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On the Origin of Fractional Shapiro Steps in Systems of Josephson Junctions with Few Degrees of Freedom

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Abstract

We investigate the origin of fractional Shapiro steps in arrays consisting of a few overdamped Josephson junctions. We show that when the symmetry reduces the equations to that of a single junction equation, only integer steps appear. Otherwise, fractional steps will appear when the evolution equations contain second (or higher) order derivatives or non-sinusoidal terms. We make a point of distinguishing the last two possibilities in the generation of the fractional steps.

1 Introduction

In thermodynamics we learn about the significance of a *cyclic* process, as an attempt to delay the production of entropy and instead to produce useful work. This physical basis is at work in the design of any man-made as well as naturally occurring engine. A rechargeable battery is an example of the former and the living cell of the latter. What we deal with is a nearly reversible cycle that returns the system to (nearly) its beginning state, to make the laptop work or the cell to live another day. In a cyclic process, the arrow of time is made circular!

However, the nutrition that keeps a cell going is not enough to keep the heart beating properly. The 10^4 special set of cells, called pacemaker cells, communicate with each other and go in their cycles in unison to trigger the rest of the heart and create the pumping action.[1] This synchronous character is a recurring theme in nature. It is now known that drugs that affect the firing synchrony of neuronal signals in the brain have disruptive effects on memory.[2] This is but one aspect of the significance of neuronal synchrony. In physics one can find more familiar instances of synchrony; the Moon rotates once on its axis as it cycles around the Earth, keeping the same side facing us. Two pendulum clocks hanging on the wall will become synchronous, with their oscillations running out of phase.[3]

The first models that describe the mutual entrainment of a collection of oscillators where synchronization occurs due to the interaction of the oscillators without any externally imposed drive are due to Winfree and Kuramoto.[4, 5] Winfree studied among other things the circadian rhythms of biological clocks, and Kuramoto put forth a mathematical model of mean field nature that could be solved exactly.

In the Kuramoto model, a collection of nonlinear oscillators are coupled with each other. The natural frequency of each oscillator is pulled or pushed depending on its relative phase with other oscillators and adjusted according to the strength of the coupling. Kuramoto showed that beyond a critical coupling (in the sense of a dynamic transition) the oscillators pull together forming a large cluster characterized by a synchrony in motion. As the coupling increases even further, the synchronization becomes more and more complete.[6, 7]

The Kuramoto model can be realized in the laboratory, thanks to the Josephson junction. In a Josephson junction, two superconducting islands are weakly coupled across a thin barrier. [8] Each island keeps its (macroscopic) superconducting phase, and Josephson showed that the tunneling current of the Cooper (electron) pairs depends on the phase difference between the islands. In a current controlled system, a current through the junction imposes a phase difference. When the current exceeds a critical value, a voltage develops across the junction. Josephson showed that this voltage is proportional to the rate of change of the phase difference with time. Thus the junction is an element that runs in a cycle and whose frequency can be controlled.

Shapiro showed that a Josephson junction can tune itself to the oscillations of the external source when the average frequency (or voltage) of the junction is close to an integer multiple of the frequency of the external source, in this case a radio frequency drive.[9]

Putting Josephson junctions in series couples the oscillators, and a series array of overdamped junctions (i.e. junctions with small capacitance) with a parallel load can become synchronized [10]. This model is described by a variation of the Kuramoto model [11]. The mutual entrainment occurs due to the coupling (the load) and does not require an external frequency of a master drive. However, it can also help understand the response of the system to an external source. We recently showed [12] that in case of arrays of a few junctions the mutual entrainment among the junctions can be interpreted as a kind of normal mode for the system. The external drive will predominantly excite these frequencies. In this way a more general feature, fractional Shapiro steps, can appear. This would mean, tuning to a fraction of the frequency of the external source.

In this paper we shall follow the above ideas and look more closely at the origin of the fractional Shapiro steps. The systems we study have only a few degrees of freedom and consist of overdamped junctions. In Sec. II we will introduce the elements of the model. In Sec. III we consider a special kind of a load, namely the inductive load, and the role it plays in the formation of the fractional steps. Sec. IV is devoted to a particular arrangement of the junctions that will bring out another mechanism leading to the fractional steps. The conclusions follow in Sec. V.

