Goodness-of-fit Test for Normally Distributed AR(1) Disturbances of the Multiple Linear Regression model

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Abstract. We suggest the modified Anderson-Darling(AD) test procedures for testing normality of the AR(1) disturbances of the multiple linear regression model. The asymptotic null distribution of the transformed sample is obtained, and an algorithm is given to approximate the critical values of the test statistic for the finite sample size. The power of the test against various alternative distributions of the model disturbances is illustrated by Monte Carlo simulation.

Introduction

The multiple linear regression model is used to describe the relationship between a dependent variable and several independent variables. The generic form of the linear regression model is

$y = \beta_1 x_1 + \cdots + \beta_p x_p + \varepsilon$,

where y is the dependent variable, $x_1, \cdots, x_p$ are independent variables, ε is a random disturbance.

The first-order autoregressive disturbance ε_i, or AR(1) process, is commonly defined as

$\varepsilon_i = \rho \varepsilon_{i-1} + v_i$,

where the v_i are usually assumed to be normally distributed with zero mean and variance σ_v^2. The AR(1) disturbance term ε_i in the current time period is directly related to the disturbance term in the previous time period. The normal AR(1) model is the one most widely used and studied[1].

Tests based on the empirical distribution function(EDF) are uaually more powerful than the Pearson chi-square test[2]. The simulation show that the Anderson-Darling(AD) test is on the whole better than other EDF tests [3]. The critical values of the modified AD test test can be approximated by Monte Carlo simulation for the finite sample sizes. In this paper, we propose the test statistics for testing normality of the AR(1) disturbance terms of the linear regression models.

The paper is organized as follows. In Section 2, we introduce the multiple linear regression model. In Section 3, the asymptotic null distribution of the transformed sample is obtained, the test statistic for normal disturbances is proposed. The algorithm to estimate the critical values is presented. In Section 4, power simulation of the test statistic is given. Section 5 contains some concluding comments. The proof of Theorem 1 is postponed to Appendix.

The multiple linear regression model with AR(1) disturbances

The multiple linear regression model with a first-order autoregressive disturbance is represented in the form as

$$y_i = \beta_1 x_{i1} + \cdots \beta_p x_{ip} + \varepsilon_i,\ \ \varepsilon_i = \rho \varepsilon_{i-1} + v_i, i = 1,2,\cdots,n,$$
$$E(v_i) = 0, E(v_i^2) = \sigma_v^2, Cov(v_i, v_j) = 0, i \neq j, \quad (1)$$

where $\beta_1, \cdots,\ \beta_p, \rho, \sigma_v^2$ are unknown parameters.

By repeated substitution,we have

$$\varepsilon_i = v_i + \rho v_{i-1} + \rho^2 v_{i-2} + \cdots.$$

When $|\rho| < 1$, the AR(1) process $\{\varepsilon_i\}$ is stationary. Thus

$$Var(\varepsilon_i) = \tfrac{\sigma_v^2}{1-\rho^2}, Cov(\varepsilon_i, \varepsilon_{i-j}) = \tfrac{\sigma_v^2}{1-\rho^2}\rho^j = \sigma_\varepsilon^2 \rho^j.$$

Let

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}, \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}, \ \mathrm{x}_j = \begin{pmatrix} x_{j1} \\ \vdots \\ x_{jp} \end{pmatrix}, j = 1, \cdots, n, X = \begin{pmatrix} \mathrm{x}_1' \\ \vdots \\ \mathrm{x}_n' \end{pmatrix}.$$

