Boundary Algebra and Exact Solvability of the Asymmetric Exclusion Process

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Abstract

We consider a lattice driven diffusive system with $U_q(su(2))$ invariance in the bulk. Within the matrix product states approach the stationary probability distribution is expressed as a matrix product state with respect to a quadratic algebra. Boundary processes amount to the appearance of parameter dependent linear terms in the algebraic relations and lead to a reduction of the bulk symmetry. We find the boundary quantum group of the process to be a tridiagonal algebra, the linear covariance algebra for the bulk $U_q(su(2))$ symmetry, which allows for the exact solvability.

1   Introduction

Stochastic interacting particle systems [1] received a lot of attention since they provide a way of modelling phenomena like traffic flow [2], kinetics of biopolymerization [3], interface growth [4]. Among these, the asymmetric simple exclusion process (ASEP) has become a paradigm in nonequilibrium physics due to its simplicity, rich behaviour and wide range of applicability.

   The asymmetric exclusion process is an exactly solvable model of a lattice diffusion system of particles interacting with a hard core exclusion, i.e. the lattice site can be either empty or occupied by a particle. As a stochastic process it is described in terms of a probability distribution $P(s_i, t)$ of a stochastic variable $s_i = 0, 1$ at a site $i = 1, 2, ...L$ of a linear chain. A state on the lattice at a time t is determined by the occupation numbers s_i and a transition to another configuration s_i' during an infinitesimal time step dt is given by the probability $\Gamma(s, s')dt$. Due to probability conservation $\Gamma(s, s) = -\sum_{s' \neq s} \Gamma(s', s)$. The rates $\Gamma \equiv \Gamma_{jl}^{ik}$, $i, j, k, l = 0, 1$ are assumed to be independent from the position in the bulk. For diffusion processes the transition rate matrix becomes simply $\Gamma_{ki}^{ik} = g_{ik}$. At the boundaries, i.e. sites 1 and L additional processes can take place with rates L_i^j and R_i^j ($i, j = 0, 1$). In the set of occupation numbers $(s_1, s_2, ..., s_L)$ specifying a configuration of the system $s_i = 0$ if a site i is empty, $s_i = 1$ if there is a particle at a site i. Particles hop to the left with probability $g_{01}dt$ and to the right with probability $g_{10}dt$. The event of exchange happens if out of two adjacent sites one is a vacancy and the other is occupied by a particle. The symmetric simple exclusion process is known as the lattice gas model of particle hopping between nearest-neighbour sites with a constant rate $g_{01} = g_{10} = g$. The partially asymmetric simple exclusion

process with hopping in a preferred direction is the driven diffusive lattice gas of particles moving under the action of an external field. The process is totally asymmetric if all jumps occur in one direction only, and partially asymmetric if there is a different non-zero probability of both left and right hopping. The number of particles in the bulk is conserved and this is the case of periodic boundary conditions. In the case of open systems, the lattice gas is coupled to external reservoirs of particles of fixed density. The most interesting examples (see [5] for a review) are phase transitions inducing boundary processes [6] when a particle is added with probability αdt and/or removed with probability γdt at the left end of the chain, and it is removed with probability βdt and/or added with probability δdt at the right end of the chain.

The time evolution of the model is governed by the master equation for the probability distribution of the stochastic system $\frac{dP(s,t)}{dt} = \sum_{s'} \Gamma(s,s')P(s',t)$ which can be mapped to a Schroedinger equation in imaginary time for a quantum Hamiltonian with nearest-neighbour interaction in the bulk and single-site boundary terms $\frac{dP(t)}{dt} = -HP(t)$ where $H = \sum_j H_{j,j+1} + H^{(L)} + H^{(R)}$. The probability distribution thus becomes a state vector in the configuration space of the quantum spin chain and the ground state of the Hamiltonian, in general non-Hermitian, corresponds to the steady state of the stochastic dynamics. As known the open ASEP is related to the integrable spin 1/2 XXZ quantum spin chain through the similarity transformation $\Gamma = -qU_\mu^{-1}H_{XXZ}U_\mu$ [7] with $q = \frac{g_{01}}{g_{10}} \neq 1$. H_{XXZ} is the Hamiltonian of the $U_q(su(2))$ invariant quantum spin chain H_{XXZ}^{qg} with anisotropy Δ and with added non diagonal boundary terms B_1 and B_L (which depend on the ASEP boundary parameters)

$$H_{XXZ} = -1/2 \sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y - \Delta \sigma_i^z \sigma_{i+1}^z + 1/2(q-q^{-1})(\sigma_{i+1}^z - \sigma_i^z) + \Delta) + B_1 + B_L \quad (1.1)$$

