Iterative Algorithms for System of Generalized Mixed Equilibrium Problems[1]

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Abstract. The purpose of this paper is to present an iterative algorithm for system of generalized mixed equilibrium problems in Banach space. We prove convergence theorems of the iterative algorithm for finding a common element of the fixed point set of relatively nonexpansive mappings and the solution set of system of generalized mixed equilibrium problems in Banach space under some suitable conditions. Our results extend and improve previous result of [1-3].

Introduction and Preliminaries

Let E be a Banach space whose norm is denoted by $\|\cdot\|$. Let C be a nonempty closed subset of E. Let $F,G:C\times C\to R$ be two nonlinear bifunctions. Let J be the normalized duality mapping $Jx=\left\{x^*\in E^*:\left\langle x,x^*\right\rangle=\|x\|\|x^*\|,\|x\|=\|x^*\|\right\}, x\in E,$ where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ is the generalized duality pairing between E and E^*. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

We know the following[4]: If E is uniformly convex, then it is reflexive; If E is uniformly smooth if and only if E^* is uniformly convex; If E^* is uniformly convex, then J is uniformly norm-to-norm continuous on each bounded subset of E. If E is a smooth , strictly convex and reflexive Banach space and C is a nonempty closed convex subset of E. we denote by ϕ the function defined by $\phi(x,y)=\|x\|^2-2\langle x,Jy\rangle+\|y\|^2$, for all $x,y\in E$. (1.1)

Following Alber [5], the generalized projection $\prod_C:E\to C$ is a map that assigns to an arbitrary point $x\in E$ the minimum point of the functional $\phi(x,y)$, that is, $\prod_C x=\bar{x}$, where $\bar{x}$ is the solution to the minimization problem $\phi(\bar{x},x)=\inf_{y\in C}\phi(y,x)$. (1.2)

Existence and uniqueness of the operator $\prod_C$ follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J. It is obvious from the definition of function ϕ that $\left(\|y\|-\|x\|\right)^2\le\phi(y,x)\le\left(\|y\|+\|x\|\right)^2,\ \forall x,y\in E$. (1.3)

Define a function $V:E\times E^*\to R$ as follows:

$$V\left(x,x^*\right)=\|x\|^2-2\left\langle x,x^*\right\rangle+\left\|x^*\right\|^2,\forall\left(x,x^*\right)\in E\times E^*, \qquad (1.4)$$

then, it is obvious that $V\left(x,x^*\right)=\phi\left(x,J^{-1}\left(x^*\right)\right)$ and $V\left(x,J(y)\right)=\phi(x,y)$.

If E is a Hilbert space, then $\phi(x,y)=\|x-y\|^2$. If E is a reflexive, strictly convex and smooth Banach space, then for all $x,y\in E$, $\phi(x,y)=0$ if and only if $x=y$. It is sufficient to show that if $\phi(x,y)=0$ then $x=y$. This implies that $\langle x,Jy\rangle=\|x\|^2=\|Jy\|^2$. From the definition of J, one has $Jx=Jy$. Therefore, we have $x=y$ (see [4] for more details).

The generalized mixed equilibrium problem (for short, denoted by GMEP) is to find $x\in C$ such that

$$F(x,y)+\langle y-x,Ax\rangle\geq 0,\forall y\in C, \tag{1.5}$$

the solution set to GMEP(1.5) is denoted by GMEP(F, A).

In this paper, a system of generalized mixed equilibrium problems (for short, denoted by SGMEP) is introduced, that is, find an $x\in C$, such that

$$\begin{cases} F(x,y)+\langle y-x,Ax\rangle\geq 0,\forall y\in C,\\ G(x,y)+\langle y-x,Bx\rangle\geq 0,\forall y\in C,\end{cases} \tag{1.6}$$

The solution set to SGMEP(1.6) is denoted by SGMEP(F, G, A, B).

GMEP(1.5) also includes variational inclusion problems, variational inequality problems, complementarity problems, and Nash equilibria problems as special cases. Nowadays, many authors studied the mixed equilibrium problems and proposed different iterative algorithms, for details, see references [1,6,7]. In this paper, we propose an iterative algorithm for finding a common element of the fixed point set of relatively nonexpansive mappings in Banach space under some suitable conditions. The result obtained here extend and improve the corresponding results of [1-3].

In order to obtain our main results, we need the following definitions and lemmas.

Definition 1.1 Let $A:C\to E^*$ be monotone mappings if $\langle Ax-Ay,x-y\rangle\geq 0,\forall x,y\in E$.

