.e., that

$$\lim_{a\to\infty} \int_C \hat{\mu}^2 \phi \, \mathrm{d}\mathbf{x} = \phi(\xi) \tag{2.7}$$

for every compactly supported smooth function ϕ. The maximum ξ being nondegenerate, this follows from the standard argument behind the proof of Laplace's method.

The existence of the limit $\lim_{a\to\infty} \frac{E}{a}$ is easy, as the ground state energy of H^{sc} has been shifted to zero and $H - H^{\mathrm{sc}}$ depends linearly on a. The limit defining χ must also exist due to the polynomial dependence of F on a^{-1}.

The self-adjointness of H^{sc} and Eq. (2.7) readily imply that

$$\int_C \hat{\mu}\, H^{\mathrm{sc}}(\Psi)\, \mathrm{d}\mathbf{x} = \int_C H^{\mathrm{sc}}(\hat{\mu})\, \Psi \, \mathrm{d}\mathbf{x} = 0$$

$$\int_C \hat{\mu}\, \mathsf{h}\Psi \, \mathrm{d}\mathbf{x} = \int_C \hat{\mu}^2 \left[\mathsf{h}F + \mathscr{O}(a^{-1})\right] \mathrm{d}\mathbf{x} = \mathsf{H}\chi + \mathscr{O}(a^{-1}).$$

If we now use that $H = H^{\mathrm{sc}} + 2a\mathsf{h}(\mathbf{x})$ and Eq. (2.7) to write

$$\int_C \hat{\mu}\, H^{\mathrm{sc}}(\Psi)\, \mathrm{d}\mathbf{x} + 2a \int_C \hat{\mu}\mathsf{h}\Psi \, \mathrm{d}\mathbf{x} = \int_C \hat{\mu}\, H(\Psi)\, \mathrm{d}\mathbf{x} = E\chi + \mathscr{O}(1),$$

substitute the former equations into this identity and take the limit $a \to \infty$, we readily derive Eq. (2.6). ■

Remark 1. Lemma 1 asserts that χ, when nonzero, is an eigenvector of the spin chain (2.3) with energy E.

Remark 2. An analogous result for models with hyperbolic or trigonometric potentials can be similarly established by replacing the space $\mathbb{C}[\mathbf{x}, a^{-1}]$ for $\mathbb{C}[\mathbf{z}, a^{-1}]$, where $z_i = \mathrm{e}^{2x_i}$ or $z_i = \mathrm{e}^{2\mathrm{i}x_i}$ in each case. The key point for the application of the above Lemma is the polynomial dependance of Ψ on a, which is shared by Calogero–Sutherland models and a wide range of QES spin systems.

Remark 3. Physically, the main idea is that $|\Psi|$ becomes sharply peaked at ξ, so that the particles 'freeze" at the point ξ (which can be proven to be a global minimum of the potential U_2, and thus a stable critical point of the classical movement). A detailed semi-rigorous justification of the classical freezing trick along these lines can be found in Ref. [8].

3 Models with near-neighbors interactions

In this section we shall show how the freezing lemma can be used to obtain some families of eigenvectors of the spin chain Hamiltonian (1.4). The discussion presented in this section is mainly based on Refs. [5, 7, 8].

Let Σ be the Hilbert space of the internal degrees of freedom of N particles of spin $M \in \frac{1}{2}\mathbb{N}$ and consider the domain

$$C = \left\{ \mathbf{x} \in \mathbb{R}^N : x_1 < \cdots < x_N \right\}.$$

We will be interested in the spin Hamiltonian densely defined on $L^2(C) \otimes \Sigma$ given by

$$H_{\mathrm{NN}} = -\sum_i \partial_{x_i}^2 + a^2 r^2 + \sum_i \frac{2a^2}{(x_i - x_{i-1})(x_i - x_{i+1})} + \sum_i \frac{2a}{(x_i - x_{i+1})^2} \,(a - S_{i,i+1}) \tag{3.1}$$

and its scalar counterpart

$$H_{\mathrm{NN}}^{\mathrm{sc}} = H|_{S_{i,i+1} \to 1} \,. \tag{3.2}$$

Here and in what follows a is a positive constant and we use the customary notation $r^2 = \sum_i x_i^2$. This model can be understood as a QES short-range version of the usual spin Calogero system.