2   Matrix product state approach to ASEP

The idea of the matrix product ansatz [5, 8] is that the stationary probability distribution is expressed as a product of (or a trace over) matrices that form a representation of a quadratic algebra. Without loss of generality one can choose the right probability rate $g_{10} = 1$ and the left probability rate $g_{01} = q$. The quadratic algebra of the ASEP has the form

$$D_1 D_0 - qD_0 D_1 = x_0 D_1 - D_0 x_1, \qquad x_0 + x_1 = 0 \quad (2.1)$$

where $0 < q < 1$ and x_0, x_1 are representation dependent parameters. The totally asymmetric process corresponds to $q = 0$.

For systems with periodic boundary conditions, the stationary probability distribution is related to the expression

$$P(s_1,....s_L) = Tr(D_{s_1} D_{s_2}...D_{s_L}). \quad (2.2)$$

When boundary processes are considered the normalized stationary probability distribution is expressed as a matrix element in the auxiliary vector space

$$P(s_1,....s_L) = \frac{\langle w|D_{s_1} D_{s_2}...D_{s_L}|v\rangle}{Z_L}, \quad (2.3)$$

with respect to the vectors $|v\rangle$ and $\langle w|$, defined by the boundary conditions

$$(\beta D_1 - \delta D_0)|v\rangle = x_0|v\rangle \quad (2.4)$$
$$\langle w|(\alpha D_0 - \gamma D_1) = \langle w|(-x_1)$$

The normalization factor to the stationary probability distribution is

$$Z_L = \langle w|(D_0 + D_1)^L|v\rangle \tag{2.5}$$

These relations simply mean that one associates with an occupation number s_i at position i a matrix $D_{s_i} = D_1$ if a site $i = 1, 2, ..., L$ is occupied and $D_{s_i} = D_0$ if a site i is empty.

The advantage of the matrix-product ansatz is that once the representations of the diffusion algebra and the boundary vectors are known, one can also evaluate all the relevant physical quantities such as the mean density at a site i $\langle s_i \rangle = \frac{\langle w|(D_0+D_1)^{i-1}D_1(D_0+D_1)^{L-i}|v\rangle}{Z_L}$, the two-point correlation function $\langle s_i s_j \rangle = \frac{\langle w|(D_0+D_1)^{i-1}D_1(D_0+D_1)^{j-i-1}D_1(D_0+D_1)^{L-j}|v\rangle}{Z_L}$ and higher correlation functions. The current J through a bond between site i and site $i+1$, has a very simple form $J = x_0 \frac{Z_{L-1}}{Z_L}$. The algebraic matrix state approach (MPA) is the equivalent formulation of recursion relations derived for the ASEP in earlier works [9, 10] which could not be readily generalized to other models. In most applications one uses infinite dimensional representations of the quadratic algebra. Finite dimensional representations [7, 11] impose a constraint on the model parameters. The MPA was generalized to many-species models [5, 12] and to dynamical MPA [13].

For a process with only incoming particles at the left boundary and only outgoing particles at the right one ($\delta = \gamma = 0$ in (2.4)) the quadratic algebra is solved [14] by a pair of deformed oscillators and the solution is related to q-Hermite [14] and Al-Salam-Chihara polynomials [15]. In the general case of four boundary parameters the exact solution was achieved through relation to the Askey-Wilson polynomials [16].

Led by the idea of the major importance of the boundary conditions for the ASEP steady state behaviour we consider the algebraic properties of the boundary operators. In the next section we construct the boundary operators using the representation of the $U_q(su(2))$ bulk symmetry and find that they generate a tridiagonal Askey-Wilson algebra whose irreducible modules are given in terms of the Askey-Wilson polynomials.