Definition 1.2[1] A point $x\in C$ is a fixed point of a mapping $T:C\to C$ if $Tx=x$. The set of fixed points of T is denoted by T_{Fix}, that is, $T_{Fix}=\{x\in C|Tx=x\}$.

Definition 1.3[1] A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$, the set of asymptotic fixed points of T is denoted by $\hat{T}_{Fix}$.

Definition 1.4[1] A mapping T from C into itself is said to be relatively nonexpansive if $\hat{T}_{Fix}=T_{Fix}$ and $\varphi(p,Tx)\leq\varphi(p,x)$ for all $x\in C$ and $P\in T_{Fix}$.

Lemma 1.1[7] Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed convex subset of E and let $x\in E$. Then

$$\phi(y,\Pi_C x)+\phi(\Pi_C x,x)\leq\phi(y,x),\ \forall y\in C. \tag{1.7}$$

Lemma 1.2 Define $T_r:E\to C$ as follows:

$$T_r(x)=\left\{z\in C,F(z,y)+\langle y-z,Az\rangle+\frac{1}{r}\langle y-z,Jz-Jx\rangle\geq 0.\forall y\in C\right\}. \tag{1.8}$$

Then $$\phi(p,T_r(z))+\phi(T_r(z),z)\leq\phi(p,z),\forall p\in(T_r)_{Fix},z\in E \tag{1.9}$$

Lemma 1.3[7] Let C be a nonempty closed convex subset of a smooth Banach space E, and $x\in E$. Then, $x_0=\prod_C x$ if and only if

$$\langle x_0-y,Jx-Jx_0\rangle\geq 0,\ \forall y\in C. \tag{1.10}$$

Lemma 1.4[8] Let E be a strictly convex and smooth real Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n,y_n)\to 0$, and either $\{x_n\}$ or $\{y_n\}$ is bounded. Then $\|x_n-y_n\|\to 0$.

For solving SGMEP(1.6), let $F:C\times C\to R$ be a bifunction and $\phi:C\to R$ be a convex and lower semi-continuous function satisfying the following conditions:

(A1) $F(x,x)=0$, for all $x\in C$;

(A2) F is monotone, i.e., $F(x,y)+F(y,x)\leq 0$, for all $x,y\in C$;

(A3) For each $x,y,z\in C$, $\limsup_{t\to\infty}F(tz+(1-t)x,y)\leq F(x,y)$;

(A4) For each $x \in C,\ y \mapsto F(x,y)$ and $y \mapsto G(x,y)$ are convex and lower semicontinuous.

2. Main Results

Theorem 2.1 Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E. Let $F,G: C\times C \to R$ be two bifunctions satisfying conditions (A1)-(A4), Let $A,B: C \to E^*$ be continuous and monotone mappings, and let T be a relatively nonexpansive mapping from C into itself with $\Gamma = T_{Fix} \cap SGMEP(F,G,A,B) \neq \varnothing$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and $C_0 = C$,

$$\begin{aligned}
&F(u_n,y)+\langle y-u_n, Au_n\rangle+\frac{1}{r_n}\langle y-u_n, Ju_n-Jx_n\rangle \geq 0, \forall y\in C,\\
&G(y_n,y)+\langle y-y_n, By_n\rangle+\frac{1}{s_n}\langle y-y_n, Jy_n-Ju_n\rangle \geq 0, \forall y\in C,\\
&z_n=\Pi_C J^{-1}\left(\eta_n Jx_n+(1-\eta_n)Jy_n\right), \qquad (2.1)\\
&w_n=J^{-1}\left(\alpha_n Jx_n+(1-\alpha_n)JTz_n\right),\\
&C_{n+1}=\left\{z\in C_n : \phi(z,w_n)\leq \alpha_n\phi(z,x_n)+(1-\alpha_n)\phi(z,z_n)\leq \phi(z,x_n)\right\},\\
&x_{n+1}=\Pi_{C_{n+1}} x_0, \forall n\geq 0, x_0\in C.
\end{aligned}$$

Where $r_n, s_n > 0$, $\eta_n, \alpha_n \in (0,1)$, $\limsup_{n\to\infty}\alpha_n < 1$. Then the sequence $\{x_n\}$ generated by (2.1) converges strongly to the point $\Pi_\Gamma x_0$.

Proof we first show that $\{x_n\}$ is bounded. Let $p \in \Gamma = T_{Fix} \cap SGMEP(F,G,A,B)$.