Let

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{pmatrix}, \ \Gamma = \Gamma(\rho) = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{pmatrix}, \tag{2}$$

we have $\Gamma'\Gamma = (1-\rho^2)\Sigma^{-1}$ (see [1]). Then the model (1) can be written as

$$Y = X\beta + \varepsilon, \ \sigma_\varepsilon^2 = \sigma_v^2 / (1-\rho^2) \tag{3}$$

$$\mathrm{E}(\varepsilon) = 0, Cov(\varepsilon) = \sigma_\varepsilon^2 \Sigma,$$

where Σ is defined in(2). Let

$$\tilde{Y}(\rho) = \Gamma Y, \tilde{X}(\rho) = \Gamma X, \tilde{\varepsilon}(\rho) = \Gamma \varepsilon.$$

Then the model (3) can be written as

$$\tilde{Y}(\rho) = \tilde{X}(\rho)\beta + \tilde{\varepsilon}(\rho), \tag{4}$$

$$\mathrm{E}(\tilde{\varepsilon}(\rho)) = 0, Cov(\tilde{\varepsilon}(\rho)) = \sigma_v^2 I_n$$

Thus the model (4) becomes a classical regression model. Let $v(\rho) = (v_1, \cdots, v_n)'$ and $\varepsilon_1 = (1/\sqrt{1\text{-}\rho^2})v_1$, where v_i' are defined in (1). We have

$$\tilde{\varepsilon}(\rho) = v(\rho). \tag{5}$$

Goodness of fit test for the normal distribution of AR(1) disturbances

The transformed sample and the test statistic

Lemma1[1]. Consider the model(4). Let $rank(X) = p$ and let $\theta = (\rho, \beta, \sigma_v^2)$. Let $v(\rho) \sim N(0, \sigma_v^2 I_n)$, where $v(\rho)$ is defined in (5). Then

(a). The log-likelihood function of $\tilde{Y}(\rho)$ is

$$\ln L = -\frac{n}{2}[\ln(2\pi) + \ln\sigma_v^2] - \frac{1}{2\sigma_v^2}[\tilde{\varepsilon}(\rho)]'\tilde{\varepsilon}(\rho),$$

where $\tilde{Y}(\rho)$, $\tilde{X}(\rho)$ and $\tilde{\varepsilon}(\rho)$ are defined in (4).

(b). With probability tending to 1 as $n \to \infty$, there exists a solution $\hat{\theta}_n$ of the likelihood equations such that $\hat{\theta}_n$ is consistent for estimating θ.

Lemma2[1]. Consider the model(4). Suppose the conditions of Lemma1 hold. Let $f_{y_n, y_{n-1}, \cdots, y_2 | y_1}$ denote the conditional probability density of $(y_n, y_{n-1}, \cdots, y_2)'$ given y_1, where $\mathrm{Y} = (y_1, y_2, \cdots, y_n)'$ is defined by (3). Let

$$\tilde{Y}=\begin{pmatrix}\tilde{y}_1\\ \vdots\\ \tilde{y}_n\end{pmatrix}=\begin{pmatrix}\tilde{y}_1\\ \tilde{Y}_2\end{pmatrix},\tilde{X}=\begin{pmatrix}\tilde{x}_1'\\ \vdots\\ \tilde{x}_n'\end{pmatrix}=\begin{pmatrix}\tilde{x}_1'\\ \tilde{X}_2\end{pmatrix},\tilde{\varepsilon}=\begin{pmatrix}\tilde{\varepsilon}_1\\ \vdots\\ \tilde{\varepsilon}_n\end{pmatrix}=\begin{pmatrix}\tilde{\varepsilon}_1\\ \tilde{E}_2\end{pmatrix}, \tag{6}$$

where $\tilde{X}_2$ is $(n-1)\times p$ matrix, $\tilde{Y}_2$ and $\tilde{\varepsilon}_2$ are $(n-1)\times 1$ matrics. Then

(a). The conditional log-likelihood function of $(y_n, y_{n-1},\cdots,y_2)'$ given y_1 is

$$LnL_1=-\frac{n-1}{2}[\ln(2\pi)+\ln\sigma_v^2]-\frac{1}{2\sigma_v^2}\sum_{i=2}^{n}[(y_i-\rho y_{i-1})-(x_i-\rho x_{i-1})'\beta]^2$$
$$=-\frac{n-1}{2}[\ln(2\pi)+\ln\sigma_v^2]-\frac{1}{2\sigma_v^2}\tilde{E}_2'\tilde{E}_2,$$

where $x_i, i=1,\cdots,n$ are defined by (3).