3 The tridiagonal boundary algebra

In the general case of incoming and outgoing particles at both boundaries there are four operators $\beta D_1, -\delta D_0, -\gamma D_1, \alpha D_0$ and one needs an addition rule to form two linear independent boundary operators acting on the dual boundary vectors. From the quadratic algebra (2.1) two relations follow

$$\beta D_1 \alpha D_0 - q\alpha D_0 \beta D_1 = x_1 \beta \alpha D_0 - \alpha \beta D_1 x_0 \tag{3.1}$$

and

$$\gamma D_1 \delta D_0 - q\delta D_0 \gamma D_1 = x_1 \gamma \delta D_0 - \delta \gamma D_1 x_0 \tag{3.2}$$

To find a solution of these quadratic relations we emphasize the equivalence of the ASEP to the integrable $su_q(2)$ spin $1/2$ XXZ and use the $U_q(su(2))$ algebra in the form of a deformed (u, v) algebra, $(u = -v < 0)$ with the defining commutation relations

$$[N, A_\pm] = \pm A_\pm \qquad [A_-, A_+] = uq^N + vq^{-N} \tag{3.3}$$

and a central element

$$Q = A_+ A_- + \frac{vq^N - uq^{1-N}}{1-q} \tag{3.4}$$

This representation allows one to write the two linearly independent boundary operators $B^R = \beta D_1 - \delta D_0, B^L = -\gamma D_1 + \alpha D_0$ in the form

$$\beta D_1 - \delta D_0 = \tag{3.5}$$
$$-\frac{x_1 \beta}{\sqrt{1-q}} q^{N/2} A_+ - \frac{x_0 \delta}{\sqrt{1-q}} A_- q^{N/2} - \frac{x_1 \beta q^{1/2} + x_0 \delta}{1-q} q^N - \frac{x_1 \beta + x_0 \delta}{1-q}$$
$$\alpha D_0 - \gamma D_1 =$$
$$\frac{x_0 \alpha}{\sqrt{1-q}} q^{-N/2} A_+ + \frac{x_1 \gamma}{\sqrt{1-q}} A_- q^{-N/2} + \frac{x_0 \alpha q^{-1/2} + x_1 \gamma}{1-q} q^{-N} + \frac{x_0 \alpha + x_1 \gamma}{1-q}$$

We separate the shift parts from the boundary operators. Denoting the corresponding rest operator parts by A and A^* we write the left and right boundary operators in the form

$$\beta D_1 - \delta D_0 = A - \frac{x_1 \beta + x_0 \delta}{1-q} \tag{3.6}$$
$$\alpha D_0 - \gamma D_1 = A^* + \frac{x_0 \alpha + x_1 \gamma}{1-q}$$

The operators A and A^* defined by

$$A = \beta D_1 - \delta D_0 + \frac{x_1 \beta + x_0 \delta}{1-q} \tag{3.7}$$
$$A^* = \alpha D_0 - \gamma D_1 - \frac{x_0 \alpha + x_1 \gamma}{1-q}$$

and their q-commutator

$$[A, A^*]_q = q^{1/2} A A^* - q^{-1/2} A^* A \tag{3.8}$$

in the representation (3.5) form a closed linear algebra

$$[[A, A^*]_q, A]_q = -\rho A^* - \omega A - \eta \tag{3.9}$$
$$[A^*, [A, A^*]_q]_q = -\rho^* A - \omega A^* - \eta^*$$

where the representation dependent structure constants are given by

$$-\rho = x_0 x_1 \beta \delta q^{-1} (q^{1/2} + q^{-1/2})^2, \qquad -\rho^* = x_0 x_1 \alpha \gamma q^{-1} (q^{1/2} + q^{-1/2})^2 \tag{3.10}$$

$$-\omega = (x_1 \beta + x_0 \delta)(x_1 \gamma + x_0 \alpha) - (x_1^2 \beta \gamma + x_0^2 \alpha \delta)(q^{1/2} - q^{-1/2})Q \tag{3.11}$$