At this point we need to introduce some additional notation. Let us denote the elements of the standard basis of Σ by $|s_1 \dots s_N\rangle$, where $s_i = -M, -M+1, \dots, M$, and define the projector Λ in $L^2(C) \otimes \Sigma$ to be the total symmetrizer under particle permutations, that is, under the simultaneous exchange of the i-th and j-th coordinates and spins, for any $1 \leqslant i, j \leqslant N$. We shall also define the subspace $\Sigma' \subset \Sigma$ of spin states $|s\rangle \in \Sigma$ such that $\sum_i |s_{i,i+1}^+\rangle$ is symmetric, $|s_{ij}^\pm\rangle$ being defined by

$$\Lambda\big(g_\pm(x_i,x_j)|s\rangle\big) = \sum_{i<j} g_\pm(x_i,x_j)|s_{ij}^\pm\rangle\,, \tag{3.3}$$

where $g_\pm$ is an arbitrary smooth function such that $g_\pm(x,y) = \pm g_\pm(x,y)$. A complete characterization of this subspace can be found in Ref. [7].

The main result in Ref. [7] concerning the spectrum of H can be summarized in the following

Theorem 1. *Let*

$$\alpha = N\Big(a + \frac{1}{2}\Big) - \frac{3}{2}\,, \qquad \beta \equiv \beta(m) = 1 - m - N\Big(a + \frac{1}{2}\Big)\,, \qquad t = \frac{2r^2}{N\bar{x}^2} - 1\,,$$
$$E_0 = Na(2a+1)\,, \qquad \mu = \mathrm{e}^{-\frac{a}{2}r^2} \prod_i |x_i - x_{i+1}|^a\,,$$

where $\bar{x} = \frac{1}{N}\sum_i x_i$ is the center of mass coordinate. The Hamiltonian H_0 possesses the following families of spin eigenfunctions with eigenvalue $E_{lm} = E_0 + 2a(2l+m)$, with $l \geqslant 0$ and m as indicated in each case:

$$\Psi_{lm}^{(0)} = \mu \bar{x}^m L_l^{-\beta}(ar^2) P_{\lfloor \frac{m}{2} \rfloor}^{(\alpha,\beta)}(t)\, \Phi^{(0)}\,, \qquad m \geqslant 0\,,$$
$$\Psi_{lm}^{(1)} = \mu \bar{x}^{m-1} L_l^{-\beta}(ar^2) P_{\lfloor \frac{m-1}{2} \rfloor}^{(\alpha+1,\beta)}(t)\, \big(\Phi^{(1)} - \bar{x}\Phi^{(0)}\big)\,, \qquad m \geqslant 1\,,$$

$$\Psi^{(2)}_{lm} = \mu\,\bar{x}^{m-2} L_l^{-\beta}(ar^2)\bigg[P^{(\alpha+2,\beta)}_{\lfloor\frac{m}{2}\rfloor-1}(t)\big(\Phi^{(2)} - 2\bar{x}\Phi^{(1)}\big) + \bar{x}^2\bigg(P^{(\alpha+2,\beta)}_{\lfloor\frac{m}{2}\rfloor-1}(t) - \frac{2(\alpha+1)}{2\lfloor\frac{m-1}{2}\rfloor+1}P^{(\alpha+1,\beta)}_{\lfloor\frac{m}{2}\rfloor-1}(t)\bigg)\Phi^{(0)}\bigg], \quad m \geqslant 2,$$

$$\widetilde{\Psi}^{(2)}_{lm} = \mu\,\bar{x}^{m-2} L_l^{-\beta}(ar^2)\bigg[P^{(\alpha+2,\beta)}_{\lfloor\frac{m}{2}\rfloor-1}(t)\big(\widetilde{\Phi}^{(2)} - 2\bar{x}\Phi^{(1)}\big) + \bar{x}^2\bigg(P^{(\alpha+2,\beta)}_{\lfloor\frac{m}{2}\rfloor-1}(t) + \frac{2(\alpha+1)}{\big(2\lfloor\frac{m-1}{2}\rfloor+1\big)(N-1)}P^{(\alpha+1,\beta)}_{\lfloor\frac{m}{2}\rfloor-1}(t)\bigg)\Phi^{(0)}\bigg], \quad m \geqslant 2,$$