(b). Let $\hat{\rho}$ denote the conditional MLE of ρ in LnL_1. For a given value of ρ, the conditional MLE of β and σ_v^2 are the usual ones, i.e.,

$$\hat{\beta}(\rho)=(\tilde{X}_2'\tilde{X}_2)^{-1}\tilde{X}_2'\tilde{Y}_2,\hat{\sigma}_v^2=\tilde{Q}(\rho)/(n-p-1), \tag{7}$$

$$\tilde{Q}(\rho)=[\hat{\tilde{E}}_2(\rho)]'\hat{\tilde{E}}_2(\rho),\hat{\tilde{E}}_2(\rho)=\tilde{Y}_2-\tilde{X}_2\hat{\beta}(\rho). \tag{8}$$

Remark1. The conditional MLE of ρ and β can be obtained by minimizing $\tilde{E}_2'\tilde{E}_2$ in LnL_1 in Lemma2. Thus the conditional MLE of ρ and β do not depend on σ_v^2.

Theorem1. Let $v(\rho)$ be defined in (5) and let $v(\rho)\sim N(0,\sigma_v^2 I_n)$. Let $\hat{\rho}$, $\hat{\beta}(\cdot)$ and $\hat{\sigma}_v^2(\cdot)$ be defined in
Lemma2. Let ε_1, x_1 and y_1 are defined in (3) and let

$$\hat{v}=(\hat{v}_1,\cdots,\hat{v}_n)'=(\hat{v}_1,\hat{V}_2)=\tilde{Y}(\hat{\rho})-\tilde{X}(\hat{\rho})\hat{\beta}(\hat{\rho}),\quad \hat{\sigma}_v^2(\hat{\rho})=\hat{V}_2'(\hat{\rho})\hat{V}_2(\hat{\rho})/(n-p-1), \tag{9}$$

$$Z=(z_1,\cdots,z_n)'=\hat{v}/\hat{\sigma}_v(\hat{\rho}),\hat{\sigma}_v(\hat{\rho})=\sqrt{\hat{\sigma}_v^2(\hat{\rho})}. \tag{10}$$

Then

(a).The asymptotic distributions of $z_i, i=1,\cdots,n$ are $N(0,1)$,which we write as $z_i \overset{a}{\sim} N(0,1), i=1,\cdots,n$.

(b). The $z_i, i=1,\cdots,n$ are asymptotically independent.

Let $\tilde{F}$ be the unknown distribution function of $\tilde{\varepsilon}$ in the model (4). We want to test the hypothesis

$$H_0:\tilde{F}=\tilde{F}_0, \tag{11}$$

where $\tilde{F}_0$ denotes the distribution function of $N(0,\sigma_v^2 I_n)$. By Theorem1., the goodness-of-fit test for $N(0,\sigma_v^2 I_n)$ can be translated into the goodness-of-fit test for $z_i \overset{a}{\sim} N(0,1), i=1,\cdots,n$. Let $z_{(i)}, i=1,\cdots,n$ be the order statistics of $Z=(z_1,\cdots,z_n)'$ in (10). The computational formula of the modified AD for testing $z_i \overset{a}{\sim} N(0,1)$ is given by [4]

$$\varsigma(Z)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)[\ln\Phi(z_{(i)})+\ln(1-\Phi(z_{(n+1-i)}))], \tag{12}$$

where $\Phi(\cdot)$ denotes the distribution function of $N(0,1)$.