$$\eta = q1/2(q^{1/2} + q^{-1/2}) \left(x_0 x_1 \beta \delta (x_1 \gamma + x_0 \alpha)Q - \frac{(x_1 \beta + x_0 \delta)(x_1^2 \beta \gamma + x_0^2 \alpha \delta)}{q^{1/2} - q^{-1/2}} \right) \tag{3.12}$$
$$\eta^* = q1/2(q^{1/2} + q^{-1/2}) \left(x_0 x_1 \alpha \gamma (x_1 \beta + x_0 \delta)Q + \frac{(x_0 \alpha + x_1 \gamma)(x_0^2 \alpha \delta + x_1^2 \beta \gamma)}{q^{1/2} - q^{-1/2}} \right)$$

Relations (3.9) are the well known Askey-Wilson relations

$$A^2A^* - (q+q^{-1})AA^*A + A^*A^2 = \rho A^* + \omega A + \eta \tag{3.13}$$
$$A^{*2}A - (q+q^{-1})A^*AA^* + AA^{*2} = \rho^*A + \omega A^* + \eta^*$$

for the shifted boundary operators A, A^*. The algebra (3.9) was first considered in the works of Zhedanov [17, 18] and recently discussed in a more general framework of a tridiagonal algebra [19, 20]. It is an associative algebra with a unit generated by a (tridiagonal) pair of operators A, A^* and defining relations

$$[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \rho A^*] = 0 \tag{3.14}$$
$$[A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(AA^* + A^*A) - \rho^*A] = 0$$

In the general case a tridiagonal pair is determined by the sequence of scalars $\beta, \gamma, \gamma^*, \rho, \rho^*$ from a field K. (We note that we keep the conventional notations for the scalars in (3.14) - β and γ should not be confused with the boundary rates.) Tridiagonal pairs have been classified according to the dependence on the scalars [19]. Examples are the q-Serre relations with $\beta = q + q^{-1}$ and $\gamma = \gamma^* = \rho = \rho^* = 0$ and the Dolan-Grady relations [21] with $\beta = 2, \gamma = \gamma^* = 0, \rho = k^2, \rho^* = k^{*2}$. The AW relations considered in [22, 23] for the XXZ chain correspond to $\rho = \rho^*, \eta = \eta^* = 0$.

Tridiagonal pairs are determined up to an affine transformation

$$A \to tA + c, \qquad A^* \to t^*A^* + c^* \tag{3.15}$$

where t, t^*, c, c^* are some scalars. The affine transformation can be used to bring a tridiagonal pair in a reduced form with $\gamma = \gamma^* = 0$.

The (shifted) boundary operators of the asymmetric exclusion process obeying the Askey-Wilson algebra (3.9) form a tridiagonal pair with $\beta = q + q^{-1}, \gamma = \gamma^* = 0$, and ρ, ρ^* following from $\rho, \rho^*, \omega, \eta, \eta^*$ as given by eqs.(3.10 - 3.12). The Askey-Wilson algebra possesses some important properties that allow to obtain its ladder representations, spectra, overlap functions (for details see [17, 20]). Namely, there exists a basis f_r with respect to which A is diagonal $Af_r = \lambda_r f_r$ and the operator A^* is tridiagonal $Af_r = a_{r+1}f_{r+1} + b_r f_r + c_{r-1}f_{r-1}$. The diagonal eigenvalues satisfy a quadratic equation

$$\lambda_{r+1}^2 + \lambda_r^2 - (q+q^{-1})\lambda_r\lambda_{r+1} - \rho = 0 \tag{3.16}$$