$$\Psi^{(3)}_{lm} = \mu\,\bar{x}^{m-3} L_l^{-\beta}(ar^2)\bigg[\frac{2}{3N}P^{(\alpha+3,\beta)}_{\lfloor\frac{m-3}{2}\rfloor}(t)\sum_i x_i^3 + \bar{x}^3\varphi_m(t)\bigg]\Phi^{(0)}, \qquad m \geqslant 3,$$

$$\widehat{\Psi}^{(3)}_{lm} = \mu\,\bar{x}^{m-3} L_l^{-\beta}(ar^2)\bigg[P^{(\alpha+3,\beta)}_{\lfloor\frac{m-3}{2}\rfloor}(t)\big(\widehat{\Phi}^{(3)} - 2\bar{x}\Phi^{(2)}\big) + 2\bar{x}^2\bigg(P^{(\alpha+3,\beta)}_{\lfloor\frac{m-3}{2}\rfloor}(t) + \frac{2(\alpha+2)}{2\lfloor\frac{m}{2}\rfloor-1}P^{(\alpha+2,\beta)}_{\lfloor\frac{m-3}{2}\rfloor}(t)\bigg)\Phi^{(1)} - 2\bar{x}^3\bigg(\frac{1}{3}P^{(\alpha+3,\beta)}_{\lfloor\frac{m-3}{2}\rfloor}(t) + \frac{1}{2\lfloor\frac{m}{2}\rfloor-1}P^{(\alpha+2,\beta)}_{\lfloor\frac{m-3}{2}\rfloor}(t) + \big(1-(-1)^m\big)\frac{2\alpha+3}{2m(m-2)}P^{(\alpha+1,\beta)}_{\lfloor\frac{m-3}{2}\rfloor}(t)\bigg)\Phi^{(0)}\bigg], \quad m \geqslant 3,$$

$$\Psi^{(4)}_{lm} = \mu\,\bar{x}^{m-4} L_l^{-\beta}(ar^2)\bigg[\frac{3}{2\big(\lfloor\frac{m-3}{2}\rfloor+\frac{1}{2}\big)}\bar{x}^2P^{(\alpha+3,\beta)}_{\lfloor\frac{m}{2}\rfloor-2}(t)\,\Phi^{(2)} + \bigg(\frac{3}{2}\bar{x}^3\phi_m(t) - \frac{1}{N}P^{(\alpha+4,\beta)}_{\lfloor\frac{m}{2}\rfloor-2}(t)\sum_i x_i^3\bigg)\Phi^{(1)} + \bigg(\frac{1}{N}\bar{x}P^{(\alpha+4,\beta)}_{\lfloor\frac{m}{2}\rfloor-2}(t)\sum_i x_i^3 + \frac{3}{2}\bar{x}^4\chi_m(t)\bigg)\Phi^{(0)}\bigg], \quad m \geqslant 4.$$

Here

$$\Phi^{(k)} = \Lambda(x_1^k|s\rangle), \qquad \widetilde{\Phi}^{(2)} = \Lambda(x_1x_2|s\rangle), \qquad \widehat{\Phi}^{(3)} = \Lambda(x_1x_2(x_1-x_2)|s\rangle),$$

where the spin state $|s\rangle$ is symmetric under S_{12} and belongs to Σ' for the eigenfunction $\widetilde{\Psi}^{(2)}_{lm}$, and is antisymmetric under S_{12} for the eigenfunction $\widehat{\Psi}^{(3)}_{lm}$. The functions φ_m, ϕ_m and χ_m are polynomials given explicitly by

$$\varphi_m = \frac{m+2\alpha+2}{m-1}P^{(\alpha+2,\beta-2)}_{\frac{m}{2}} - P^{(\alpha+3,\beta-1)}_{\frac{m}{2}-1} - \frac{4\alpha+7}{m-1}P^{(\alpha+2,\beta-1)}_{\frac{m}{2}-1} + \frac{1}{3}P^{(\alpha+3,\beta)}_{\frac{m}{2}-2},$$