2 The Model

We begin by introducing the resistive and capacitively shunted junction (RCSJ) model of the Josephson junction, driven by an external current. In this model, the junction is in parallel with a resistor and also a capacitor and is described by a second order differential equation:[13]

$$\frac{\hbar C}{2e}\frac{d^2\phi}{dt^2}+\frac{\hbar}{2eR}\frac{d\phi}{dt}+I_c\sin\phi=I_{ext} \tag{2.1}$$

The parameters C, R and I_c are the capacitance, normal resistance and critical current of the junction, respectively. It is convenient to put this equation in dimensionless form by measuring time in units of $\hbar/2eRI_c$ and current in units of I_c, with the result

$$\beta_c \frac{d^2\phi}{dt^2} + \frac{d\phi}{dt} + \sin\phi = I_{ext} \tag{2.2}$$

Here $\beta_c = 2eR^2CI_c/\hbar$ is known as the McCumber parameter. In analogy with a damped driven pendulum this parameter shows the strength of the inertial term compared with damping. For large values of the McCumber parameter the junction is underdamped, whereas for small values of β_c the junction is in the overdamped regime.

This paper will focus on the overdamped limit, and in general assume $\beta_c = 0$. In this limit Eq. 2.2 has an analytic solution for the case of an extremely overdamped junction driven by a constant current. When the normalized external current is less than unity, there is a fixed point for which the phase is constant and the junction voltage is zero. The fixed point disappears, and the equation undergoes a saddle-node bifurcation when the external current exceeds unity. In this case the phase will grow without bound; we will say that the junction is in the rotating state. Since the voltage across the junction is proportional to the rate of the change of the phase, in the rotating state a voltage with non-zero average develops across the junction.

When the external current has an ac component, the junction can be synchronized by the external frequency. This fixes the voltage, so that as a function of the dc current the voltage exhibits plateaus, called Shapiro steps [9]. In general we can index the steps by integers m and n according to

$$2e < V >= \frac{n}{m}\hbar\omega \tag{2.3}$$

where $< V >$ denotes the time average voltage across the junction and ω is the frequency of the ac part of the driving current. The cases with $m = 1$ are called integer steps; otherwise we deal with synchronization of the subharmonic type, or fractional steps. Renne and Polder [15] and Waldram and Wu [16] have analyzed the single junction overdamped equation and shown that only integer steps can occur for it.

Fractional steps can occur for an underdamped junction. This has been investigated by many authors numerically and analytically. For example Azbel and Bak studied the case with small inertial term and a train of pulses as the periodic drive, and showed that subharmonic synchronization is possible.[17]

In this article we show that in circuits consisting of overdamped Josephson junctions and possibly linear elements, fractional steps can appear both as a result of the appearance of the second (or higher) order derivatives in the equations, or by non-sinusoidal terms developing in the equations of motion.

It is also shown that whenever the symmetry of the array reduces the equations to that of the single (overdamped) junction, only integer steps appear in the characteristics.

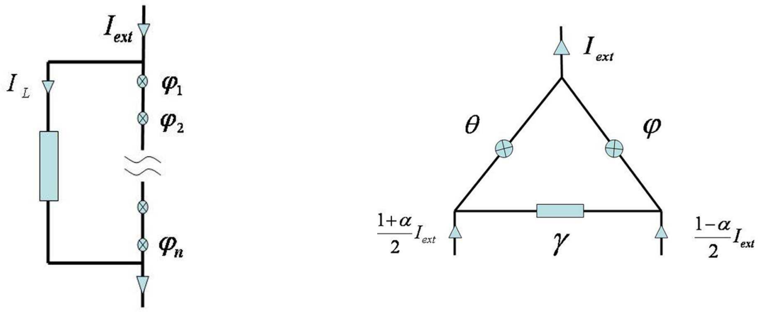

Figure 1: A serial array with a linear load (left) and two parallel junctions coupled by an inductance or third junction (right).