which yields the spectrum

$$\lambda_r = q^{-r} + \frac{\rho q^r}{(q-q^{-1})^2} \tag{3.17}$$

The algebra possesses a duality property. Due to the duality property the dual basis exists in which the operator A^* is diagonal $A^*f_p^* = \lambda_p^* f_p^*$ and the operator A is tridiagonal $Af_s^* = a_{s+1}^* f_{s+1}^* + b_s^* f_s^* + c_{s-1}^* f_{s-1}^*$ where λ_p^* satisfies the quadratic equation (3.16) with $-\rho$ replaced by $-\rho^*$. The overlap function of the two basis $\langle s|r \rangle = \langle f_s^*|f_r \rangle$ can be expressed in terms of the Askey-Wilson polynomials. To obtain the explicit form of the infinite-dimensional representation we make use of the rescaling property to bring the Askey-Wilson algebra (3.9) in a form with a known (basic) representation. For the purpose we first divide the first relation in eq.(2.4) by β and the

second one by α, which amounts to a tridiagonal pair following from the structure constants $\rho/\beta^2, \rho^*/\alpha^2, \omega/\alpha\beta, \eta/\alpha\beta^2, \eta^*/\alpha^2\beta$. For further convenience we denote

$$-\frac{\gamma}{\alpha} = ac \qquad -\frac{\delta}{\beta} = bd \tag{3.18}$$

Besides, we set $x_0 = -x_1 = s$, where s is a free parameter from the algebraic relation $x_0 + x_1 = 0$. We rescale the generators $A \equiv \frac{1}{\beta}A$ and $A^* \equiv \frac{1}{\alpha}A^*$ as follows

$$A \to (q^{-1/2} - q^{1/2})\frac{1}{q^{-1/2}s\sqrt{bd}}A, \qquad A^* \to (q^{-1/2} - q^{1/2})\frac{\sqrt{bd}}{s}A^* \tag{3.19}$$

The tridiagonal relations for the transformed operators read

$$[A, A^2A^* - (q+q^{-1})AA^*A + A^*A^2 + (q-q^{-1})^2A^*] = 0 \tag{3.20}$$
$$[A^*, A^{*2}A - (q+q^{-1})A^*AA^* + AA^{*2} + abcdq^{-1}q - q^{-1})^2A] = 0$$

where $abcd = \frac{\gamma}{\alpha}\frac{\delta}{\beta}$. Let $p_n = p_n(x; a, b, c, d)$ denote the nth Askey-Wilson polynomial [24] depending on four parameters a, b, c, d

$$p_n =_4 \Phi_3 \left(\begin{array}{c} q^{-n}, abcdq^{n-1}, ay, ay^{-1} \\ ab, ac, ad \end{array} \Big| q; q \right) \tag{3.21}$$

with $p_0 = 1$, $x = y + y^{-1}$ and $0 < q < 1$. Then, there is a basic representation of this algebra in the space of symmetric Laurent polynomials $f[y]$ with a basis $(p_0, p_1, ...)$ as follows

$$Af[y] = (y + y^{-1})f[y], \qquad A^*f[y] = \mathscr{D}f[y] \tag{3.22}$$

where $\mathscr{D}$ is the second order q-difference operator [24] having the Askey-Wilson polynomials p_n as eigenfunctions.

$$\mathscr{D}p_n = \lambda_n^* p_n, \qquad \lambda_n^* = q^{-n} + abcdq^{n-1} \tag{3.23}$$

and the operator A^* is represented by an infinite-dimensional matrix $\mathrm{diag}(\lambda_0^*, \lambda_1^*, \lambda_2^*, ...)$. The operator $Ap_n = xp_n$ is represented by a tridiagonal matrix

$$\mathscr{A} = \begin{pmatrix} a_0 & c_1 & & \\ b_0 & a_1 & c_2 & \\ & b_1 & a_2 & \cdot \\ & & \cdot & \cdot \end{pmatrix} \tag{3.24}$$

whose matrix elements are obtained from the three term recurrence relation for the Askey-Wilson polynomials