$$\phi_m = P^{(\alpha+4,\beta-1)}_{\frac{m}{2}-1} - 2P^{(\alpha+3,\beta-1)}_{\frac{m}{2}-1} - \frac{m+2\alpha+3}{(m-1)(m-3)}P^{(\alpha+2,\beta-1)}_{\frac{m}{2}-1} - \frac{1}{3}P^{(\alpha+4,\beta)}_{\frac{m}{2}-2} + \frac{m+2\alpha-1}{m-3}P^{(\alpha+3,\beta)}_{\frac{m}{2}-2},$$

$$\chi_m = \frac{3m+2\alpha}{(m-1)(m-3)}P^{(\alpha+2,\beta-1)}_{\frac{m}{2}-1} + \frac{2m-7}{m-3}P^{(\alpha+3,\beta-1)}_{\frac{m}{2}-1} - P^{(\alpha+4,\beta-1)}_{\frac{m}{2}-1} - \frac{m+2\alpha+2}{(m-1)(m-3)}P^{(\alpha+2,\beta)}_{\frac{m}{2}-2} - \frac{m+2\alpha}{m-3}P^{(\alpha+3,\beta)}_{\frac{m}{2}-2} + \frac{1}{3}P^{(\alpha+4,\beta)}_{\frac{m}{2}-2},$$

for even m, and

$$\varphi_m = 2P^{(\alpha+2,\beta-1)}_{\frac{m-1}{2}} - P^{(\alpha+3,\beta-1)}_{\frac{m-1}{2}} + \frac{1}{3}P^{(\alpha+3,\beta)}_{\frac{m-3}{2}} + \frac{m+2\alpha+2}{m(m-2)}P^{(\alpha+1,\beta)}_{\frac{m-3}{2}} - \frac{m+2\alpha+2}{m-2}P^{(\alpha+2,\beta)}_{\frac{m-3}{2}},$$

$$\phi_m = P^{(\alpha+4,\beta-1)}_{\frac{m-3}{2}} - \frac{2m-5}{m-2}P^{(\alpha+3,\beta)}_{\frac{m-3}{2}} - \frac{1}{3}P^{(\alpha+4,\beta)}_{\frac{m-5}{2}} + \frac{m+2\alpha-1}{m-2}P^{(\alpha+3,\beta)}_{\frac{m-5}{2}},$$

$$\chi_m = \frac{2m-3}{m(m-2)}P^{(\alpha+2,\beta-1)}_{\frac{m-3}{2}} + \frac{2(m-3)}{m-2}P^{(\alpha+3,\beta-1)}_{\frac{m-3}{2}} - P^{(\alpha+4,\beta-1)}_{\frac{m-3}{2}} - \frac{m+2\alpha+1}{m(m-2)}P^{(\alpha+2,\beta)}_{\frac{m-5}{2}} - \frac{m+2\alpha}{m-2}P^{(\alpha+3,\beta)}_{\frac{m-5}{2}} + \frac{1}{3}P^{(\alpha+4,\beta)}_{\frac{m-3}{2}},$$

for odd m.

In the statement of this theorem, L^k_n and $P^{(a,b)}_n$ respectively denote the Laguerre and Jacobi polynomials. As a consequence of Theorem 1, the normalized ground state function of $H^{\mathrm{sc}}_{\mathrm{NN}}$ is given by $\hat{\mu} = \mu/\|\mu\|$. Gerschgorin's theorem [12, 15.814] enables us to show that the Hessian of $\log\mu$ is in fact positive definite in C. With some more work it is in fact possible [8] to prove the following

Proposition 1. *The normalized ground state function of $H^{\mathrm{sc}}_{\mathrm{NN}}$ has a unique critical point ξ in C, which is a hyperbolic maximum. Moreover,*

$$\sum_i \xi_i = 0, \qquad \sum_i \xi_i^2 = N. \tag{3.4}$$

A short computation shows that the sites' equation (1.5) simply means that ξ is a critical point of the ground state function $\hat{\mu}$. Furthermore, H_{NN} and $H^{\mathrm{sc}}_{\mathrm{NN}}$ are indeed of the form (2.1)-(2.2), with

$$\mathsf{h}(\mathbf{x}) = \sum_i (x_i - x_j)^{-2}(1 - S_{ij}),$$

and clearly $\mathsf{H}_{\mathrm{NN}} = \mathsf{h}(\xi)$. Hence one can resort to Lemma 1 to obtain some eigenvalues and eigenvectors of the spin chain (1.4) in closed form.