3 The case of the inductive (linear) load

When a single junction is in parallel with a linear RL load, the Josephson equation is coupled with a linear first order differential equation:

$$\frac{d\phi}{dt} + \sin\phi + I_L = I_{ext} \tag{3.1}$$

$$\frac{dI_L}{dt} + \frac{1}{\tau}I_L = \frac{\rho}{\tau}\frac{d\phi}{dt} \tag{3.2}$$

where I_L is the load current, and I_{ext} is the external current which could have a dc and an ac component. The current and time variables are normalized as before and $\tau = (L/R_L)(2eRI_c/\hbar)$ where L and R_L are the inductance and the resistance of the load respectively. ρ is the ratio of the normal resistance of the junction R to the resistance of the load. The second equation is linear and can be solved for I_L. Substituting the result in Eq. 3.1 results in a second order equation:

$$\tau\frac{d^2\phi}{dt^2} + \frac{d\phi}{dt}(1+\rho+\tau\cos\phi) + \sin\phi = I_{ext} + \tau\frac{dI_{ext}}{dt} \tag{3.3}$$

This model can have subharmonic steps in the current-voltage characteristic because it is governed by a second-order differential equation. This equation with $\rho = 0$ has been considered by some authors as the model for the single junction with an intrinsic inductance [18].

An interesting extension of the circuit replaces the single junction by a serial array of junctions. Most of the study on serial arrays is focused on the stability of the in-phase solution [10, 11, 19]. In the in-phase state all the junctions in the array evolve in complete synchrony with each other. It has been shown that when the load is inductive, the in-phase solution for a serial array is stable. Here we show that the response of the serial array to a RF excitation can reveal the synchrony among the junctions.

The circuit consists of N identical overdamped junctions in series, parallel with a RL load. The equations of motion are:

$$\frac{d\phi_i}{dt} + \sin\phi_i + I_L = I_{ext}, \qquad i = 1,2,\ldots,N \tag{3.4}$$

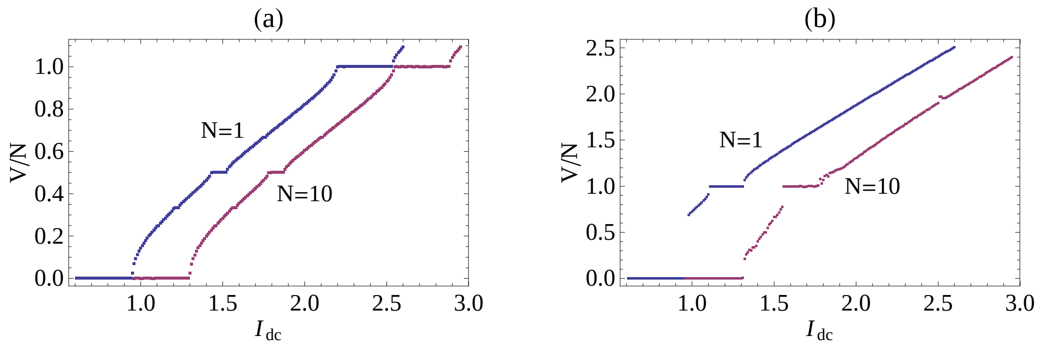

Figure 2: (a) I-V characteristics of a single junction with an inductive load with $\tau = 4$ and $\rho = 1$ and a serial array with $\tau = 4$, $N = 10$ and $\rho = 1/10$. The latter has been shifted along the horizental axis. (b) The same plots as (a) but with a capacitive load. The amplitude and frequency of the ac signal are 0.5 and 1 in normalized units, respectively. The similarity in the two plots in (a) is an indication of the serial junctions being in phase; whereas in (b) the in phase solution is not stable which results in the difference of the plot for a single junction with that of the array.

$$\frac{dI_L}{dt} + \frac{1}{\tau} I_L = \frac{\rho}{\tau} \sum_{i=1}^{N} \frac{d\phi_i}{dt}. \tag{3.5}$$

For the in-phase solution $\phi_1 = \phi_2 = ... = \phi_N$ and the dynamics of the array is reduced to those described by Eqs. 3.1 and 3.2 with $N\rho$ in place of ρ. So in presence of the RF signal, if the parameters are changed appropriately, the characteristics will resemble those of the single junction, as can be seen in Fig. 2. It is known that for a serial array without load or with a capacitive load, the in-phase solution is not stable [10]. We checked the behavior of the array with a capacitive load and found it to be different from that of the single junction with the same load.