$$xp_n = b_n p_{n+1} + a_n p_n + c_n p_{n-1}, \qquad p_{-1} = 0 \tag{3.25}$$

The explicit form of the matrix elements of A reads

$$a_n = a + a^{-1} - b_n - c_n \tag{3.26}$$

$$b_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})} \tag{3.27}$$

$$c_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})} \tag{3.28}$$

The basis is orthogonal with the orthogonality condition for the Askey-Wilson polynomials $P_n = a^{-n}(ab,ac,ad;q)_n p_n$: $\int_{-1}^{1} \frac{w(x)}{2\pi\sqrt{1-x^2}} P_m(x;a,b,c,d|q) P_n(x;a,b,c,d|q) dx = h_n \delta_{mn}$, where $w(x)$ the weight function and h_n is a proper normalization [24]. Returning to the original operators we readily obtain their representation by making use of the proper scale transformations. A representation π with $(q^{-1/2} - q^{1/2})\frac{1}{\beta}A$ diagonal and $(q^{-1/2} - q^{1/2})\frac{1}{\alpha}A^*$ tridiagonal is realized in a space with basis

$$(p_0, p_1, p_2, ...)^t \tag{3.29}$$

The diagonal matrix is $\mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, ...)$ with λ_n being the eigenvalue

$$\lambda_n = \kappa q^{-n} + \frac{s^2 bd}{\kappa} q^{n-1} \tag{3.30}$$

of $\frac{s}{\sqrt{ac}}\mathscr{D}$. We have written λ_n in this form redefining $\frac{s}{\sqrt{ac}}$ as a new parameter κ and treating it independent on ac. λ_n in eq.(3.30) is the general solution of the quadratic equation (3.16). The tridiagonal matrix is obtained from the transposed matrix $\mathscr{A}$ upon multiplication by $s^2 q^{-1/2}\kappa$. The dual representation π^* has a basis

$$(p_0, p_1, p_2, ...) \tag{3.31}$$

with respect to which $(q^{-1/2} - q^{1/2})\frac{1}{\alpha}A^*$ is diagonal with diagonal elements

$$\lambda_n^* = \kappa^* q^{-n} + \frac{s^2 ac}{\kappa^*} q^{n-1} \tag{3.32}$$

which are the eigenvalues of the rescaled $\frac{s}{\sqrt{bd}}\mathscr{D}^*$. Once again we write λ^* in this form by redefining $\frac{s}{\sqrt{bd}}$ as a new independent parameter κ^*. λ^* in eq.(3.32) is the general solution of the spectrum defining quadratic equation (3.16) with $-\rho$ replaced by $-\rho^*$. The matrix $(q^{-1/2} - q^{1/2})\frac{1}{\beta}A$ is tridiagonal and its matrix elements are found upon multiplication of the matrix $\mathscr{A}$ by $s^2 q^{-1/2}\kappa^*$. As a consequence of this one obtains the corresponding representations for the left and right boundary operators by shifting the diagonal elements of the rescaled A, A^* according to (3.7). The result reads explicitly: In a representation π the right boundary operator $D_1 + bdD_0$ is represented by a diagonal infinite dimensional matrix with eigenvalues

$$\lambda_n(s) = \frac{q^{1/2}}{1-q}\left(\kappa q^{-n} + \frac{s^2 bd}{\kappa}q^{n-1}\right) + \frac{s}{1-q}(1+bd) \tag{3.33}$$

The left boundary operator $D_0 + acD$ is tridiagonal whose representing matrix is

$$\pi(D_0 + acD) = s^2 q^{-1/2}\kappa\mathscr{A}^t + \frac{s}{1-q}(1+ac) \tag{3.34}$$

In the dual representation π^* the operator $D_0 + acD_1$ is diagonal with eigenvalues

$$\lambda_n^*(s) = \frac{q^{1/2}}{1-q}\left(\kappa^* q^{-n} + \frac{s^2 ac}{\kappa^*}q^{n-1}\right) + \frac{s}{1-q}(1+ac) \tag{3.35}$$

and $D_1 + bdD_0$ is tridiagonal with representing matrix

$$\pi^*(D_1 + bdD_0) = s^2 q^{-1/2} \kappa^* \mathscr{A} + \frac{s}{1-q}(1+bd) \tag{3.36}$$

The formulae (3.33-3.36) define the ladder representation (resp. the dual representation) of the tridiagonal pair in a Hilbert space with an inner product, (the auxiliary Hilbert space of the ASEP).