It follows from Lemma 1.1 and (1.9), that $\phi(p,u_n)\leq\phi(p,x_n), \phi(p,y_n)\leq\phi(p,u_n)$. (2.2)

In virtue of (1.4) and the definition of V, we have

$$\begin{aligned}
\phi(p,z_n)&=\phi\left(p,\Pi_C J^{-1}\left(\eta_n Jx_n+(1-\eta_n)Jy_n\right)\right)\leq\phi\left(p,J^{-1}\left(\eta_n Jx_n+(1-\eta_n)Jy_n\right)\right)\\
&=V\left(p,\eta_n Jx_n+(1-\eta_n)Jy_n\right)\leq\eta_n\phi(p,x_n)+(1-\eta_n)\phi(p,y_n)\leq\phi(p,x_n). \qquad (2.3)
\end{aligned}$$

It follows from (1.4), (2.3) and definition 1.4, we have

$$\begin{aligned}
&\phi(p,w_n)=\phi\left(p,J^{-1}\left(\alpha_n Jx_n+(1-\alpha_n)JTz_n\right)\right)=V\left(p,\alpha_n Jx_n+(1-\alpha_n)JTz_n\right)\\
&\leq\alpha_n V(p,Jx_n)+(1-\alpha_n)V(p,JTz_n)\leq\alpha_n\phi(p,x_n)+(1-\alpha_n)\phi(p,z_n)=\phi(p,x_n).
\end{aligned} \qquad (2.4)$$

Hence, for $\forall n\geq 0$, we get $p\in C_{n+1}$, and also $\{x_n\}$ is well defined.

On the other hand, from Lemma 1.3 and $x_n=\Pi_{C_n} x_0$,we have

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \langle x_n - p, Jx_0 - Jx_n\rangle\geq 0, \forall p,z\in C_n. \qquad (2.5)$$

Noting that Lemma 1.3 implies that for all $p\in\Gamma\subseteq C_n$ and $n\geq 1$, we have

$$\phi(x_n,x_0)=\phi\left(\Pi_{C_n} x_0, x_0\right)\leq\phi(p,x_0)-\phi(p,x_n)\leq\phi(p,x_0), \qquad (2.6)$$

then, the sequence $\{\phi(x_n,x_0)\}$ is bounded. And it follows from $\left(\|x_n\|-\|x_0\|\right)^2\leq\phi(x_n,x_0)$ that $\{x_n\}$ is bounded. Furthermore, it is easy to see that $\{y_n\}$ $\{z_n\}$, $\{w_n\}$ are all bounded. Since $x_n=\Pi_{C_n} x_0$ and $x_{n+1}=\Pi_{C_{n+1}} x_0\in C_{n+1}\subset C_n$, we have $\phi(x_n,x_0)\leq\phi(x_{n+1},x_0)$, $\forall n\in N$. (2.7)

Thus $\{\phi(x_n,x_0)\}$ is nondecreasing. Hence, the limit of $\{\phi(x_n,x_0)\}$ exists. By the construction of C_n, for any positive integer $m\geq n$, one has that $C_m\subset C_n$ and $x_m=\Pi_{C_m} x_0\in C_n$. And

$$\phi(x_m,x_n)=\phi(x_m,\Pi_{C_n}x_0)\le\phi(x_m,x_0)-\phi(\Pi_{C_n}x_0,x_0)=\phi(x_m,x_0)-\phi(x_n,x_0). \qquad (2.8)$$

Letting $m,n\to\infty$ in (2.8), we get $\phi(x_m,x_n)\to 0$. It follows from Lemma 1.4 that $\|x_m-x_n\|\to 0\ (m,n\to\infty)$. That is, $\{x_n\}$ is a Cauchy sequence, then let $\lim_{n\to\infty}x_n=u\in C$.

Next, by the construction of C_{n+1}, we know $\alpha_n\phi(z,x_n)+(1-\alpha_n)\phi(z,z_n)\le\phi(z,x_n)$, for each $\alpha_n\in(0,1)$,one has $\phi(z,z_n)\le\phi(z,x_n)$ .Sinc $x_{n+1}=\Pi_{C_{n+1}}x_0\in C_{n+1}$, we have $\phi(x_{n+1},z_n)\le\phi(x_{n+1},x_n)$.