Theorem 2. *If $|s\rangle \in \Sigma$ and $|s'\rangle \in \Sigma'$, the states*

$$\chi_0 \equiv \chi_0(|s\rangle) = \Lambda|s\rangle, \tag{3.5a}$$

$$\chi_1 \equiv \chi_1(|s\rangle) = \sum_i \xi_i\,|s_i\rangle, \tag{3.5b}$$

$$\chi_2 \equiv \chi_2(|s'\rangle) = \sum_i \xi_i^2|s'_i\rangle + (N-1)\sum_{i<j}\xi_i\xi_j\,|s'^{+}_{ij}\rangle, \quad S_{12}|s'\rangle = |s'\rangle, \tag{3.5c}$$

$$\chi_3 \equiv \chi_3(|s\rangle) = \sum_{i<j}\xi_i\xi_j(\xi_i - \xi_j)\,|s^-_{ij}\rangle + 2\sum_i \xi_i|s_i\rangle, \quad S_{12}|s\rangle = -|s\rangle, \tag{3.5d}$$

satisfy the equations

$$(\mathsf{H} - i)\chi_i = 0, \qquad i = 0,1,2,3.$$

Proof. The proof consists in applying Lemma 1 to suitably chosen pairs (Ψ, F). First, it is easy to show that $\chi_0 \in \ker \mathsf{H}$ directly. However, we prefer to prove it using the same freezing trick argument as in the other cases to illustrate the application of the lemma in the easiest case. To this end, consider the functions

$$\Psi = \frac{\Psi_{00}^{(0)}}{\|\hat{\mu}\|} = \hat{\mu}\,\Lambda|s\rangle\,, \qquad F = \Lambda|s\rangle\,,$$

which trivially satisfy the hypotheses in Lemma 1 with $E = 0$. Since in this case

$$\mathsf{E} = 0\,, \qquad \chi = \Lambda|s\rangle\,,$$

from Proposition 1 it follows that χ_0, if nonzero, is an eigenvector of H with eigenvalue 0.

The next case is slightly more complicated. With the notation of Theorem 1, let us set

$$\Psi = \frac{\Psi_{01}^{(1)}}{\|\mu\|} = \hat{\mu}\big(\Phi^{(1)} - \bar{x}\Phi^{(0)}\big)\,, \qquad F = \Phi^{(1)} - \bar{x}\Phi^{(0)}\,.$$

It is not difficult to show that (Ψ, F) satisfies the hypotheses in Lemma 1 with $E = 2a$. Since $\mathsf{E} = 1$ in this case, the state

$$\chi = \lim_{a\to\infty} F(\xi) = \sum_i \xi_i |s_i\rangle$$

is either zero or an eigenvector of H with energy 1. Here we have used the identity $\sum_i \xi_i = 0$, cf. Eq. (3.4).

The other two states are obtained in a similar manner starting with the functions

$$\Psi = \frac{\Psi_{02}^{(2)} + (N-1)\widetilde{\Psi}_{02}^{(2)}}{\|\mu\|} = \hat{\mu}\left(\Phi^{(2)} + (N-1)\widetilde{\Phi}^{(2)} - 2N\bar{x}\Phi^{(1)} + (N+2)\bar{x}^2\Phi^{(0)}\right),$$

$$F = \frac{\Psi}{\hat{\mu}}\,, \qquad E = 4a\,, \qquad |s'\rangle \in \Sigma'\,, \qquad S_{12}|s'\rangle = |s'\rangle\,,$$

and

$$\Psi = \frac{\widehat{\Psi}_{03}^{(3)} - 4\Psi_{03}^{(1)} - \frac{8}{3}\Psi_{03}^{(0)}}{\|\mu\|} = \hat{\mu}\left[\widehat{\Phi}^{(3)} - 2\bar{x}\Phi^{(2)} + \frac{2r^2}{N}\Phi^{(1)} + \bar{x}\left(\frac{r^2}{2N} - \frac{4\bar{x}^2}{3}\right)\Phi^{(0)}\right],$$