To solve the ASEP boundary problem we choose the left and right boundary vectors to be to be of the form

$$\langle w| = h_0^{-1/2}(p_0, 0, 0, ...) \qquad |v\rangle = h_0^{-1/2}(p_0, 0, 0, ...)^t \tag{3.37}$$

where h_0 is a normalization from the orthogonality condition. These vectors belong to the two dual representations of the tridiagonal boundary algebra and are the eigenvectors of the corresponding diagonal operator B^R and B^L. The eigenvalue equations have the form

$$(D_1 - \frac{\delta}{\beta}D_0)|v\rangle - \frac{s}{\beta}|v\rangle = 0 \tag{3.38}$$

$$\langle w|(D_0 - \frac{\gamma}{\alpha}D_1) - \langle w|\frac{s}{\alpha} = 0$$

It follows from the above relations that the constants κ, κ^* obey the quadratic equations

$$\kappa^2 + \frac{1}{\beta}(1 - \delta - (1-q))\kappa - \frac{\delta}{\beta} = 0 \tag{3.39}$$

$$(\kappa^*)^2 + \frac{1}{\alpha}(1 - \gamma - (1-q))\kappa^* - \frac{\gamma}{\alpha} = 0$$

with solutions

$$\kappa_\pm = \frac{-(\beta - \delta - (1-q)) \pm \sqrt{(\beta - \delta - (1-q))^2 + 4\beta\delta}}{2\beta} \tag{3.40}$$

$$\kappa_\pm^* = \frac{-(\alpha - \gamma - (1-q)) \pm \sqrt{(\alpha - \gamma - (1-q))^2 + 4\alpha\gamma}}{2\alpha}$$

Hence the boundary eigenvalue equations are satisfied for the corresponding roots (3.40) which (in this representation) are uniquely identified with the four parameters of the Askey-Wilson polynomials

$$a = \kappa_+^*, \quad b = \kappa_+, \quad c = \kappa_-^*, \quad d = \kappa_- \tag{3.41}$$

We can further show that each boundary operator and the transfer matrix operator generate isomorphic AW algebras. This allows for the calculation of the relevant physical quantities in terms of the Askey-Wilson polynomials.

4 Discussion and conclusion

We have constructed the boundary operators of the open ASEP as linear covariance elements for the $U_q(su(2))$, which is the invariance algebra of the integrable XXZ chain. It is known [25] that the bulk driven diffusive system with reflecting boundaries can be mapped to the spin $1/2$ $U_q(su(2))$-invariant quantum spin chain. Within the matrix product approach the bulk process is

described by a quadratic algebra with no linear x-dependent terms $D_1D_0 - qD_0D_1 = 0$. The stationary probability distribution, i.e. the ground state of the $U_q(su(2))$ invariant Hamiltonian H_{XXZ}^{qg}, corresponds to the q-symmetrizer of the Young diagram with one row and L boxes [26]. The presence of the boundary processes (i.e. the nondiagonal boundary terms in the Hamiltonian) reduces the $U_q(su(2))$ bulk invariance and amounts to the appearance of linear terms in the quadratic algebra. The boundary conditions define the boundary operators which carry a residual symmetry of the process. It is expressed in the fact that the boundary operators are constructed in terms of the $U_q(su(2))$ generators, as seen from the explicit formulae (3.5). With $A_{\pm}, N$ being the generators of a finite dimensional $U_q(su(2))$ representation, it can be verified from eq.(3.5) that $\alpha D_0 - \gamma D_1$ commutes with $H(q)^{qg}$ and $\beta D_1 - \delta D_0$ commutes with $H(-q^{-1})^{qg}$, related to $H(q)^{qg}$ by a gauge transformation. Thus the boundary operators constructed as the linear covariant objects of the bulk $U_q(su(2))$ symmetry acquire a very important physical meaning - they can be interpreted as the two nonlocal conserved charges of the open ASEP. Such non local boundary symmetry charges were originally obtained for the sine Gordon model [27] and generalized to affine Toda field theories [28] and derived from spin chain point of view as commuting with the transfer matrix for a special choice of the boundary conditions [29]. In particular, the left boundary operator $\alpha D_0 - \gamma D_1$ in the finite dimensional representation (26) is analogous to the one boundary Temperley-Lieb algebra centralizer in the "nondiagonal" spin $1/2$ representation [30].