The simplest two dimensional array consists of two junctions in parallel forming a loop which is known as the dc SQUID [20]. When the quantum mechanical phase is constant across each grain, the junctions effectively behave like a single junction and no fractional steps exist in the characteristics. As shown in Figure 1b, the inductance of the loop serves to couple two junctions linearly [14]

$$\frac{d\theta}{dt} + \sin(\theta - \pi f) + \frac{1}{\beta_L}(\theta - \phi) = \frac{1+\alpha}{2} I_{ext} \tag{3.6}$$

$$\frac{d\phi}{dt} + \sin(\phi + \pi f) + \frac{1}{\beta_L}(\phi - \theta) = \frac{1-\alpha}{2} I_{ext} \tag{3.7}$$

The coupling constant is $1/\beta_L$ with $\beta_L = 2\pi L_0 I_c/\Phi_0$ where L_0 is the loop self-inductance and $\Phi_0 = h/2e$ is the quantum of flux. (β_L is the normalized inductance of the loop. [14]) The frustration f is defined as Φ_{ext}/Φ_0 where Φ_{ext} is the external magnetic flux through the loop. The equations have been written in dimensionless form as before and α is a parameter that sets the possible asymmetry in current division; that is, $\alpha = 0$ is the symmetric case. It can be noted that when no external field is present ($f = 0$) and the current is divided symmetrically ($\alpha = 0$), the equations are invariant under permutation of θ and ϕ. This permutation symmetry is broken when either of these conditions is removed.

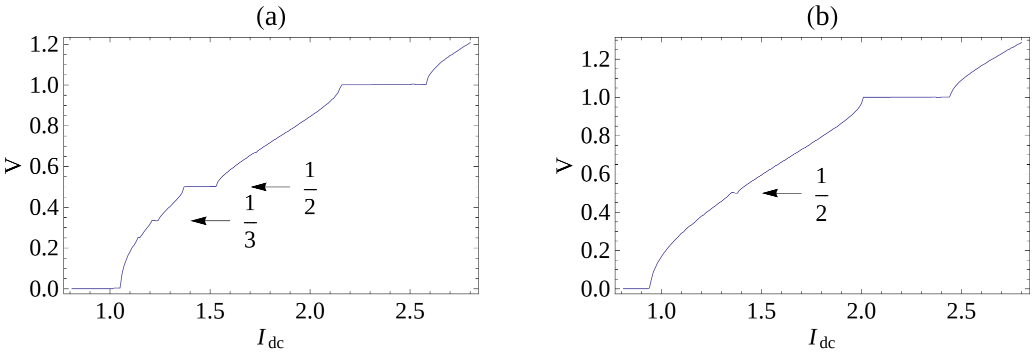

Figure 3: Characteristics of an inductive SQUID with symmetric bias, $f = 1/3$ with $\beta_L = 1$ (a) and $\beta_L = 5$ (b). The subharmonic Shapiro steps appear when the symmetry is broken by the applied magnetic field but they are narrower for larger values of the coupling constant. The amplitude and frequency of the ac signal are 0.5 and 1 in normalized units, respectively.

We have shown the results of the numerical investigation in Fig. 3. For a symmetric array, the coupling is effectively inactive and the equations reduce to that of a single junction: no subharmonic appears. Asymmetry (regardless of how it is imposed) makes the equations differ from the single junction equation and this is clear in the appearance of the subharmonic steps. When the coupling constant is large (i.e. β_L is small), however, the fractional steps disappear despite of the asymmetry of the array (see Fig. 4). In this limit the two phases are strongly stuck together and the two phase differences remain small: for very large coupling the behavior approaches that of the simple SQUID without inductance.