$$F = \frac{\Psi}{\hat{\mu}}\,, \qquad E = 6a\,, \qquad S_{12}|s\rangle = -|s\rangle$$

in each case. Indeed, with the former set of functions we immediately arrive at

$$\chi = \sum_i \xi_i^2 |s_i'\rangle + (N-1)\sum_{i<j} \xi_i \xi_j |s_{ij}'^{+}\rangle\,,$$

while in the latter case we obtain

$$\chi = \sum_{i<j} \xi_i \xi_j (\xi_i - \xi_j)|s_{ij}^{-}\rangle + \frac{2}{N}\sum_{i,j} \xi_i \xi_j^2 |s_i\rangle\,.$$

Using Eq. (3.4), the result follows. ∎

Remark 4. If $M = 1/2$, one can prove that $\chi_3(|s\rangle)$ always vanishes.

Remark 5. It is interesting to note that the three lowest frequencies of the small oscillations near the equilibrium of the classical potential U_2 are also integers [15].

4 Integer energies in spin chains with near-neighbors interactions

In Theorem 2 we have explicitly constructed several eigenvectors of $\mathsf{H}_{\rm NN}$ whose corresponding energies are integer numbers. Such eigenvectors can be rightly termed algebraic, as they are in fact obtained from the algebraic eigenfunctions of $H_{\rm NN}$.

The fact that these eigenvalues are integers merely reflects that the energies of the corresponding algebraic eigenfunctions of H are of the form

$$E_k = E_0 + \gamma a k + \mathscr{O}(1)\,, \qquad k \in \mathbb{N}\,. \tag{4.1}$$

This expression is ubiquitous in the theory of rational QES models, and therefore one can interpret the existence of integer energies as a reflection of the potential QES character of the associated dynamical spin model. As discussed in Ref. [8], this model (and many other amenable to an exact treatment using Lemma 1) can be regarded as QES spin chains.

Although we have not been able to obtain a rigorous proof thereof, there is very strong numerical evidence (based on computations performed for up to $N = 20$ spins) of the validity of the following fact:

Conjecture 1. *The algebraic eigenvectors* (3.5a)–(3.5c) *span the three lowest levels of the spin chain* (1.4). *When $M \geqslant 1$ the fourth algebraic energy, $\mathsf{E} = 3$, is however the fifth lowest level of the chain, and its whole eigenspace is spanned by the states of the form* (3.5d). *The algebraic energies are singled out in the spectrum of H as being the only integer ones, i.e.,*

$$\mathrm{spec}(\mathsf{H}) \cap \mathbb{Z} = \{0,1,2,3\}\,.$$

It is well known that all the eigenvalues of the PF chain are integers. Since the spin chain (1.4) is obtained from the latter by keeping only the terms with nearest-neighbors interactions both in the definition of the potential and in the equations for the chain sites, one would be tempted to believe that the number of integer energies of the spin chains

$$\mathsf{H}_n = \sum_{i<j\leqslant i+n \mod N} (\xi_i - \xi_j)^{-2}(1 - S_{ij})\,, \qquad 1 \leqslant n \leqslant N\,, \tag{4.2}$$

which satisfy $\mathsf{H}_1 = \mathsf{H}_{\rm NN}$ and $\mathsf{H}_N = \mathsf{H}_{\rm PF}$, should in fact increase with n. Quite remarkably, numerical simulations seem to show that this is not the case. In particular, it seems unlikely that the solvability properties of either the chains (4.2) or their associated spin models significantly improve as $n < N$ increases.

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References

[1] BEISERT N and STAUDACHER M, Long-range $\mathfrak{psu}(2,2|4)$ Bethe ansätze for gauge theory and strings, *Nucl. Phys. B* 727 (2005), 1–62.

[2] BERENSTEIN D and CHERKIS S, Deformations of $N = 4$ SYM and integrable spin chain models, *Nucl. Phys. B* 702 (2004), 49–85.

[3] BETHE H, Zur Theorie der Metalle I. Eigenwerte und Eigenfunktionen der linearen Atomkette, *Z. Physik* 71 (1931), 205–226.

[4] CALOGERO F, Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials, *J. Math. Phys.* 12 (1971), 419–436.