Noting $\lim_{n\to\infty}\phi(x_{n+1},x_n)=0$ that $\lim_{n\to\infty}\phi(x_{n+1},z_n)=0$, by Lemma 1.4, we have

$$\lim_{n\to\infty}\|x_{n+1}-z_n\|=0. \qquad (2.9)$$

It follows from (1.4), the definition of ϕ and definition 1.4, we have

$$\begin{aligned}\phi(x_n,w_n)&=\phi(x_n,J^{-1}(\alpha_nJx_n+(1-\alpha_n)JTz_n))=V(x_n,\alpha_nJx_n+(1-\alpha_n)JTz_n)\\&\le\alpha_nV(x_n,Jx_n)+(1-\alpha_n)V(x_n,JTz_n)\le\alpha_n\phi(x_n,x_n)+(1-\alpha_n)\phi(x_n,z_n)=(1-\alpha_n)\phi(x_n,z_n).\end{aligned} \qquad (2.10)$$

So $\lim_{n\to\infty}\|x_n-w_n\|=0$. From (2.1) , we know

$$\begin{aligned}\|Jx_{n+1}-Jw_n\|&=\|Jx_{n+1}-\alpha Jx_n-(1-\alpha_n)JTz_n\|=\|\alpha_n(Jx_{n+1}-Jx_n)+(1-\alpha_n)(Jx_{n+1}-JTz_n)\|\\&=\|(1-\alpha_n)(Jx_{n+1}-JTz_n)-\alpha_n(Jx_n-Jx_{n+1})\|\ge(1-\alpha_n)\|Jx_{n+1}-JTz_n\|-\alpha_n\|Jx_n-Jx_{n+1}\|.\end{aligned}$$

Hence $\qquad (1-\alpha_n)\|Jx_{n+1}-JTz_n\|\le(\|Jx_{n+1}-Jw_n\|+\alpha_n\|Jx_n-Jx_{n+1}\|)$.

In view of the condition $\limsup_{n\to\infty}\alpha_n<1$, we have $\lim_{n\to\infty}\|Jx_{n+1}-JTz_n\|=0$. (2.11)

Since J is uniformly norm-to-norm continuous on bounded set, we have $\lim_{n\to\infty}\|x_{n+1}-Tz_n\|=0$ .Since $\|z_n-Tz_n\|\le\|z_n-x_{n+1}\|+\|x_{n+1}-Tz_n\|$, then $\lim_{n\to\infty}\|z_n-Tz_n\|=0$. (2.12)

From (2.9) and (2.12) that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$ .By the definition of and $x_n\to u\,(n\to\infty)$, we have $Tu=u$, which implies that $u\in T_{Fix}$ .Next, let us prove $u\in SGMEP(F,G,A,B)$. From (2.3), we know $\phi(p,z_n)\le\eta_n\phi(p,x_n)+(1-\eta_n)\phi(p,y_n)$, Noting that

$$\phi(u_n,x_n)\le\phi(p,x_n)-\phi(p,u_n)\le\phi(p,x_n)-\phi(p,y_n)\le\frac{1}{1-\eta_n}\left(\|x_n\|^2-\|z_n\|^2+2\langle p,Jz_n-Jx_n\rangle\right),$$

from (2.9), we have $\lim_{n\to\infty}\phi(u_n,x_n)=0$,meaning that $\lim_{n\to\infty}\|u_n-x_n\|=\lim_{n\to\infty}\|y_n-u_n\|=0$.(2.13)

From (2.13)and definition 1.1, we get $F(y,u)+\langle u-y,Ay\rangle\le 0,\ \forall y\in C$.

Let $t\in(0,1]$, $y\in C$, set $y_t=ty+(1-t)u\in C$, we get

$$\begin{aligned}0&=F(y_t,y_t)+\langle y_t-y_t,Ay_t\rangle=F(y_t,ty+(1-t)u)+\langle ty+(1-t)u-y_t,Ay_t\rangle\\&\le t[F(y_t,y)+\langle y-y_t,Ay_t\rangle]+(1-t)[F(y_t,u)+\langle u-y_t,Ay_t\rangle]\le F(y_t,y)+\langle y-y_t,Ay_t\rangle\end{aligned} \qquad (2.14)$$

Let $t\to 0$, then $y_t\to u$. So we get $F(u,y)+\langle y-u,Au\rangle\ge 0$, $\forall y\in C$ .Similarly, we can get that $G(u,y)+\langle y-u,Bu\rangle\ge 0$, $\forall y\in C$. That is, $u\in GMEPS(F,G,A,B)$. Finally, we claim that $u=\Pi_\Gamma x_0$. From Lemma 1.3 and $x_n=\Pi_C x_0$, we get $\langle Jx_0-Jx_n,x_n-z\rangle\ge 0$, $\forall z\in C_n$ .Since $\Gamma\subset C_n$, we have $\langle Jx_0-Jx_n,x_n-p\rangle\ge 0$, $\forall p\in\Gamma$ .Which implies $\langle Jx_0-Ju,u-p\rangle\ge 0$, $\forall p\in\Gamma$.

By again Lemma 1.3, we can conclude that $u=\Pi_\Gamma x_0$, which completes the proof.

Remark 2.1 Theorem 2.1 extend the corresponding results of [1-3].

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