We have used the deformed (u, v) algebra for the solution of the boundary problem to include and generalize previously known solutions of the MPA. The (u, u) algebra, known as deformed oscillator algebra $c_q u(2)$ was considered in [31] in relation to known solutions in terms of deformed oscillators. It is important to once again emphasize the representation dependence of the Askey-Wilson algebra (as well as of the MPA bulk quadratic algebra (2.1)). Using any of the particular forms of the deformed (u, v) algebra we obtain the AW algebra as its linear covariance algebra. The functional dependent structure constants $\rho, \rho^*, \omega, \eta, \eta^*$ in eqs.(3.10-3.12) carry the information of the corresponding algebra and in particular, this reflects in different spectra of the diagonal (tridiagonal) operators and different Askey-Wilson polynomials. This is the formal mathematical difference between the deformed general oscillator algebra $c_q u(2)$ used in [31] and the $U_q(su(2))$ case in the present paper. Namely, the spectrum of the diagonal operators for $c_q u(2)$ with positive structure constants ρ, ρ^* is of the form $\sim cosh$, while for $U_q(su(2))$ with negative structure constants ρ, ρ^*, it is $\sim sinh$. Hence one has different identifications of the AW four parameters with the boundary rates which, in our opinion, may enrich the variety of physical applications and is worth considering.

There is one very important difference between the $U_q(su(2))$ case and the $c_q u(2)$ one. It lies in the fact that $U_q(su(2))$ is the invariance of the ASEP in the bulk which is broken by the boundary processes with incoming and outgoing particles at both boundaries. The presence of boundary processes breaks the bulk invariance and destroys the integrability of the (equivalent) quantum spin chain. With suitably chosen boundary conditions a remnant of the $U_q(su(2))$ quantum bulk symmetry can survive. It is the purpose of our consideration to show that the reduction of the bulk invariance gives rise to the boundary symmetry which remains as the linear covariance algebra of the bulk $U_q(su(2))$ symmetry. Thus the boundary Askey-Wilson algebra whose structure constants depend on the finite dimensional $U_q(su(2))$ representations is the residual symmetry of the open ASEP and this has important physical consequences, in particular, in relation to Bethe ansatz integrability. The Bethe solution of the open ASEP [32] was achieved through the mapping to the $U_q(su(2))$ integrable XXZ quantum spin chain with most general non diagonal boundary terms, provided a particular constraint was satisfied. The $c_q u(2)$ algebra has only infinite dimensional

representation as opposed to $U_q(su(2))$ which (for generic q) has finite dimensional representations only. This mathematical difference in the choice of the representation to form the boundary algebra implies a physical consequence which turns to be the key in relation to Bethe ansatz integrability. The suitably chosen representation dependent boundary algebra manifests itself in the extent to which integrability is preserved. In the $c_q u(2)$ case the exact solvability of the model is achieved in the stationary state. With $U_q(su(2))$ one can further employ Bethe ansatz to obtain exact results for the approach to stationarity at large times and to completely determine the spectrum of the transfer matrix. As commented in [33] the way one can satisfy the condition for the Bethe ansatz solution of the ASEP implies additional symmetries. In our opinion, the linear covariance Askey-Wilson algebra of the bulk $U_q(su(2))$, whose generators are interpreted as the two nonlocal conserved charges of the ASEP, is the hidden symmetry behind Bethe ansatz solvability.

To summarize, we have considered the open asymmetric exclusion process which is equivalent to the integrable XXZ spin chain with bulk $U_q(su(2))$ symmetry. Within the matrix product ansatz the boundary processes amount to the presence of linear terms in the quadratic algebra and lead to a reduction of the bulk symmetry. The boundary operators generate a tridiagonal Askey-Wilson algebra, which is the linear covariance algebra of the bulk $U_q(su(2))$ symmetry. It is the symmetry that survives and allows for the exact solvability in the stationary state and provides the framework for employing Bethe ansatz to determine the dynamical properties of the open process.

5 Acknowledgments

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