[5] ENCISO A, FINKEL F, GONZÁLEZ-LÓPEZ A and RODRÍGUEZ M A, Haldane–Shastry spin chains of BC_N type, *Nucl. Phys. B* 707 (2005), 553–576.

[6] ENCISO A, FINKEL F, GONZÁLEZ-LÓPEZ A and RODRÍGUEZ M A, Solvable scalar and spin models with near-neighbors interactions, *Phys. Lett. B* 605 (2005), 214–222.

[7] ENCISO A, FINKEL F, GONZÁLEZ-LÓPEZ A and RODRÍGUEZ M A, Exchange operator formalism for N-body spin models with near-neighbors interactions, *J. Phys. A: Math. Theor.* 40 (2007), 1857–1883.

[8] ENCISO A, FINKEL F, GONZÁLEZ-LÓPEZ A and RODRÍGUEZ M A, A novel quasi-exactly solvable spin chain with nearest-neighbors interactions, *Nucl. Phys. B* 789 (2008) 452–482.

[9] FINKEL F and GONZÁLEZ-LÓPEZ A, Global properties of the spectrum of the Haldane–Shastry spin chain, *Phys. Rev. B* 72 (2005), 174411(6).

[10] FREYHULT L, KRISTJANSEN C and MÅNSSON T, Integrable spin chains with $\mathrm{U}(1)^3$ symmetry and generalized Lunin–Maldacena backgrounds, *J. High Energy Phys* 008, (17 pp.).

[11] GORSKY A, Spin chains and gauge-string duality, *Theor. Math. Phys.* 142 (2005), 153–165.

[12] GRADSHTEYN I S and RYZHIK I M, Table of Integrals, Series, and Products (Academic Press, San Diego, 2000), sixth edition.

[13] HALDANE F D M, Exact Jastrow–Gutzwiller resonating-valence-bond ground state of the spin-$1/2$ antiferromagnetic Heisenberg chain with $1/r^2$ exchange, *Phys. Rev. Lett.* 60 (1988), 635–638.

[14] HEISENBERG W, Zur Theorie des Ferromagnetismus, *Z. Physik* 49 (1928), 619–636.

[15] KHARE A, LORIS I and SASAKI R, Affine Toda-Sutherland systems, *J. Phys. A: Math. Gen.* 37 (2004), 1665–1679.

[16] MINAHAN J and ZAREMBO K, The Bethe-ansatz for $\mathcal{N} = 4$ super Yang–Mills, *J. High Energy Phys.* 013, (29 pp.).

[17] POLYCHRONAKOS A P, Lattice integrable systems of Haldane–Shastry type, *Phys. Rev. Lett.* 70 (1993), 2329–2331.

[18] POLYCHRONAKOS A P, Exact spectrum of $\mathrm{SU}(n)$ spin chain with inverse-square exchange, *Nucl. Phys. B* 419 (1994), 553–566.

[19] M. REED and B. SIMON, Methods of Modern Mathematical Physics (Academic Press, San Diego, 1978

[20] ROIBAN R and VOLOVICH A, Yang–Mills correlation functions from integrable spin chains, *J. High Energy Phys.* 032, (24 pp.).

[21] SHASTRY B S, Exact solution of an $S = 1/2$ Heisenberg antiferromagnetic chain with long-ranged interactions, *Phys. Rev. Lett.* 60 (1988), 639–642.

[22] SHIFMAN M A, New findings in quantum mechanics (partial algebraization of the spectral problem), *Int. J. Mod. Phys. A* 4 (1989), 2897–2952.

[23] SUTHERLAND B, Exact results for a quantum many-body problem in one dimension, *Phys. Rev. A* 4 (1971), 2019–2021.

[24] SUTHERLAND B, Exact results for a quantum many-body problem in one dimension. II, *Phys. Rev. A* 5 (1972), 1372–1376.

[25] TURBINER A V, Quasi-exactly solvable problems and $\mathfrak{sl}(2)$ algebra, *Comm. Math. Phys.* 118 (1988), 467–474.

[26] USHVERIDZE A G, Quasi-Exactly Solvable Models in Quantum Mechanics (Institute of Physics Publishing, Bristol, 1994).