The algorithm to estimate the critical values

Let $-1<\rho<1$. Choose the values of ρ=-1(Δ)1 with the implied estimates of the other parameters that minimizing $[\tilde{E}_2]^{'}\tilde{E}_2$, where $\tilde{E}_2$ is defined in (6). The obtained value of ρ is denoted by $\breve{\rho}$. Let $\Delta\to 0$, then $\breve{\rho}\to\hat{\rho}$, the conditional MLE of ρ. Let $\Delta=0.01$. The algorithm to estimate the critical values of $\varsigma(Z)$ in (12) consists of the following steps:

(1). Generate $\tilde{\varepsilon}_*=(\tilde{\varepsilon}_{1*},\tilde{E}_{2*}^{'})^{'}=(\tilde{\varepsilon}_{1*},\cdots,\tilde{\varepsilon}_{n*})^{'}$ from the normal distribution $N(0,I_n)$.

(2). Compute

$$\hat{V}_{2*}=[\mathrm{I}_{n-1}-\mathrm{P}_{\tilde{X}_2^{'}}(\breve{\rho})]\tilde{E}_{2*}, \tag{13}$$

where

$$\mathrm{P}_{\tilde{X}_2^{'}}(\breve{\rho})=\tilde{X}_2(\tilde{X}_2^{'}\tilde{X}_2)^{-1}\tilde{X}_2^{'},\tilde{X}_2=\tilde{X}_2(\breve{\rho}),$$

where $\tilde{X}_2(\cdot)$ is defined in (7), but with ρ replaced by $\breve{\rho}$.

(3). Compute

$$\hat{\sigma}_{v*}^2(\breve{\rho})=\frac{1}{n-p-1}\tilde{V}_{2*}^{'}\tilde{V}_{2*},\hat{\sigma}_{v*}(\breve{\rho})=\sqrt{\hat{\sigma}_{v*}^2(\breve{\rho})}. \tag{14}$$

(4). Compute

$$\hat{v}_{1*}=\tilde{y}_1(\breve{\rho})-[\tilde{\mathrm{x}}_1(\breve{\rho})]^{'}\hat{\beta}(\breve{\rho})=\tilde{\varepsilon}_{1*}-\tilde{\mathrm{x}}_1^{'}(\tilde{\mathrm{X}}_2^{'}\tilde{\mathrm{X}}_2)^{-1}\tilde{\mathrm{X}}_2^{'}\tilde{E}_{2*}. \tag{15}$$

(5). Compute

$$Z_*=(\mathrm{z}_{1*},\mathrm{z}_{2*},\cdots,\mathrm{z}_{n*})^{'}=\hat{v}_*/\hat{\sigma}_{v*}(\breve{\rho}), \tag{16}$$

where $\hat{v}_*=(\hat{v}_{1*},\hat{V}_{2*}^{'})^{'}$.

(6). Compute $\varsigma_*=\varsigma(\mathrm{Z}_*(\tilde{\varepsilon}_*))$, where $\varsigma(\cdot)$ is defined in (12).

Doing these N times gives a sample of replicates $\varsigma_{1*},\cdots,\varsigma_{N*}$. Let $\varsigma_{(1*)},\cdots,\varsigma_{(N*)}$ be the order statistics, the critical values for ς can be estimated from $\varsigma_{(1*)},\cdots,\varsigma_{(N*)}$. Since

$$\varsigma_*=\varsigma(\mathrm{Z}_*(\tilde{\varepsilon}_*))=\varsigma(\mathrm{Z}_*(\sigma_v\tilde{\varepsilon}_*)), \tag{17}$$

so that we can take $\sigma_v=1$ in step 1 above. The proofs of (15) and (17) are postponed to Appendix.

Power simulations

The proportion of times that the values of $\varsigma(Z)$ in (12) exceeded the critical value is recorded as the empirical power of the test . We will use the economic data in a given region in [5] to illustrate the power of the test statistic $\varsigma(Z)$. The data are given in Table1. The dependent variable is y, the total cost of the class A goods for export (TCG). The independent variable is x_2, the gross national product (GNP). Let $x_1\equiv 1$.