To get a better intuition about the dynamics of this array it is instructive to rewrite the equations in terms of the coordinates $\gamma = \theta - \phi$ and $\xi = \theta + \phi$:

$$\frac{d\gamma}{dt} + \frac{2}{\beta_L}\gamma = \alpha I_{ext} - 2\sin(\gamma/2 - \pi f)\cos(\xi/2) \tag{3.8}$$

$$\frac{d\xi}{dt} = I_{ext} - 2\cos(\gamma/2 - \pi f)\sin(\xi/2) \tag{3.9}$$

When $\alpha = f = 0$, the array is symmetric and $\gamma = 0$ is a stable fixed point of Eq. 3.8 regardless of ξ. Eqs. 3.8 and 3.9 are then decoupled, and Eq. 3.9 a single junction equation. When either of the parameters f or α are non-zero there is no fixed point for γ. Eq. 3.8 can be looked upon as a first order equation with a bounded forcing term, so γ must also be bounded and not a "rotating" phase variable (this is seen by the constraint placed on the equation by the second term on the left hand side). This allows us to linearize Eq. 3.8 around $\gamma = 2\pi f$:

$$\frac{d\gamma}{dt} + \frac{1}{\beta_L}(2 + \beta_L\cos(\xi/2))\gamma = \alpha I_{ext} + 2\pi f\cos(\xi/2) \tag{3.10}$$

A solution of integral form for γ in terms of ξ can be found. Expanding cosine in Eq. 3.9 up to quadratic term and substituting γ yields:

$$\frac{d\xi}{dt} + 2\sin(\xi/2)\{1 - \frac{1}{8}[2\pi f - \frac{1}{u(t)}\int dt' u(t')(\alpha I_{ext} + 2\pi f \cos(\xi/2))]^2\} = I_{ext} \tag{3.11}$$

Where $u(t)$ is the integrating factor of the Eq. 3.10:

$$u(t) = \exp[\int dt'(2 + \cos(\xi/2))/\beta_L] \tag{3.12}$$

Eq. 3.11 can be transformed into a third order differential equation. Keeping higher orders of the expansion of $\cos(\gamma/2)$ increases the order of the differential equation further. We claim that the appearance of subharmonic resonances in these differential equations is again due to their higher order, as was the case for those describing the serial array.

4 The triangular array and the nonsinusoidal equation

An interesting modification of the inductive SQUID is the triangular array of Josephson junctions. For a symmetric bias current this array behaves similar to the SQUID discussed above, but new interesting features arise when the array is biased unsymmetrically. We begin with the latter case to show that fractional steps now appear due to a different reason. We will then return to the symmetric bias to show that the equations resemble partly the equations of the inductive SQUID. For an arbitrary current division, defined as before by α, the equations of motion are

$$\dot{\theta} = \frac{3+\alpha}{6} I_{ext} - \frac{1}{3}\sin(\phi + \pi f) - \frac{2}{3}\sin(\theta - \pi f) - \frac{1}{3}\sin(\theta - \phi). \tag{4.1}$$

$$\dot{\phi} = \frac{3-\alpha}{6} I_{ext} - \frac{2}{3}\sin(\phi + \pi f) - \frac{1}{3}\sin(\theta - \pi f) + \frac{1}{3}\sin(\theta - \phi), \tag{4.2}$$

Since three junctions here form a loop, the phase difference across the third junction (which we have labeled γ) is just the difference between θ and ϕ. As in the case of the inductive SQUID, when both f and α are zero, the equations show a permutation symmetry for θ and ϕ. This in turn means that γ is equal to zero: the coupling is inactive, and the equations of motion are the equations of two uncoupled junctions.

When the current is fed totally unsymmetrically ($\alpha = 1$ with $f = 0$), there is an exact solution $\theta = 2\phi$ for the equations 4.1 and 4.2. When uncoupled, this solution results in the “nonsinusoidal” term, $\sin(2\phi)$. Using Eq. 4.2, we get

$$\dot{\phi} = \frac{1}{3} I_{ext} - \frac{1}{3}\sin(\phi) - \frac{1}{3}\sin(2\phi). \tag{4.3}$$

Indeed this circuit can be considered as a serial array of junctions, consisting of ϕ and γ junctions, coupled by a parallel non-linear load, played by another Josephson junction. The above solution is valid when the in-phase solution is stable for the serial junctions. Numerical results show that this solution for two junctions in series is stable, but for more than two junctions; having a single junction as the load, the in-phase solution is not stable[12].