Table1 The TCG and GNP Data in a region

Year	y_i	x_{i1}	x_{i2}
1967	4010	1	22418
1968	3711	1	22308
1969	4004	1	23319
1970	4151	1	24180
1971	4569	1	24893
1972	4582	1	25310
1973	4697	1	25799
1974	4753	1	25866
1975	5062	1	26868
1976	5669	1	28134
1977	5628	1	29091
1978	5736	1	29450
1979	5946	1	30705
1980	6501	1	32372
1981	6549	1	33152

1982	6705	1	33764
1983	7104	1	34411
1984	7609	1	35429
1985	8100	1	36200

Let the significance level $\alpha = 0.05$. Using the ordinary least squares(OLS) estimator of the $\beta = (\beta_1, \beta_2)'$, the sample coefficient of determination R^2 is 0.9816. However, the AR(1) disturbance has been found by the Durbin-Watson test. Thus, we consider the model (1) with $p = 2$ and the sample size $n = 19$. The conditional MLE $\breve{\rho}$ in (13) is 0.6. Using the modified AD test statistic ς in (12) with $n = 19$ and the estimated critical value, the null hypothesis H_0 in (11) is accepted, i.e., $v_1, v_2, \cdots, v_n$ in (1) are i.i.d. $\sim N(0, \sigma_v^2)$, where

$$v_1 = \sqrt{1-\rho^2} \cdot \varepsilon_1, v_i = \varepsilon_i - \rho\varepsilon_{i-1}, i = 2, \cdots, \mathrm{n}. \quad (18)$$

The power simulation of ς in (12) is given in Table2. We consider testing the null hypothesis H_0 in (11) against 14 alternatives. These alternatives include the distribution such as Student $T(\mathrm{s}), \mathrm{s} = 1,2,3$, Beta, Exponential, Normal and Chi-square. Samples of sizes $n = 19$ with replications of 20000 are generated.

Table2 Simulated powers of ς

Alternative	$T(1)$	$T(2)$	$T(3)$	$B(0.5,0.5)$	$B(1,3)$	$N(1,1)$	$N(10,1)$
	0.7679	0.4133	0.2468	0.3928	0.2979	0.0751	0.9998
Alternative	$Exp(0.5)$	$Exp(1)$	$Exp(2)$	$Lap(0,1)$	$Chi2(1)$	$N(0,2)$	$N(0,9)$
	0.5809	0.5770	0.5796	0.1894	0.8121	0.0485	0.0512

From Table2, the following statements can be asserted:

(1) The powers of the test statistic ς are nearly 0.05 against the $N(0,2)$ and $N(0,9)$ distributions. These results are consistent with the significance level α=0.05. The $N(0,2)$ and $N(0,9)$ distributions belong to the $N(0,\tau^2)$ family, thus the powers of ς are around 0.05.

(2) The powers of the test statistic ς against the $N(1,1)$ and $N(10,1)$ distributions are 0.0751 and 0.9998, respectively. These results are consistent with the fact that the $N(\mu,1), \mu = 1,10$ do not belong to the $N(0,\tau^2)$ family and the power of ς increases with increasing the parameter $\mu(\mu = 1 \to 10)$.

(3) ς offers good power against the other alternatives, such as the $T(1)$ and the $Chi2(1)$ distributions.

Conclusions

When regression model is applied to time series data, the residuals are frequently autocorrelated. Based on the modified Anderson-Darling test, we present the goodness-of-fit test for the normal distribution of the AR(1) disturbances of the multiple linear regression model. The asymptotically normal distribution of the transformed sample is obtained under the null hypothesis. The critical values of the test statistic ς can be estimated by Monte Carlo simulation. One advantage of the test is that ς is constructed by using the data matrix (X,Y) in (3). Thus ς reflects the inner relation between the observation vector Y and the design matrix X. Therefore the test statistic ς will have good power against a wide range of alternatives.

A kth-order autoregressive disturbance takes the form $\varepsilon_i = \varphi_1\varepsilon_{i-1} + \varphi_2\varepsilon_{i-2} + \cdots + \varphi_k\varepsilon_{i-k} + v_i$, where v_i is a white noise process. The goodness-of-fit tests for the AR(k) disturbances of the linear regression model need further investigation.

Appendix

Proof of Theorem1 (a). Let $\hat{\theta}_{n-1}=(\hat{\rho},\hat{\beta}(\hat{\rho}),\hat{\sigma}_v^2(\hat{\rho}))$ denote the conditional MLE of $\theta=(\rho,\beta,\sigma_v^2)$ in LnL_1. By lemma1, $\hat{\theta}_{n-1}\xrightarrow{p}\theta, n\to\infty$, where $\xrightarrow{p}$ denotes convergence in probability. Thus

$$\hat{v}_i=\tilde{y}_i(\hat{\rho})-[\tilde{x}_i(\hat{\rho})]'\hat{\beta}(\hat{\rho})\to\tilde{y}_i(\rho)-[\tilde{x}_i(\rho)]'\beta, n\to\infty, \quad (19)$$

where $\tilde{x}_i(\cdot)$ and $\tilde{y}_i(\cdot)$ are defined in (6). Since

$$v(\rho)/\sigma_v \sim N(0,I_n), \hat{\sigma}_v(\hat{\rho})\to\sigma_v, n\to\infty, \quad (20)$$

where $v(\rho)$ and $\hat{\sigma}_v(\hat{\rho})$ are defined in (5) and (10), respectively. We have

$$z_i=\hat{v}_i/\hat{\sigma}_v(\hat{\rho})\xrightarrow{d}v_i/\sigma_v \sim N(0,1), n\to\infty,$$

where $\xrightarrow{d}$ denotes convergence in distribution. Proof of Theorem1 (b). By (20), $v_1,\cdots,v_n$ are independent, thus $z_1,\cdots,z_n$ are asymptotically independent.

Proof of (15).

$$\begin{aligned}\hat{v}_{1*}&=\tilde{y}_1(\breve{\rho})-[\tilde{x}_1(\breve{\rho})]'\hat{\beta}(\breve{\rho})\\&=(\tilde{x}_1'\beta+\tilde{\varepsilon}_{1*})-\tilde{x}_1'(\tilde{X}_2'\tilde{X}_2)^{-1}\tilde{X}_2'(\tilde{X}_2'\beta+\tilde{E}_{2*})\\&=\tilde{\varepsilon}_{1*}-\tilde{x}_1'(\tilde{X}_2'\tilde{X}_2)^{-1}\tilde{X}_2'\tilde{E}_{2*}.\end{aligned} \quad (21)$$

Here $\tilde{x}_1=\tilde{x}_1(\breve{\rho}), \tilde{X}_2=\tilde{X}_2(\breve{\rho})$.

Proof of (17). By (13), $\hat{v}_{1*}(\sigma_v\cdot\tilde{\varepsilon}_*)=\sigma_v\hat{v}_{1*}(\tilde{\varepsilon}_*)$, $\hat{V}_{2*}(\sigma_v\cdot\tilde{\varepsilon}_*)=\sigma_v\hat{V}_{2*}(\tilde{\varepsilon}_*)$. Thus by (14),

$$\hat{\sigma}_{v*}^2(\sigma_v\cdot\tilde{\varepsilon}_*)=\sigma_v^2\hat{\sigma}_{v*}^2(\tilde{\varepsilon}_*). \quad (22)$$

By (16), we have $Z_*(\sigma_v\tilde{\varepsilon}_*)=Z_*(\tilde{\varepsilon}_*)$. Hence ς_* in (17) does not depend on σ_v, the desired result is obtained.

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