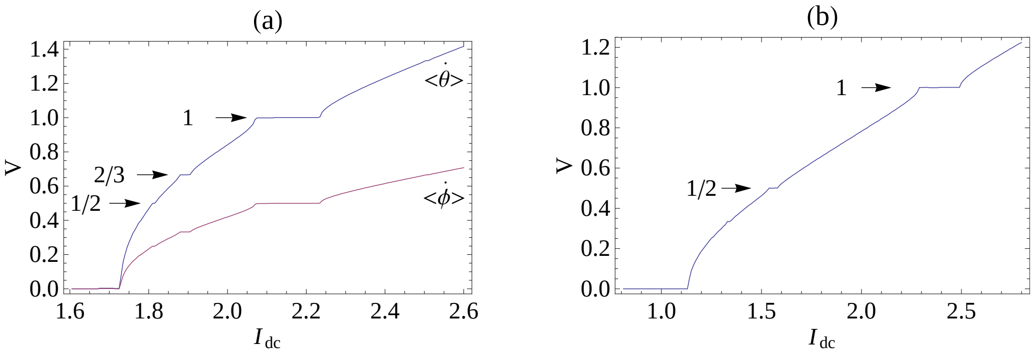

Figure 4: I-V characteristics of a triangular plaquette for unsymmetric bias $\alpha = 1$ with zero f (a) and symmetric bias with $f = 1/3$ (b). The amplitude and frequency of the ac signal are 0.5 and 1 in normalized units.

Equation 4.3 shows fractional Shapiro steps as well as integer steps in the characteristics. We note that in this case in contrast with the previous arrangements, the uncoupled differential equation is of the first order. So here the non-sinusoidal term is responsible for the appearance of subharmonic synchronization.

For the other non-zero values of the current division factor α, there is no exact relation between θ and ϕ and the equations can not be decoupled. But the numerical results again show that in the absence of an ac external current there is an internal synchronization between two degrees of freedom which can survive and result in factional steps when the ac signal is added. [12]

At the end we return to the case when $\alpha = 0$ and the symmetry is removed by an external magnetic field, that is $f \neq 0$. Let us to rewrite the Eqs. 4.1 and 4.2 in terms of the variables γ and ξ introduced before

$$3\frac{d\gamma}{dt} + 2\sin(\gamma/2)[2\cos(\gamma/2) + \cos\pi f\cos(\xi/2)] = 2\cos(\frac{\gamma}{2})\cos(\frac{\xi}{2})\sin\pi f, \tag{4.4}$$

$$\frac{d\xi}{dt} = I_{ext} - 2\sin(\xi/2)\cos(\gamma/2 - \pi f). \tag{4.5}$$

The last equation is similar to Eq. 3.9. We have arranged the first equation to be compared with the single junction equation. For $f = 0$, as for the linear coupled SQUID, $\gamma = 0$ is a solution and the two equations are uncoupled. For non-zero f the right hand side of Eq. 4.4 can be considered as the forcing term, which is proportional to $\sin(\pi f)$. Since there is not any constant drive, γ will be oscillating and for $f \in (0, 1/2)$ the amplitude of the oscillations grows with increasing f. It means that with larger f the two equations are more strongly coupled. The larger coupling causes the width of the fractional steps to grow with f.

As for the origin of the fractional steps in this case, we note that since γ experiences oscillations, the equations can be linearized around $\gamma = 2\pi f$ and the results are similar to the arguments after Eq. 3.12. Again one of the equations can be solved and substitution of the result in the other equation leads to higher order derivatives which are responsible for the fractional steps.

5 Conclusion

We have investigated two situations leading to the appearance of the fractional Shapiro steps. One mechanism is a change in the equations of motion that leads to the presence of higher harmonics of the phase difference, e.g. $\sin(2\phi)$. Another mechanism is a change that introduces higher order derivatives. The pattern of the fractional steps in both cases is similar. We can expect that in general, if the system of few degrees of freedom is such that symmetry will not reduce it to the equation of a single junction, fractional steps appear.

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